1. (a) Give the $\epsilon - \delta$ definition of f(x) is continuous at x = a.

   (b) Use the definition of limit to evaluate $\lim_{x\to 1} x^3 + 2x - 1$.

   (c) Prove that if $f : \mathbb{R} \to \mathbb{R}$ and the left hand and right hand limit of f(x) at x = a both exist and are equal, then the limit at a exists.

   (d) Prove that if the derivative of f(x) exists at x = a, then f(x) is continuous at x = a. Prove or disprove the converse.

2. (a) Find the quadratic approximation to $\sqrt{37}$, and explain exactly what this approximation is giving you.

   (b) Find $\frac{dy}{dx}$ if

      i. $y = \ln(\sqrt{cosx^3})$

      ii. $xy^2 + x^3y = \cos(xy)$

      iii. $y = e^{x^4} \sin x \ln x \tan x$

      iv. $y = \ln 3t, x = t^3 + t^2$

   (c) Given that the derivative of $y = x^r$ is rx^{r-1}, for r any integer prove the same formula for r any rational. Then defining $\ln x$ as an integral show how one defines x^r for any real number r and derive the same formula.

3. (a) State and prove Rolle's Theorem. State the Mean Value Theorem.

   (b) Solve the following

      i. $(x^2 + 1)\frac{dy}{dx} = y$

      ii. $\frac{dy}{dx} + 3y = \exp 3x$

   (c) A rectangle is to be inscribed inside a semicircle of radius 2 cm. What are the dimensions of the rectangle of largest possible area?

4. (a) Let f(x) be defined on [0,1] as 1 on all rational numbers and 0 on all irrationals. Explain why this function is not Riemann integrable.

   (b) Integrate the following.

      i. $\int x^2 \ln x dx$

ii. $\int x \ln x^2 dx$

iii. $\int \sin^4 x \cos^2 x dx$

iv. $\displaystyle\int \frac{x+1}{x^2+x+1} dx$

v. $\displaystyle\int \frac{x^2+x+1}{(x-1)^2(x-2)} dx$

5. (a) Find the area of the region bounded by $y = x^2$ and $y = x + 1$.

(b) How is $\int_0^1 \frac{1}{x^2} dx$ defined?

(c) Find the volume of the solid of revolution gotten by revolving the region bounded by $y = x^2, x = 0$, and $y = 1$ about the y-axis, first by the method of disks, and then by the method of cylindrical shells.

6. (a) Define $\lim_{n\to\infty} a_n = L$, and $\sum_{n=1}^{\infty} a_n = S$.

(b) Do the following series converge or diverge? Give reasons.

i. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3-2}$

ii. $\displaystyle\sum_{n=1}^{\infty} \frac{n}{2^n}$

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$

(c) Prove that if $\sum_{n=1}^{\infty} a_n = a$ and $\sum_{n=1}^{\infty} b_n = b$ then $\sum_{n=1}^{\infty}(a_n + b_n) = a + b$.

(d) Find for what values of x the following power series converges absolutely, conditionally, or diverges.

$$\sum_{n=1}^{\infty} \frac{(x-3)^n}{n!}$$

MA 1123 Annual Exam 2015.

Q1. (a) $f(x)$ is continuous at $x=a$ means
$\forall \epsilon > 0 \ \exists \delta > 0$ s.t. $|x-a| < \delta$
$\Rightarrow |f(x) - f(a)| < \epsilon$

(b) $\lim_{x \to 1} x^3 + 2x - 1 = 2.$

$x^3 + 2x - 1 - 2 = x^3 + 2x - 3$

$f(1) = 0 \Rightarrow x - 1 \mid x^3 + 2x - 3$

$$\begin{array}{r|l} & x^2 + x + 3 \\ \hline x-1 & x^3 + 2x - 3 \\ & x^3 - x^2 \\ \hline & x^2 + 2x - 3 \\ & x^2 - x \\ \hline & 3x - 3 \end{array}$$

$|x^3 + 2x - 3| = |x-1| |x^2 + x + 3|.$

if $|x-1| < 1$ then $0 < x < 2$

& $|x^2 + x + 3| < 4 + 2 + 3 = 9.$

Let $\delta = \min(\epsilon/9, 1)$. if $|x-1| < \delta$

then $|x-1| |x^2 + 2x - 3| < \epsilon/9 \cdot 9 = \epsilon.$

1 (c) if $\lim_{x \to a^-} f(x) = L = \lim_{x \to a^+} f(x)$

$\lim_{x \to a^-} f(x) = L$ means $\forall \epsilon > 0, \exists \delta_1 > 0$

s.t $0 < a - x < \delta_1 \Rightarrow |f(x) - L| < \epsilon$.

$\lim_{x \to a^+} f(x) = L$ means $\forall \epsilon > 0, \exists \delta_2 > 0$

s.t $0 < x - a < \delta_2 \Rightarrow |f(x) - L| < \epsilon$.

Let $\delta = \min(\delta_1, \delta_2)$ then.

$0 < |x - a| < \delta \Rightarrow |f(x) - L| < \epsilon$

True $\forall \epsilon$, i.e. $\lim_{x \to a} f(x) = L$

(d)

$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$ exists

But $f(a+h) - f(a) = \frac{f(a+h) - f(a)}{h} \cdot h$.

$\therefore \lim_{h \to 0} (f(a+h) - f(a)) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \cdot \lim_{h \to 0} h$

$= f'(a) \cdot 0 = 0$

i.e. $f(x)$ is cont. at $x = a$

But if $f(x) = |x|$, $f(x)$ is cont. at $x = 0$, but $f'(0)$ does not exist.

2 (a) Let $f(x) = \sqrt{x} = x^{1/2}$

Let $x_0 = 36$.

then $f(\cancel{x_0}) \sim f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$

is the quadratic approx.

$f(x_0) = 6 \quad f'(x) = \frac{1}{2} x^{-1/2} \quad f'(x_0) = \frac{1}{12}$

$f''(x) = -\frac{1}{4} x^{-3/2} \quad f''(x_0) = -\frac{1}{4} \cdot \frac{1}{216}$.

So $\sqrt{37} \sim 6 + \frac{1}{12}(1) - \frac{1}{1728}(1)^2$

$= 6.08276$.

(b) (i) $y = \ln \sqrt{\cos x^3}$.

$u = x^3 \qquad \frac{du}{dx} = 3x^2$

$w = \cos u. \qquad \frac{dw}{du} = -\sin u$

$v = \sqrt{\cancel{\cos u}\, w} \qquad \frac{dv}{dw} = \frac{1}{2} w^{-1/2}$

$y = \ln v \qquad \frac{dy}{dv} = \frac{1}{v}$.

$$\frac{dy}{dx} = \frac{dy}{dv} \cdot \frac{dv}{dw} \cdot \frac{dw}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{\sqrt{\cos x^3}} \cdot \frac{1}{2} \frac{1}{\sqrt{\cos x^3}} \cdot -\sin x^3 \cdot 3x^2$$

$$= -\frac{3}{2} x^2 \tan x^3.$$

2 (ii) $$xy^2 + x^3 y = \cos xy$$

$$y^2 + x \cdot 2y\frac{dy}{dx} + 3x^2 y + x^3\frac{dy}{dx} = \sin xy\left(y + x\frac{dy}{dx}\right).$$

$$(2xy + x^3 + x\sin xy)\frac{dy}{dx} = y\sin xy - 3x^2y - y^2$$

$$\frac{dy}{dx} = \frac{x\sin xy - 3x^2 y - y^2}{2xy + x^3 + x\sin xy}.$$

(iii)

$$y = e^{x^4}\sin x \ln x \tan x.$$

$$\frac{dy}{dx} = 4x^3 e^{x^4}\sin x \ln x \tan x$$

$$+ e^{x^4}.\cos x.\ln x.\tan x$$

$$+ e^{x^4}\sin x.\frac{1}{x}\tan x$$

$$+ e^{x^4}\sin x \ln x.\sec^2 x.$$

(iv) $$y = \ln 3t, \quad x = t^3 + t^2$$

$$\frac{dy}{dx} = dy/dt \,/\, dx/dt$$

$$= \frac{3}{3t} \,/\, 3t^2 + 2t$$

$$= \frac{1}{3t^3 + 2t^2}$$

2 (c) $y = x^{n/m} \iff y^m = x^n \quad m, n \in \mathbb{Z}$

$$m y^{m-1} \frac{dy}{dx} = n x^{n-1}$$

$$\frac{dy}{dx} = \frac{n}{m} x^{n-1} (x^{n/m})^{1-m}$$

$$= \frac{n}{m} x^{n-1} x^{n/m - n}$$

$$= \frac{n}{m} x^{n/m - 1}$$

Defining $\ln x = \int_1^x \frac{1}{t}\, dt \quad x > 0$

then $\frac{d(\ln x)}{dx} = \frac{1}{x}$

Defining $\exp x = \ln^{-1} x$, the inverse function

then if $e = \exp 1$ we can show that $e^x = \exp x$, all rational x

So we define $e^x = \exp x$, <u>all x</u>

then $\frac{d}{dx}(e^x) = e^x$.

Next if $a > 0$ we define

$$a^x = e^{x \ln a}$$

Now to differentiate x^a, a real $x > 0$

$$x^a = e^{a \ln x}$$

$$\frac{d}{dx}(x^a) = e^{a \ln x} \cdot \frac{d}{dx}(a \ln x)$$

$$= e^{a \ln x} \cdot \frac{a}{x} = a \frac{x^a}{x} = a x^{a-1}$$

3 (a) Rolle's Theorem. Let $f(x)$ be cont. on $[a, b]$ and differentiable on (a, b), then if $f(a) = f(b) = 0$ $\exists$ $a < c < b$ s.t $f'(c) = 0$.

Proof. Case 1 $f(x) = 0$ $\forall$ c in (a, b).

$\Rightarrow f(x) = \text{constant}$

$\Rightarrow f'(c) = 0$ $\forall$ c in (a, b)

Case 2 $\exists$ d in (a, b) with $f(d) > 0$.

Since f cont. on $[a, b]$ the extreme value theorem f has an absolute max. on $[a, b]$. Since $f(a) = f(b) = 0$ + $f(d) > 0$ this absol. max. cannot be at a or b. Say absol. max is at c, then by an earlier theorem c is a critical point. Then since $f'(c)$ exists we must have $f'(c) = 0$.

Case 3 $\exists$ d in (a, b) with $f(d) < 0$. Now $\exists$ absol. min. etc ....

Mean Value Theorem. Let $f(x)$ be cont. on $[a, b]$ and differentiable on (a, b) then $\exists$ $a < c < b$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

3 (b) (i) $(x^2+1)\frac{dy}{dx} = y.$

Separate the variables $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x^2+1}$

$$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{x^2+1}\,dx$$

$$\int \frac{1}{y}\,dy = \tan^{-1}x + c.$$

$$\ln y = \tan^{-1}x + c.$$

$$y = e^{\tan^{-1}x + c}$$

$$= Ae^{\tan^{-1}x}$$

(ii) $\frac{dy}{dx} + 3y = e^{3x}.$

Find the integrating factor

$$\mu = e^{\int P(x)\,dx}$$

$$= e^{\int 3\,dx} = e^{3x}.$$

$$e^{3x}\frac{dy}{dx} + 3e^{3x}y = e^{6x}.$$

$$\frac{d(e^{3x}y)}{dx} = e^{6x}.$$

$$e^{3x}y = \frac{1}{6}e^{6x} + c.$$

$$y = \frac{1}{6}e^{3x} + ce^{-3x}$$

3 (c)

$$A(x) = 2x\sqrt{4-x^2}$$

$$\frac{dA}{dx} = 2\sqrt{4-x^2} + \frac{2x}{\sqrt{4-x^2}} \cdot \tfrac{1}{2} \cdot -2x.$$

$$= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} = 0$$

$$2(4-x^2) - 2x^2 = 0$$

$$8 = 4x^2$$

$$2 = x^2$$

$$\sqrt{2} = x$$

length $= 2\sqrt{2}$.
height $= \sqrt{2}$.

4 (a) $f(x) = 1$ if x is rational
$= 0$ if x is irrational.

Take any partition and Riemann sum $\sum_{i=1}^{n} f(x_i')(x_i - x_{i-1})$.

then if we choose all x_i' rational.
we get $\sum (x_i - x_{i-1}) = 1 = (1-0)$.
and if we choose all x_i' irrational
we get sum $= 0$. So there is
no limit as $\|P\| \to 0$.

4 (b) (i) $\int x^2 \ln x \, dx$

$u = \ln x \quad \frac{dv}{dx} = x^2$

$\frac{du}{dx} = \frac{1}{x} \quad v = \frac{1}{3}x^3.$

$uv = \int u \frac{dv}{dx} dx + \int v \frac{du}{dx} dx.$

$\frac{1}{3}x^3 \ln x = \int x^2 \ln x \, dx + \int \frac{1}{3} x^2 \, dx.$

$\int x^2 \ln x \, dx = \frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 + C.$

(ii) $\int x \ln x^2 \, dx$

Let $u = x^2$

$\frac{du}{dx} = 2x$

$\int \frac{1}{2} \frac{du}{dx} . \ln u \, dx$

$= \frac{1}{2} \int \ln u \, du$

$=$

$f = \ln u \quad \frac{dg}{du} = 1.$

$\frac{df}{du} = \frac{1}{u} \quad g = u.$

$u \ln u = \int \ln u \, du + \int 1 . du.$

$\int \ln u \, du = u \ln u - u + C_1.$

$\int x \ln x^2 \, dx = \frac{1}{2}(x^2 \ln x^2 - x^2) + C_2.$

4 (iv) $\displaystyle\int \frac{x+1}{x^2+x+1}\,dx.$

$$\frac{x+1}{x^2+x+1} = \frac{x+1}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{x+\frac{1}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}} + \frac{\frac{1}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}}$$

Let $u = x+\frac{1}{2}$

$\frac{du}{dx} = 1.$

$$\int \frac{u}{u^2+\frac{3}{4}} \cdot \frac{du}{dx}\,dx + \frac{1}{2}\int \frac{1}{u^2+\frac{3}{4}}\,\frac{du}{dx}\,dx$$

$v = u^2+\frac{3}{4}$

$\frac{dv}{du} = 2u.$

$$+\frac{1}{2}\int \frac{\frac{4}{3}}{(\frac{2}{\sqrt{3}}u)^2+1}\,du.$$

$v = \frac{2}{\sqrt{3}}u.\quad \frac{dv}{du} = \frac{2}{\sqrt{3}}.$

$$\int \frac{\frac{1}{2}\frac{dv}{du}}{v}\,du + \frac{1}{2}\int \frac{\frac{2}{\sqrt{3}}\cdot\frac{dv}{du}}{v^2+1}\,du$$

$$\frac{1}{2}\ln v + \frac{1}{2}\cdot\frac{2}{\sqrt{3}}\tan^{-1} v$$

$$\frac{1}{2}\ln(u^2+\tfrac{3}{4}) + \frac{1}{\sqrt{3}}\tan^{-1}(\tfrac{2}{\sqrt{3}}u).$$

$$= \frac{1}{2}\ln((x+\tfrac{1}{2})^2+\tfrac{3}{4}) + \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{2}{\sqrt{3}}(x+\tfrac{1}{2})\right)$$

4 (b) (iii)

$$\int \sin^4 x \cos^2 x \, dx.$$

$$\sin^2 x = \frac{1}{2}(1 - \cos 2x).$$

$$\cos^2 x = \frac{1}{2}(1 + \cos 2x)$$

$$\frac{1}{8}\int (1-\cos 2x)^2(1+\cos 2x)\,dx$$

$$= \frac{1}{8}(1 - \cos^2 2x)(1 - \cos 2x)$$

$$= \frac{1}{8}(1 - \cos^2 2x - \cos 2x + \cos^3 2x]$$

$$= \frac{1}{8}\left(1 - \left(\frac{1}{2}(1 + \cos 4x)\right] - \cos 2x + \cos^3 2x\right].$$

$$\frac{1}{8}\left[\int \frac{1}{2}\,dx - \frac{1}{2}\int \cos 4x\,dx - \int \cos 2x\,dx + \int \cos^3 2x\,dx\right]$$

$u = \sin 2x.$

$\frac{1}{2}\int (1-u^2)\frac{du}{dx}\,dx.$

$$= \frac{1}{8}\left[\frac{1}{2}x - \frac{1}{8}\sin 4x - \frac{1}{2}\sin 2x + \frac{1}{2}\left(u - \frac{u^3}{3}\right)\right].$$

$$= \frac{1}{16}x - \frac{1}{64}\sin 4x - \frac{1}{16}\sin 2x + \frac{1}{16}\sin 2x - \frac{1}{48}\sin^3 2x + C.$$

$$= \frac{x}{16} - \frac{\sin 4x}{64} - \frac{\sin^3 2x}{48} + C.$$

4 (v) $$\frac{x^2+x+1}{(x-1)^2(x-2)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{C}{x-2}$$

$$x^2+x+1 = A(x-1)(x-2) + B(x-2) + C(x-1)^2$$

$x=2$ $\quad 7 = C$.

$x=1$ $\quad 3 = -B$.

$x=0$ $\quad 1 = 2A - 2B + C$

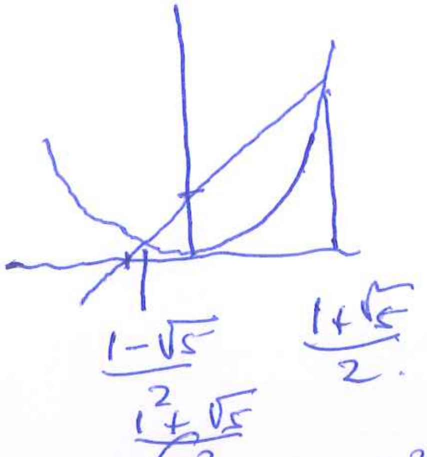
$1 = 2A + 6 + 7$

$-12 = 2A \quad A = -6$.

$$\int \frac{x^2+x+1}{(x-1)^2(x-2)}\,dx = -6\int \frac{1}{x-1}\,dx - 3\int \frac{1}{(x-1)^2}\,dx + 7\int \frac{1}{x-2}\,dx$$

$$= -6\ln|x-1| + 3\frac{1}{(x-1)} + 7\ln|x-2| + C.$$

5 (a)



$x^2 = x+1$.

$x^2 - x - 1 = 0$.

$x = \frac{1 \pm \sqrt{5}}{2}$

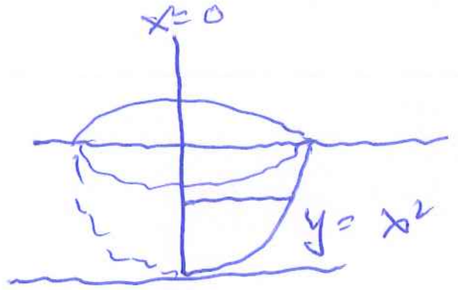
$$\text{Area} = \int_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}} x+1-x^2\,dx = \left.\frac{x^2}{2} + x - \frac{x^3}{3}\right|_{\frac{1-\sqrt{5}}{2}}^{\frac{1+\sqrt{5}}{2}}$$

$$= \frac{5\sqrt{5}}{6}.$$

5 (b) Since $\frac{1}{x^2}$ is not bounded on $(0,1)$ the Riemann integral does not exist. So we define the improper integral

$$\int_0^1 \frac{1}{x^2}\,dx = \lim_{t\to 0^+} \int_t^1 \frac{1}{x^2}\,dx$$

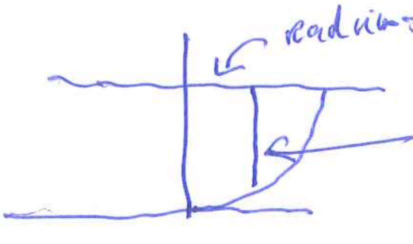
(c)



Disks – circle radius $x=\sqrt{y}$.

$$\int_0^1 \pi(\sqrt{y})^2\,dy = \pi \int_0^1 y\,dy$$

$$= \pi \frac{y^2}{2}\Big|_0^1 = \frac{\pi}{2}.$$



Cylindrical shells

$$\int_0^1 2\pi x(1-x^2)\,dx$$

$$= 2\pi \int_0^1 x - x^3\,dx$$

$$= 2\pi \left.\frac{x^2}{2} - \frac{x^4}{4}\right|_0^1$$

$$= 2\pi\left(\frac{1}{2}-\frac{1}{4}\right) = \pi/2$$

6 (a) $\lim_{n\to\infty} a_n = L$ means $\forall \epsilon > 0$

$\exists N$ s.t $n \geq N \rightarrow |a_n - L| < \epsilon$.

$\sum_{n=1}^{\infty} a_n = S$ means if $S_k = \sum_{n=1}^{k} a_n$

then $\lim_{k\to\infty} S_k = S$.

(b) (i) $a_n = \dfrac{n^2+n+1}{n^3-2}$ compare with $\frac{1}{n}$.

$$\frac{\frac{n^2+n+1}{n^3-2}}{\frac{1}{n}} = \frac{n^3+n^2+n}{n^3-2} \rightarrow 1 \text{ as } n\to\infty$$

So $\sum a_n$ does the same thing as $\sum \frac{1}{n}$ by the Limit Comparison Test.

But we know $\sum \frac{1}{n}$ diverges, hence $\sum \frac{n^2+n+1}{n^3-2}$ diverges.

(ii) $a_n = \dfrac{n}{2^n}$ $\quad a_{n+1} = \dfrac{n+1}{2^{n+1}}$

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n} \cdot \frac{1}{2} \rightarrow \frac{1}{2}$$

So $\sum \frac{n}{2^n}$ converges by the Ratio Test.

6 (b)(iii) $a_n = \frac{1}{\sqrt{n}} = \frac{1}{n^{1/2}}$.

We know from the integral test that $\frac{1}{n^p}$ diverges if $p \leq 1$.

So $\sum \frac{1}{\sqrt{n}}$ diverges

(c) given $\sum_{n=1}^{\infty} a_n = a$ so $\sum_{n=1}^{k} a_n \to a$ as $k \to \infty$

$\sum b_n = b$ so $\sum_{n=1}^{k} b_n \to b$ as $k \to \infty$

Hence $\sum_{n=1}^{k} (a_n + b_n) = \sum_{n=1}^{k} a_n + \sum_{n=1}^{k} b_n$

$\to a + b$ by the theorem on the limit of a sum

(d) $\sum \frac{(x-3)^n}{n!}$ $a_n = \frac{(x-3)^n}{n!}$

$a_{n+1} = \frac{(x-3)^{n+1}}{(n+1)!}$.

$$\frac{|a_{n+1}|}{|a_n|} = \frac{|x-3|}{n+1} \to 0 \ \forall x.$$

Hence by the Ratio Test

$\sum \frac{(x-3)^n}{n!}$ converges absolutely all x