MA1123 April 2013

1. (a) Define function, and state the vertical and horizontal line tests, if $f : \mathbb{R} \to \mathbb{R}$

   (b) Define what it means for f(x) :$\mathbb{R} \to \mathbb{R}$ to be continuous at x = a. What is a jump discontinuity?

   (c) Use the definition of limit to show $\lim_{x \to 2} x^2 = 4$

   (d) Let $f(x) = |x|$, find $f'(0)$ or show that it does not exist.

2. (a) Find the quadratic approximation to $\sqrt{8.9}$

   (b) Find $\frac{dy}{dx}$ if

      i. $y = \ln \cos^2(x^3 + 2)$

      ii. $y = x^3 \ln x \cos x \exp x$

      iii. $x^2 y + y^3 x + \cos(xy) = xy$

      iv. $x = \ln t^2, y = \cos(t^3 + t)$

   (c) Let $f(x) = x^3 - 2x^2 - 4x + 8$. Find where $f(x)$ is increasing, decreasing, concave up, concave down, has local extrema, and points of inflection. Use this information to sketch the function.

3. (a) State Rolle's Theorem and the Mean Value Theorem.

   (b) Use the Mean Value Theorem to prove that if $f'(x) = g'(x)$, for all x, then $f(x) = g(x)$ + constant

   (c) A circular swimming pool has a twelve foot radius, and can be filled to a maximum depth of 4 feet. Water enters the pool at a rate of 9 gallons per minute. How quickly is the depth of the water in the pool changing? How long does it take to fill the pool?

4. (a) State how $\int_a^b f(x)dx$ is defined in terms of Riemann Sums

   (b) Integrate the following.

      i. $\int \ln x dx$

      ii. $\int x \cos x^2 dx$

      iii. $\int x \cos x dx$

iv. $\displaystyle\int \frac{1}{x^2+x+1}dx$

v. $\displaystyle\int \frac{x}{(x-1)(x-2)}dx$

vi. $\displaystyle\int \frac{1}{(x-1)^2(x^2+1)}dx$

5. (a) Find the area of the region bounded by $y = \sin x$, $y = \cos x$ between $x = 0$ and $x = \pi$.

   (b) What is an improper integral?

   (c) Find the volume of the solid of revolution gotten by revolving the region bounded by $y = x^3, y = 0$, and $x = 1$ about the x-axis, first by the method of disks, and then by the method of cylindrical shells.

6. (a) Define $\lim_{n\to\infty} a_n = L$, and $\sum_{n=1}^{\infty} a_n = S$.

   (b) Prove that if $\lim_{n\to\infty} a_n = L_1$ and $\lim_{n\to\infty} b_n = L_2$, then $\lim_{n\to\infty}(a_n + b_n) = L_1 + L_2$.

   (c) Do the following series converge or diverge? Give reasons.

   i. $\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$

   ii. $\sum_{n=1}^{\infty} \frac{n-1}{n^2+2n-1}$

   iii. $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$

   iv. $\sum_{n=1}^{\infty} \frac{2^n}{n!}$

Solutions MA1123 April 2013

Q 1 (a) A function $f: \mathbb{R} \to \mathbb{R}$ assigns a ~~unique~~ single element of $\mathbb{R}$ to each value of $\mathbb{R}$

The vertical line test says that if a vertical line ~~there~~ crosses ~~the~~ a curve ~~graph~~ of f more than once then it f is not the graph of a function.

The horizontal line test says that if a horizontal line crosses the graph of a function more than once then the function is not 1-1

(b) $f(x)$ is continuous at $x = a$ means

(i) $\lim_{x \to a} f(x)$ exists.

(ii) $f(a)$ is defined

(iii) $\lim_{x \to a} f(x) = f(a)$

A jump discontinuity at $x = a$ is when $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ exist, but they are not equal.

1. (c) To show given $\epsilon > 0$ $\exists\, \delta > 0$

s.t $0 < |x-2| < \delta \Rightarrow |x^2-4| < \epsilon$.

$$|x^2-4| = |(x-2)(x+2)|$$
$$= |x-2||x+2|$$
$$\leq |x-2|.5 \quad \text{if } |x-2|<1.$$

Given ϵ, let $\delta = \min(\epsilon/5, 1)$.

Then $\cancel{|x^2-4|}$ $|x-2| < \delta$

$$\Rightarrow |x^2-4| < |x-2| \cdot 5$$
$$< \epsilon/5 \cdot 5 \leq \epsilon.$$

(d)

$$f'(0) = \lim_{h\to 0} \frac{|0+h|-|0|}{h}$$
$$= \lim_{h\to 0} \frac{|h|}{h}.$$

But $h > 0$, $|h|/h = +1$

so $\lim_{h\to 0^+} |h|/h = 1$

& $h<0$, $|h|/h = -1$

So $\lim_{h\to 0^-} \frac{|h|}{h} = -1$

So $f'(0) = \lim_{h\to 0} \frac{|h|}{h}$ does not exist.

2 (a) Find $\sqrt{8.9}$ by quadratic approx.

Let $f(x) = \sqrt{x}$, $x_0 = 9$.

$$f(x) \approx f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2}(x-x_0)^2$$

$f'(x) = \frac{1}{2} x^{-1/2}$ $\quad f'(9) = \frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$

$f''(x) = \frac{1}{2} \cdot -\frac{1}{2} \cdot x^{-3/2}$ $\quad f''(9) = -\frac{1}{4} \cdot \frac{1}{27} = -\frac{1}{108}$.

So $f(8.9) = 3 + \frac{1}{6}(-.1) - \frac{1}{108} \cdot \frac{1}{2}(-.1)^2$.

$= 3 - 0.016667 \;\; 0.000046$

$= \boxed{\cancel{2.983426}}$ Correct to 6 d.p.?

(b) (i) $y = \ln \cos^2(x^3+2)$.

$u = x^3+2$

$v = \cos x^3+2 = \cos u$

$w = \cos^2(x^3+2) = \cancel{\cos^2 u}\ v^2$

$y = \ln \cos^2 x^3+2) = \ln w$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

$$= \frac{1}{w} \cdot 2v \cdot -\sin u \cdot 3x^2$$

$$= \frac{-6\cos(x^3+2)\sin(x^3+2) \cdot x^2}{\cos^2(x^3+2)}$$

$$= -6x^2 \tan(x^3+2).$$

2 (b)(ii) $y = x^3 \ln x \cos x \exp x.$ 4

$$\frac{dy}{dx} = 3x^2 \ln x \cos x \exp x + x^3 \cdot \frac{1}{x} \cos x \exp x$$
$$+ x^3 \ln x (-\sin x) \exp x + x^3 \ln x \cos x \exp x.$$

$$= \exp x \left(3x^2 \ln x \cos x + x^2 \cos x + x^3 \ln x \sin x \right.$$
$$\left. + x^3 \ln x \cos x\right).$$

$$= x^2 \exp x (3 \ln x \cos x + \cos x - x \ln x \sin x + x \ln x \cos x).$$

(iii) $$x^2 y + y^3 x + \cos(xy) = xy.$$

$$2xy + x^2 \frac{dy}{dx} + 3y^2 \frac{dy}{dx} \cdot x + y^3 + \sin xy \left(y + x\frac{dy}{dx}\right) = y + x\frac{dy}{dx}$$

$$(x^2 + 3y^2\cancel{x} - x \sin xy - x)\frac{dy}{dx} = -(2xy + y^3 - y \sin xy + y)$$

$$\frac{dy}{dx} = -\frac{2xy + y^3 - y \sin xy + y}{x^2 + 3y^2\cancel{x} - x \sin xy - x}.$$

(iv) $x = \ln t^2$ $\qquad$ $y = \cos(t^3 + t)$

$$\frac{dx}{dt} = \frac{1}{t^2} \cdot 2t \qquad \frac{dy}{dt} = -\sin(t^3+t) \cdot 3t^2 + 1$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-(3t^2+1)\sin(t^3+t)}{2/t}$$

$$= -\frac{t}{2}(3t^2+1)\sin(t^3+t).$$

(c) $f(x) = x^3 - 2x^2 - 4x + 8$ 5

$f'(x) = 3x^2 - 4x - 4 = 0.$

$(3x + 2)(x - 2) = 0$

$x = -2/3 \, , 2.$

$f''(x) = 6x - 4 = 0 \quad x = 2/3.$

$f''(x) < 0$ if $6x - 4 < 0$

$6x < 4$

$x < 2/3$ concave down

$f''(x) > 0 \quad x > 2/3$ concave up.

$2/3$ is a point of inflection.

$x < -2/3 \quad f'(x) = (-)(-) = + \quad f(x) \uparrow.$

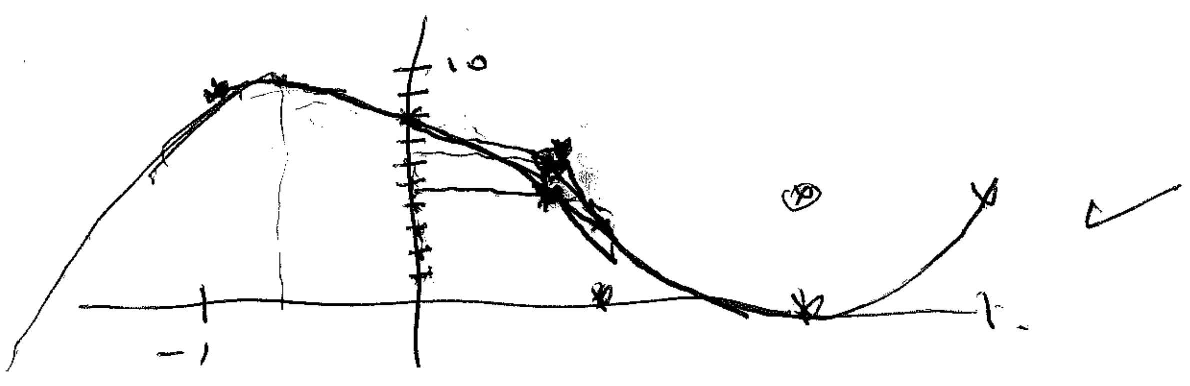
$-2/3 < x < 2 \quad f'(x) = (+)(-) = - \quad f(x) \downarrow$

$2 < x \quad f'(x) = (+)(+) = + \quad f(x) \uparrow$

$\therefore -2/3$ is a local max.

2 is a local min

x	-1	$-2/3$	0	$2/3$	1	2	3.
$f(x)$	9	$9\frac{13}{27}$	8	$4\frac{20}{27}$	3	0	5.



3. (a) Rolle's Theorem

If $f(x)$ is cont. on $[a, b]$ and $f'(x)$ exists on (a, b), then $f(a) = f(b)$ ~~so~~,

$\Rightarrow \exists\ c \quad a < c < b$ s.t. $f'(c) = 0$

Mean Value Theorem

Same hypothesis on f. $\exists\ c$ s.t.

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Let $h(x) = f(x) - g(x)$, then $h'(c) = 0$ all x.

Choose any a + b.

$\exists\ c \quad a < c < b$ s.t

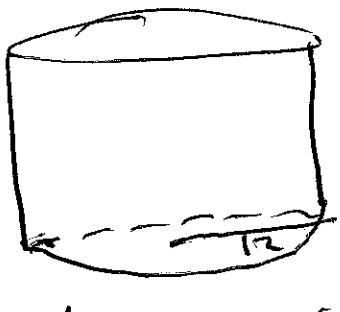
$$h'(c) = \frac{h(b) - h(a)}{b - a}$$

But $h'(c) = 0 \Rightarrow h(b) = h(a)$

$\Rightarrow h = \text{const}$

$\Rightarrow f(x) = g(x) + c.$

(c)



9 cu. ft. minute

Volume $= \pi r^2 h = \pi . 144 . h$

$$\frac{dV}{dt} = 144\pi . \frac{dh}{dt}$$

$$\frac{dh}{dt} = \frac{9}{144\pi} \text{ ft/minute} = \frac{1}{16\pi} \text{ ft/min}$$

want $V = 4 . 144\pi$

$$\text{Time} = \frac{4 . 144\pi}{9} \text{ minutes} = 64\pi \text{ minutes}$$

4. (a) Let P be any partition

$x_0 = a < x_1 < x_2 < \quad < x_n = b$

Let x_i' be any point in $[x_i, x_{i+1}]$.

Then $\sum_{i=1}^{n} f(x_i')(x_{i+1} - x_i)$ is a

Riemann sum. Let $\Delta x_i = x_{i+1} - x_i$

Then $\int_a^b f(x)dx = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(x_i')\Delta x_i$

where $\|P\| = \max_{i=1,n} \Delta x_i$

(b) (i) $\int \ln x \, dx$

$u = \ln x \quad \frac{dv}{dx} = 1$

$\frac{du}{dx} = \frac{1}{x} \quad v = x.$

$\int \ln x . 1 . dx + \int \frac{1}{x} . x dx = x \ln x.$

$\int \ln x \, dx = x \ln x - \ln x$

$= (x-1)(\ln x) . + C.$

(ii) $\int x \cos x^2 \, dx$

Let $u = x^2 \quad \frac{du}{dx} = 2x$

$\int \frac{1}{2} \frac{du}{dx} . \cos u \, dx = \frac{1}{2} \int \cos u \, du$

$= \frac{1}{2} \sin u = \frac{1}{2} \sin^2 x + c$

(vi) $$\frac{1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx+D}{x^2+1}.$$

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$$1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2.$$

Let $x=1$ $\quad 1 = 2B. \quad B = 1/2.$

$x=0$ $\quad 1 = A(-1) + B + \cancel{Cx}+D.$

$1/2 = -A + D$

$\cancel{x^2}$ x^2 term $\quad 0 = A(-1) + B + D - 2C$

$-1/2 = -A - 2C + D.$

x^3 term $\quad 0 = A + C \quad A = -C.$

$$\left.\begin{array}{l} 1/2 = -A + D. \\ -1/2 = \cancel{A} + D. \end{array}\right\} D = 0. \quad A = -1/2.$$

$$\int \frac{1}{(x-1)^2(x^2+1)} = \int \frac{-1/2\,dx}{x-1} + \int \frac{1/2\,dx}{(x-1)^2} + \int \frac{1/2\,x\,dx}{x^2+1}.$$

$$= -1/2 \ln|x-1| + 1/2 \cdot -1 \frac{1}{x-1} + \int \frac{1/2\,x\,dx}{x^2+1}$$

Let $u = x^2+1$

$\frac{du}{dx} = 2x$

$+ 1/4 \int \frac{du/dx}{u}$

$= 1/4 \ln(x^2+1).$

$$= -1/2 \ln|x-1| - 1/2 \frac{1}{x-1} + 1/4 \ln(x^2+1) + C$$

(iii) $\int x \cos x \, dx$

Let $u = x$ $\quad \frac{dv}{dx} = \cos x$

$\frac{du}{dx} = 1 \quad v = \sin x.$

$$\int x \cos x \, dx + \int 1 . \sin x \, dx = x \sin x$$

$$\int x \cos x \, dx = x \sin x + \cos x + C.$$

(iv) $$\frac{1}{x^2+x+1} = \frac{1}{x^2+x+(1/2)^2-(1/2)^2+1}$$

$$= \frac{1}{(x+1/2)^2+3/4}.$$

$$\int \frac{dx}{x^2+x+1} = \int \frac{dx}{(x+1/2)^2+(\sqrt{3}/2)^2}$$

Let $u = x + 1/2 \quad \frac{du}{dx} = 1$

$$= \int \frac{du}{u^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{u}{\sqrt{3}/2}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2}{\sqrt{3}}(x+1/2)\right) + C$$

v

$$\frac{x}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$x = A(x-2) + B(x-1).$$

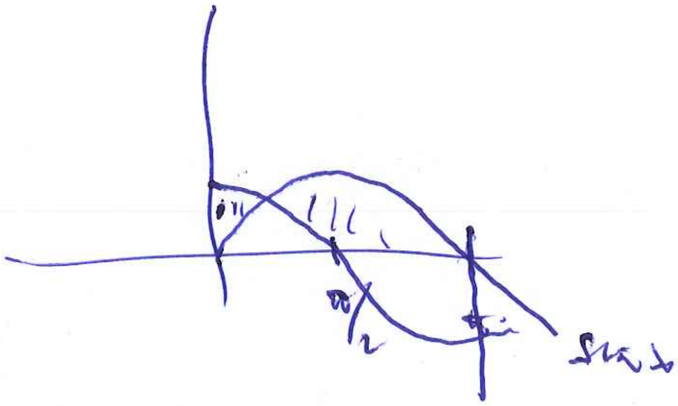
$x = 2 \quad 2 = B. \quad x = 1 \quad 1 = -A.$

$$\int \frac{x}{(x-1)(x+2)} dx = \int -\frac{1}{x-1} dx + 2\int \frac{1}{x+2} dx$$

$$= -\ln|x-1| + 2\ln|x+2|$$

$$= \ln\left(\frac{|x+2|^2}{|x-1|}\right) + C$$

5 (a)



$$\int_0^{\pi/4} \cos x - \sin x\,dx + \int_{\pi/4}^{\pi} \sin x - \cos x\,dx$$

$= \sin x + \cos x \Big|_0^{\pi/4}$

$= \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - (1) + -\cos x - \sin x \Big|_{\pi/4}^{\pi}$

$= \sqrt{2} - 1 + +1 - (-\sqrt{2})$

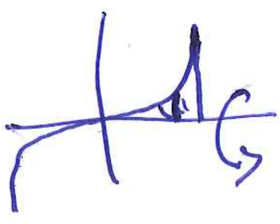
$= \sqrt{2} - 1 + 1 + \sqrt{2} = 2\sqrt{2}.$

(b). An improper integral is used when the Riemann integral does not exist. e.g. $\int_a^{\infty} f(x)\,dx = \lim_{b \to \infty} \int_a^b f(x)\,dx$

or if f(x) is $\lim_{x \to a^+} f(x) = \infty$ then $\int_a^b = \lim_{c \to a^+} \int_c$

etc.

(c)



$$\int_0^1 \pi (x^3)^2\,dx = \pi \frac{x^7}{7}\Big|_0^1 = \pi/7$$

$$\int_0^1 2\pi y\,|1 - y^{1/3}|\,dy$$

$= \int 2\pi y - 2\pi y^{4/3}\,dy$

$= 2\pi\left(\frac{1}{2} - \frac{3}{7}\right) = \pi/7 = 2\pi\left(\frac{y^2}{2} - \frac{y^{7/3}}{7/3}\right)\Big|_0^1$

6 (a) $\lim_{n\to\infty} a_n = L$ means given $\epsilon > 0$

$\exists N$ s.t. $n \geq N \Rightarrow |a_n - L| < \epsilon$

~~Lim~~ $\sum_{n=1}^{\infty} a_n = S$, means that if $S_k = \sum_{n=1}^{k} a_n$,

then $\lim_{k\to\infty} S_k = S$

(b) $\lim_{n\to\infty} a_n = L_1$ means given $\epsilon > 0$ $\exists N_1$

s.t $n \geq N_1 \Rightarrow |a_n - L_1| < \epsilon/2$

Sim $\exists N_2$ s.t $n \geq N_2 \Rightarrow |b_n - L_2| < \epsilon/2$

So if $N = \max(N_1, N_2)$, then $n \geq N$

$$\Rightarrow |a_n + b_n - (L_1 + L_2)| = |a_n - L_1 + b_n - L_2|$$
$$\leq |a_n - L_1| + |b_n - L_2|$$
$$< \epsilon/2 + \epsilon/2 = \epsilon$$

Hence $\lim_{n\to\infty} a_n + b_n = L_1 + L_2$.

(c) (i) $\sum \frac{1}{n\sqrt{n}} = \sum \frac{1}{n^{3/2}}$ converges as

it's a p-series with $p > 1$. This is shown by the integral test.

(ii) ~~$\lim_{n\to\infty} \frac{n-1}{n^2+2n-1} = \lim$~~

$$\lim \frac{\frac{1}{n}}{\frac{n-1}{n^2+2n-1}} = \lim \frac{n^2+2n-1}{n^2-n} = 1$$

$\sum \frac{1}{n}$ diverges, so $\sum \frac{n-1}{n^2+2n-1}$ diverges, 6(c) by the limit comparison test

(iii) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the Alternating Series Test

(iv) $\sum \frac{2^n}{n!}$

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} \to 0$$

Hence $\sum \frac{2^n}{n!}$ converges by the Ratio Test.