1. (a) Define what it means for a function $f : \mathbb{R} \to \mathbb{R}$ to be continuous at a point a. 3

   (b) Prove from the definition of a limit that $f(x) = x^3 + 2$ is continuous at $x = 2$. 3

   (c) Prove that the left hand and right hand limit of $f(x)$ at $x = a$ both exist and are equal, if the limit of $f(x)$ at $x = a$ exists. Is the converse true? 3 1

   (d) Let $f(x) = |x - 2|$, find $f'(2)$ or show that it does not exist. 3

   (e) Prove that the limit of a function is unique. 4

2. (a) Explain the quadratic approximation formula

$$f(x+h) \sim f(x) + f'(x)h + \frac{1}{2}f''(x)h^2. \quad 2$$

   (b) Find $\frac{dy}{dx}$ if

   i. $y = e^{\sin(\sqrt{x^3+2})}$ 2

   ii. $xy^2 + e^{x^2 y} = \sin(xy)$ 2

   iii. $y = e^{x^2}(\sin x)(\ln x)(\cos x)$ 2

   (c) Let $f(x) = \frac{(x+1)^2}{x^2+1}$. Find where $f(x)$ is increasing, decreasing, concave up, concave down, has local extrema, has points of inflection, and find any asymptotes. Use this information to sketch the function.

3. (a) State and prove Rolle's Theorem. 2 4

   (b) Use the Mean Value Theorem to prove that if $f'(x) = g'(x)$, for all x, then $f(x) = g(x)$ + constant 3

   (c) A right angled triangle whose hypotenuse is $\sqrt{3}$ metres long is revolved about one of its legs to generate a circular cone. Find the radius, height, and volume of the largest cone that can be generated in this way. 8

4. (a) Give an example of a function on [0,1] that has no Riemann Integral. And show clearly why it hasn't. 2

(b) Integrate the following.

i. $\int x^2 e^x dx$ 3

ii. $\int x e^{x^2} dx$ 3

iii. $\int (\sin^2 x)(\cos^2 x) dx$ 3

iv. $\displaystyle\int \frac{x^3+x+1}{x^2+x+1} dx$ 3

v. $\displaystyle\int \frac{x}{(x^2+2)(x+2)(x-2)} dx$ 3

5. (a) Find the area of the region bounded by $y = x$, $y = -x + 2\pi$ and $y = \sin x$. 4.

(b) What is an improper integral? 2

(c) Find the volume of the solid of revolution gotten by revolving the region bounded by $y = x^2, y = 0, x = -1$ and $x = 1$ about the x-axis, first by the method of disks, and then by the method of cylindrical shells.
6 5

6. (a) Define $\lim_{n\to\infty} a_n = L$, and $\sum_{n=1}^{\infty} a_n = S$. 2

(b) Do the following series converge or diverge? Give reasons.

i. $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3+2}$ 2

ii. $\displaystyle\sum_{n=2}^{\infty} \frac{n}{\ln n}$ 2

iii. $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n!}$ 2

(c) Prove that if $\sum_{n=1}^{\infty} a_n = S$ then $\lim_{n\to\infty} a_n = 0$. 4

(d) Find for what values of x the following power series converges absolutely, conditionally, or diverges:

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n3^n}.$$ 5.

MA 1123 – April 2014

Solutions

Q 1 (a) $f : \mathbb{R} \to \mathbb{R}$ is continuous at $x = a$

means (i) $f(a)$ is defined

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

(b) Claim $\lim_{x \to 2} x^3 + 2 = 2^3 + 2 = 10$.

ie. $\forall \epsilon > 0 \ \exists \delta > 0$ s.t.

$0 \leq |x-2| < \delta \Rightarrow |x^3+2-10| < \epsilon$.

ie $|x^3 - 8| < \epsilon$

$|x^3 - 8| = |x-2||x^2+2x+4|$

if $|x-2| < 1$, then $1 < x < 3$

and $|x^2+2x+4| \leq 9+6+4$
$= 19$.

So if $|x-2| < 1$ then $|x^3-8| < 19|x-2|$

Let $\delta = \min(1, \epsilon/19)$, then.

$0 \leq |x-2| < \delta \Rightarrow |x^3-8| < \epsilon$

(c) If $\lim_{x \to a} f(x) = L$ exists, then

$\forall \epsilon > 0 \ \exists \delta > 0$ s.t.

$$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \epsilon.$$

ie $a-\delta < x < a+\delta \rightarrow |f(x) - L| < \epsilon.$

in particular

$$0 < a-x < \delta \Rightarrow |f(x) - L| < \epsilon$$

$$\text{ie } \lim_{x \to a^-} f(x) = L$$

and

$$0 < x-a < \delta \Rightarrow |f(x) - L| < \epsilon$$

$$\text{ie. } \lim_{x \to a^+} f(x) = L.$$

Yes the converse is true, if

$$\lim_{x \to a^-} f(x) = L \ \& \ \lim_{x \to a^+} f(x) = L$$

$$\text{then } \lim_{x \to a} f(x) = L.$$

(d) $f(x) = |x-2|$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

~~$f'(2) = \lim_{h \to 0} |x-2+h$~~

$$f'(2) = \lim_{h \to 0} \frac{|2-2+h| - |2-2|}{h}$$

$$= \lim_{h \to 0} \frac{|h|}{h}$$

If $h > 0$ $\frac{|h|}{h} = 1$. So $\lim_{h \to 0^+} = 1$

+ if $h < 0$ $\frac{|h|}{h} = -1$. $\lim_{h \to 0^-} = -1$

Hence $f'(2)$ does not exist.

(e) Suppose $\lim_{x \to a} f(x) = L_1$ + $\lim_{x \to a} f(x) = L_2$

$\lim_{x \to a} f(x) = L_1 \to \forall \epsilon \ \exists \delta$ s.t

$0 < |x-a| < \delta \to |f(x) - L_1| < \epsilon$.

* Let $\epsilon = \frac{|L_1 - L_2|}{3} > 0$ + if $L_1 \neq L_2$.

then $\exists \delta_1$ s.t $0 < |x-a| < \delta_1$

$\to |f(x) - L_1| < \frac{|L_1 - L_2|}{3}$

+ $\exists \delta_2$ s.t $0 < |x-a| < \delta_2$

$\to |f(x) - L_2| < \frac{|L_1 - L_2|}{3}$.

But then if $|x-a| < \min(\delta_1, \delta_2)$

then $|f(x) - L_1| < \frac{|L_1 - L_2|}{3}$

$+ \ |f(x) - L_2| < \frac{|L_1 - L_2|}{3}$.

Then $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2|$

$\leq |L_1 - f(x)| + |f(x) - L_2|$

$< \frac{|L_1 - L_2|}{3} + \frac{|L_1 - L_2|}{3}$

So $|L_1 - L_2| < \frac{2}{3}|L_1 - L_2|$

This is a contradiction, so,

$L_1 = L_2$ from $\star$.

2 (a). The quadratic approximation formula $f(x_0+h) \approx f(x_0) + f'(x_0)h + \frac{f''(x_0)}{2}h^2$ approximates $f(x)$ at (x_0+h) by a quadratic at $x+h$ which equals $f(x)$ at $x = x_0$ and has the same first and second derivatives at $x = x_0$.

2

(b) (i) $y = e^{\sin\sqrt{x^3+2}}$

$u = x^3 + 2 \qquad v = \sqrt{u}$

$w = \sin v \qquad y = e^{w}$

$$\frac{dy}{dx} = \frac{dy}{dw} \cdot \frac{dw}{dv} \cdot \frac{dv}{du} \cdot \frac{du}{dx}$$

[Write it out!]

$$= e^{w} \cdot \cos v \cdot \tfrac{1}{2} u^{-1/2} \cdot 3x^2$$

$$= e^{\sin\sqrt{x^3+2}} \cdot \cos\sqrt{x^3+2} \cdot \tfrac{1}{2}\frac{1}{\sqrt{x^3+2}} \cdot 3x^2$$

(ii) $\underbrace{xy^2}_{} + \underbrace{e^{x^2 y}}_{} = \sin(xy)$

$$y^2 + x \cdot 2y\frac{dy}{dx} + e^{x^2y}\left(2xy + x^2\frac{dy}{dx}\right) = \cos xy\left(y + x\frac{dy}{dx}\right)$$

$$\left(2xy + x^2 e^{x^2y} - x\cos xy\right)\frac{dy}{dx} = -\left(y^2 + e^{x^2y}2xy - y\cos xy\right)$$

$$\frac{dy}{dx} = -\frac{\left(y^2 + e^{x^2y}2xy - y\cos xy\right)}{2xy + x^2 e^{x^2y} - x\cos xy}.$$

(iii) $y = e^{x^2}\sin x \ln x \cos x$

$$\frac{dy}{dx} = 2xe^{x^2}\sin x \ln x \cos x + e^{x^2}\cos x \ln x \cos x$$

$$+ e^{x^2}\sin x \cdot \frac{1}{x} \cdot \cos x + e^{x^2}\sin x \ln x(-\sin x)$$

(c) $f(x) = \dfrac{(x+1)^2}{x^2+1}$.

$$f'(x) = \frac{2(x+1)}{x^2+1} - \frac{(x+1)^2 . 2x}{(x^2+1)^2}$$

$$= \frac{2(x+1)(x^2+1) - (x+1)^2 . 2x}{(x^2+1)^2}$$

$$= \frac{2(-x^2+1)}{(x^2+1)^2} = 0.$$

$$-x^2+1 = 0 \rightarrow x = \pm 1.$$

$$f''(x) = \frac{2(-2x)}{(x^2+1)^2} - \frac{4(-x^2+1) . 2x}{(x^2+1)^3}$$

$$= \frac{-4x(x^2+1) - 4(-2x^3+2x)}{(x^2+1)^3} = 0$$

$$\rightarrow \quad 4x^3 - 12x = 0.$$

$$x = 0, \pm\sqrt{3}.$$

$f''(1) = < 0$ loc max.

$f''(-1) = > 0$ loc min.

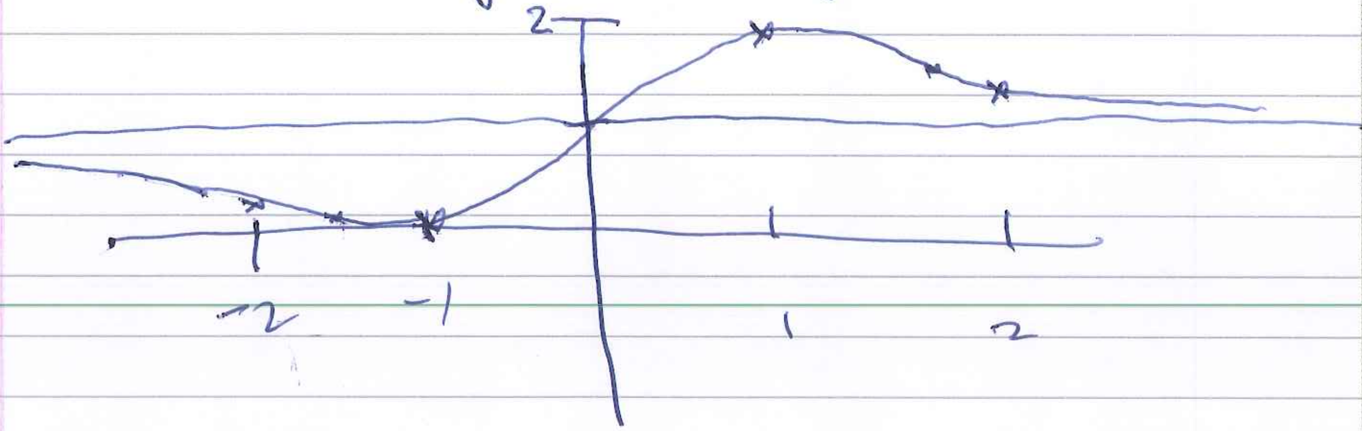
$x =$	-2	$-\sqrt{3}$	-1	0	1	$\sqrt{3}$	2
$f'(x)$	<0		0	>0	0		<0
$f(x)$	$\downarrow$			$\uparrow$			$\downarrow$
$f''(x)$	<0	0	>0	0	<0	0	>0
$f'(x)$	$\downarrow$		$\uparrow$		$\downarrow$		$\uparrow$
$f(x)$	concave down		concave up		concave down		concave up
$f(x)$	$\frac{1}{5}$	$1-\frac{\sqrt{3}}{2} = .134$	0	1	2	$1+\frac{\sqrt{3}}{2} = 1.866$	$\frac{9}{5}$

Note as $x \to -\infty$

$f(x) \to 1$ from below

as $x \to +\infty$

$f(x) \to 1$ from above



3 (a) Rolle's Theorem: If $f(x)$ is continuous on $[a, b]$ and differentiable on (a, b) and if $f(a) = f(b) = 0$ then there exists c, $a < x < c$ s.t. $f'(c) = 0$.

Proof: If $f(x) = 0$ $\forall x \in [a, b]$, then $f'(c) = 0$ $\forall a < c < b$

If $f(x_1) > 0$ some $a < x_1 < b$, we know from the extreme value theorem that $f(x)$ has an absolute maximum on $[a, b]$. Since $f(x_1) > 0$, this absolute max. cannot be at a or b, the endpoints. So by an earlier theorem it must be at a critical point, but $f'(x)$ exists all $a < x < b$, so it must be at a stationary point c with $f'(c) = 0$

If no x_1 has $f(x_1) > 0$, then some x_2 has $f(x_2) < 0$ and a similar argument using the absolute minimum gives $f'(c) = 0$.

(b) If $f'(x) = g'(x)$ $\forall x$, let $F(x) = f(x) - g(x)$

then $F'(x) = 0$ $\forall x$.

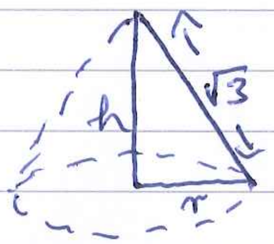
Then $\forall$ a, b by the Mean Value theorem, there exists c with

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

Hence $F(b) - F(a) = 0$ so $F(x) = \text{const}$

$\therefore$ $f(x) = g(x) + \text{constant}$

(c)



$$V = \tfrac{1}{3}\pi r^2 h.$$

$$r^2 + h^2 = 3.$$

$$h = \sqrt{3 - r^2}$$

$$V = \tfrac{1}{3}\pi r^2 \sqrt{3 - r^2}$$

$$\frac{dV}{dr} = \tfrac{1}{3}\pi \cdot 2r \cdot \sqrt{3-r^2} + \tfrac{1}{3}\pi r^2 \cdot \frac{1/2}{\sqrt{3-r^2}} \cdot -2r.$$

$$= 0$$

$$\rightarrow \tfrac{1}{3}\pi\, 2r(3-r^2) + \tfrac{1}{3}\pi r^2 . (-r) = 0$$

$$6r - 2r^3 - r^3 = 0.$$

$$6r = 3r^3$$

$$r = 0 \qquad 6/3 = r^2 \qquad r = \pm\sqrt{2}.$$

$$r = +\sqrt{2} \quad \rightarrow \quad h = \sqrt{3-2} = 1.$$

$$V = \tfrac{1}{3}\pi . 2 . 1 = 2\pi/3.$$

$r = -\sqrt{2}$ is not physical

$r = 0 \rightarrow V = 0$ not a local max,

4(a) If $f(x)$ is unbounded on (a,b) then when we go to form any Riemann sum $\sum_{i=1}^{n} f(x_i') \Delta x_i$, on at least one of the intervals in the partition we can choose $f(x_i')$ to be arbitrarily large & hence no limit of these can exist.

(b) (i) $\int x^2 e^x dx$. Integrate by Parts

$u = x^2$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 2x$ $\quad v = e^x$

$x^2 e^x = \int x^2 e^x dx + \int 2x e^x dx$.

Consider $\int x e^x dx$

$u = x$ $\quad \frac{dv}{dx} = e^x$

$\frac{du}{dx} = 1$ $\quad v = e^x$.

$x e^x = \int x e^x dx + \int e^x dx$

$\therefore \int 2x e^x dx = 2x e^x - e^x$

$\therefore \int x^2 e^x dx = x^2 e^x - 2x e^x + e^x + C$

(ii) $\int x e^{x^2} dx$ Let $u = x^2$

$\frac{du}{dx} = 2x$.

$\int x e^{x^2} dx = \int \frac{1}{2} \frac{du}{dx} \cdot e^u \cdot dx$

$= 1/2 \int e^u du = 1/2 e^u + C$

$= 1/2 e^{x^2} + C$.

(iii) $\int \sin^2 x \cos^2 x \, dx$

$$= \int \tfrac{1}{2}(1-\cos 2x)\,\tfrac{1}{2}(1+\cos 2x)\,dx$$

$$= \tfrac{1}{4}\int 1-\cos^2 2x \, dx$$

$$= \tfrac{1}{4}\int 1-\tfrac{1}{2}(1+\cos 4x)\,dx.$$

$$= \tfrac{1}{4}\int \tfrac{1}{2}-\tfrac{1}{2}\cos 4x \, dx.$$

$$= \tfrac{1}{4}\left(\tfrac{1}{2}x-\tfrac{1}{8}\sin 4x\right)+C.$$

$$= \tfrac{1}{8}x-\tfrac{1}{32}\sin 4x + C.$$

(iv) $\int \frac{x^3+x+1}{x^2+x+1}\,dx$. In $\int \frac{P(x)}{Q(x)}\,dx$

if degree $P(x) \geq$ degree $Q(x)$ we must divide,

$$\begin{array}{r|l} & x-1 \\ \hline x^2+x+1 & x^3+x+1 \\ & x^3+x^2+x \\ \hline & -x^2+1 \\ & -x^2-x-1 \\ \hline & x+2 \end{array}$$

$$\frac{x^2+x+1}{x^2+x+1} = x-1 + \frac{x+2}{x^2+x+1}$$

$$\int x-1 \, dx = \frac{x^2}{2} - x.$$

$$\int \frac{x+2}{x^2+x+1} dx = \int \frac{x+1/2}{x^2+x+1} dx + \int \frac{3/2 \, dx}{x^2+x+1}.$$

We choose $x+1/2$ because with

$$u = x^2+x+1 \qquad \frac{du}{dx} = 2x+1 = 2(x+1/2).$$

So $$\int \frac{x+1/2}{x^2+x+1} dx = \int \frac{1/2 \frac{du}{dx} \cdot dx}{u}$$

$$= 1/2 \int \frac{1}{u} du = 1/2 \ln u$$

$$= 1/2 \ln(x^2+x+1).$$

For $\int \frac{dx}{x^2+x+1}$ we complete the

square $$\frac{1}{x^2+x+1} = \frac{1}{x^2+x+1/4+3/4} = \frac{1}{(x+1/2)^2+3/4}$$

$$u = x+1/2 \qquad \frac{du}{dx} = 1$$

$$\int \frac{1}{u^2+3/4} du = \int \frac{1}{u^2+(\sqrt{3/4})^2} du$$

$$= \frac{1}{\sqrt{3/4}} \tan^{-1}\left(\frac{u}{\sqrt{3/4}}\right)$$

$$= \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right)$$

✓ $\displaystyle\int \frac{x}{(x^2+2)(x+2)(x-2)}\,dx$ Partial Fractions

$$\frac{x}{(x^2+2)(x+2)(x-2)} = \frac{Ax+B}{x^2+2} + \frac{C}{x+2} + \frac{D}{x-2}$$

$$x = (Ax+B)(x+2)(x-2) + C(x^2+2)(x-2)$$
$$+ D(x^2+2)(x+2)$$

$x = 2$

$2 = D \cdot 24 \qquad D = 1/12$

$x = -2$

$-2 = C \cdot -24 \qquad C = 1/12$

$x = 0 \qquad 0 = -4B - 4C + 4D$

But $C = D \rightarrow B = 0$.

Coeff of x^3. $\quad 0 = A + C + D$.

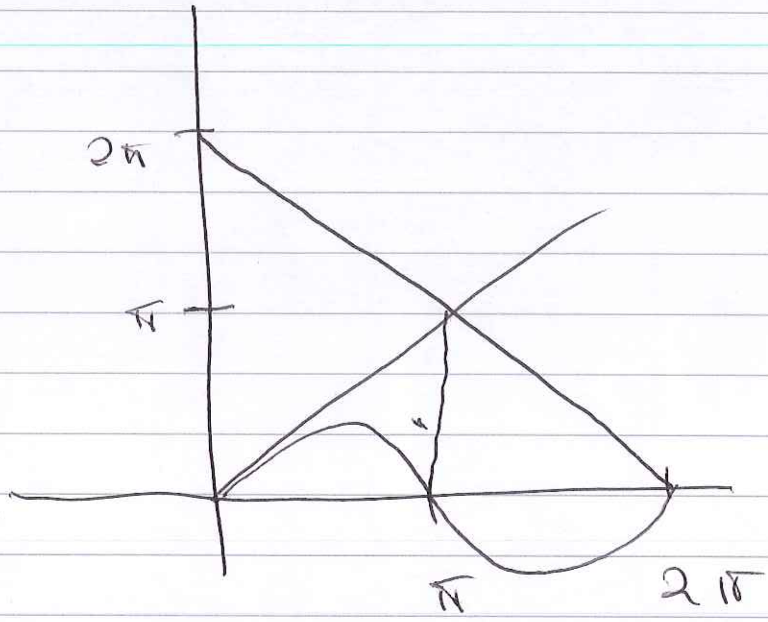
$C = D = 1/12 \rightarrow A = -1/6$

So
$$\int \frac{x}{(x^2+2)(x+2)(x-2)}\,dx = \int \frac{-1/6}{x^2+2}\,dx$$
$$+ \int \frac{1/12}{x+2}\,dx + \int \frac{1/12}{x-2}\,dx$$

$$= -\frac{1}{6}\,\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right) + \frac{1}{12}\ln|x+2|$$
$$+ \frac{1}{12}\ln|x-2|$$

5 (a).

4



$$\text{Area} = \int_0^{\pi} x - \sin x\,dx + \int_{\pi}^{2\pi} -x + 2\pi - \sin x\,dx$$

$$= \frac{x^2}{2} + \cos x \Big|_0^{\pi} + -\frac{x^2}{2} + 2\pi x + \cos x \Big|_{\pi}^{2\pi}$$

$$= \frac{\pi^2}{2} - 1 - (1). + \left[-2\pi^2 + 4\pi^2 + 1\right]$$

$$- \left[-\frac{\pi^2}{2} + 2\pi^2 - 1\right].$$

$$= \frac{\pi^2}{2} - 2 + 2\pi^2 + 1 - \frac{3\pi^2}{2} + 1.$$

$$= \pi^2$$

(b) If an integral is over an infinite integral say $[a, \infty)$, then the Riemann integral does not exist. But it may exist as an improper integral

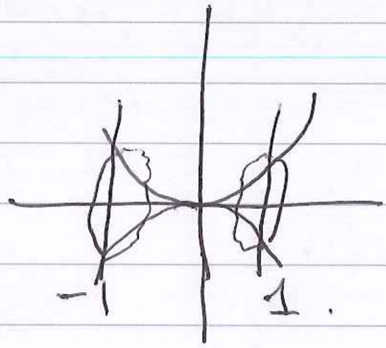
$$\int_a^\infty f(x)\,dx = \lim_{b\to\infty} \int_a^b f(x)\,dx$$

<u>OR</u> if $f(x) \to \infty$ as $x \to a$.

then $\int_a^b f(x)\,dx$ does not exist as a Riemann integral, but it may exist as an improper integral

$$\int_a^b f(x)\,dx = \lim_{c\to a^+} \int_c^b f(x)\,dx.$$

(c)



$y = x^2$

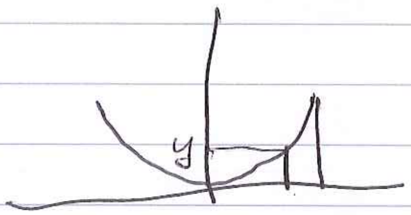
$$\text{Volume} = \int_{-1}^{1} \pi r^2 \, dx$$

$$= \int_{-1}^{1} \pi \cdot x^4 \, dx.$$

$$= \frac{\pi}{5} x^5 \Big|_{-1}^{1} = \frac{2\pi}{5}.$$

By cylindrical shells

Take top half.



$$\int_0^1 2\pi y (1 - \sqrt{y}) \, dy$$

$$= 2\pi \int_0^1 y - y^{3/2} \, dy$$

$$= 2\pi \left[y^2/_2 - \frac{y^{5/2}}{5/_2} \right]_0^1$$

$$= 2\pi \left[\tfrac{1}{2} - \tfrac{2}{5} \right] = 2\pi \cdot \tfrac{1}{10}$$

$$= \frac{\pi}{5}$$

+ Bottom half $= 2\pi/5$.

6 (a) $\lim_{n \to \infty} a_n = L$ means $\forall \epsilon > 0$

$\exists N$ s.t. $n \geq N \to |a_n - L| < \epsilon$.

$\sum_{n=1}^{\infty} a_n = S$ means $\lim_{n \to \infty} S_n = L$

where $S_n = \sum_{k=1}^{n} a_k$

(b) (i) $\sum_{n=1}^{\infty} \frac{n^2+n+1}{n^3+2}$ this goes to 0

like $1/n$, so we

use the limit comparison test

$a_n = \frac{n^2+n+1}{n^3+2}$

$\frac{a_n}{b_n} = \frac{n^3+n^2+n}{n^3+2}$

$b_n = \frac{1}{n}$

$\lim_{n \to \infty} \frac{a_n}{b_n} = 1$

$\sum b_n$ diverges $\to$ $\sum a_n$ diverges

(ii) $\sum_{n=2}^{\infty} \frac{n}{\ln n}$ $a_n = \frac{n}{\ln n} \not\to 0$ as $n \to \infty$

Hence $\sum \frac{n}{\ln n}$ diverges.

(iii) $\sum_{n=1}^{\infty} \frac{2^n}{n!}$ - Try the Ratio Test

$$\frac{a_{n+1}}{a_n} = \frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}} = \frac{2}{n+1} \to 0 \text{ as } n \to \infty$$

Hence by the Ratio Test

$\sum \frac{2^n}{n!}$ converges

(c) $\sum_{n=1}^{\infty} a_n = S$ means $\lim_{n\to\infty} S_n = S$

where $S_n = \sum_{k=1}^{n} a_k$

then $a_n = S_n - S_{n-1}$.

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1}$$

$$= S - S = 0.$$

(d) $\sum \frac{(x-2)^n}{n \cdot 3^n}$ Ratio Test $\frac{|a_{n+1}|}{|a_n|}$

$$= \frac{\frac{|x-2|^{n+1}}{(n+1) \cdot 3^{n+1}}}{\frac{|x-2|^n}{n \cdot 3^n}} = \frac{|x-2|^1}{3} \cdot \frac{n}{n+1} \to \frac{|x-2|}{3}$$

Converges absolutely if $\frac{|x-2|}{3} < 1$, $|x-2| < 3$

$\to -3 < x-2 < 3$

$-1 < x < 5$

Diverges outside this

At $x = -1$ $\quad \sum \frac{(-3)^n}{n \cdot 3^n} = \sum \frac{(-1)^n}{n}$

converges conditionally

at $x = 5$ $\quad \sum \frac{3^n}{n \cdot 3^n} = \sum \frac{1}{n}$

diverges.