## EXERCISE SET 6.5 Graphing Utility [C] CAS

Read §6.5

**1–2** Evaluate the given limit without using L'Hôpital's rule, and then check that your answer is correct using L'Hôpital's rule.

**1.** (a) $\lim_{x\to 2}\dfrac{x^2-4}{x^2+2x-8}$ (b) $\lim_{x\to +\infty}\dfrac{2x-5}{3x+7}$

**2.** (a) $\lim_{x\to 0}\dfrac{\sin x}{\tan x}$ (b) $\lim_{x\to 1}\dfrac{x^2-1}{x^3-1}$

**3–6 True–False** Determine whether the statement is true or false. Explain your answer.

**3.** L'Hôpital's rule does not apply to $\lim_{x\to -\infty}\dfrac{\ln x}{x}$.

**4.** For any polynomial $p(x)$, $\lim_{x\to +\infty}\dfrac{p(x)}{e^x}=0$.

**5.** If $n$ is chosen sufficiently large, then $\lim_{x\to +\infty}\dfrac{(\ln x)^n}{x}=+\infty$.

**6.** $\lim_{x\to 0^+}(\sin x)^{1/x}=0$

**7–43** Find the limits.

**7.** $\lim_{x\to 0}\dfrac{e^x-1}{\sin x}$

**8.** $\lim_{x\to 0}\dfrac{\sin 2x}{\sin 5x}$

**9.** $\lim_{\theta\to 0}\dfrac{\tan\theta}{\theta}$

**10.** $\lim_{t\to 0}\dfrac{te^t}{1-e^t}$

**11.** $\lim_{x\to \pi^+}\dfrac{\sin x}{x-\pi}$

**12.** $\lim_{x\to 0^+}\dfrac{\sin x}{x^2}$

**13.** $\lim_{x\to +\infty}\dfrac{\ln x}{x}$

**14.** $\lim_{x\to +\infty}\dfrac{e^{3x}}{x^2}$

**15.** $\lim_{x\to 0^+}\dfrac{\cot x}{\ln x}$

**16.** $\lim_{x\to 0^+}\dfrac{1-\ln x}{e^{1/x}}$

**17.** $\lim_{x\to +\infty}\dfrac{x^{100}}{e^x}$

**18.** $\lim_{x\to 0^+}\dfrac{\ln(\sin x)}{\ln(\tan x)}$

**19.** $\lim_{x\to +\infty} xe^{-x}$

**20.** $\lim_{x\to \pi^-}(x-\pi)\tan\frac{1}{2}x$

**21.** $\lim_{x\to +\infty} x\sin\dfrac{\pi}{x}$

**22.** $\lim_{x\to 0^+}\tan x\ln x$

**23.** $\lim_{x\to \pi/2^-}\sec 3x\cos 5x$

**24.** $\lim_{x\to \pi}(x-\pi)\cot x$

**25.** $\lim_{x\to +\infty}(1-3/x)^x$

**26.** $\lim_{x\to 0}(1+2x)^{-3/x}$

**27.** $\lim_{x\to 0}(e^x+x)^{1/x}$

**28.** $\lim_{x\to +\infty}(1+a/x)^{bx}$

**29.** $\lim_{x\to 1}(2-x)^{\tan[(\pi/2)x]}$

**30.** $\lim_{x\to +\infty}[\cos(2/x)]^{x^2}$

**31.** $\lim_{x\to 0}(\csc x-1/x)$

**32.** $\lim_{x\to 0}\left(\dfrac{1}{x^2}-\dfrac{\cos 3x}{x^2}\right)$

**33.** $\lim_{x\to +\infty}(\sqrt{x^2+x}-x)$

**34.** $\lim_{x\to 0}\left(\dfrac{1}{x}-\dfrac{1}{e^x-1}\right)$

**35.** $\lim_{x\to +\infty}[x-\ln(x^2+1)]$

**36.** $\lim_{x\to +\infty}[\ln x-\ln(1+x)]$

**37.** $\lim_{x\to 0^+} x^{\sin x}$

**38.** $\lim_{x\to 0^+}(e^{2x}-1)^x$

**39.** $\lim_{x\to 0^+}\left[-\dfrac{1}{\ln x}\right]^x$

**40.** $\lim_{x\to +\infty} x^{1/x}$

**41.** $\lim_{x\to +\infty}(\ln x)^{1/x}$

**42.** $\lim_{x\to 0^+}(-\ln x)^x$

**43.** $\lim_{x\to \pi/2^-}(\tan x)^{(\pi/2)-x}$

**44.** Show that for any positive integer $n$

(a) $\lim_{x\to +\infty}\dfrac{\ln x}{x^n}=0$ (b) $\lim_{x\to +\infty}\dfrac{x^n}{\ln x}=+\infty$.

### FOCUS ON CONCEPTS

**45.** (a) Find the error in the following calculation:

$$\lim_{x\to 1}\frac{x^3-x^2+x-1}{x^3-x^2}=\lim_{x\to 1}\frac{3x^2-2x+1}{3x^2-2x}=\lim_{x\to 1}\frac{6x-2}{6x-2}=1$$

(b) Find the correct limit.

**46.** (a) Find the error in the following calculation:

$$\lim_{x\to 2}\frac{e^{3x^2-12x+12}}{x^4-16}=\lim_{x\to 2}\frac{(6x-12)e^{3x^2-12x+12}}{4x^3}=0$$

(b) Find the correct limit.

**47–50** Make a conjecture about the limit by graphing the function involved with a graphing utility; then check your conjecture using L'Hôpital's rule.

**47.** $\lim_{x\to +\infty}\dfrac{\ln(\ln x)}{\sqrt{x}}$

~~**48.**~~ $\lim_{x\to 0^+} x^x$

**49.** $\lim_{x\to 0^+}(\sin x)^{3/\ln x}$

**50.** $\lim_{x\to (\pi/2)^-}\dfrac{4\tan x}{1+\sec x}$

**51–54** Make a conjecture about the equations of horizontal asymptotes, if any, by graphing the equation with a graphing utility; then check your answer using L'Hôpital's rule.

**51.** $y=\ln x-e^x$

**52.** $y=x-\ln(1+2e^x)$

**53.** $y=(\ln x)^{1/x}$

**54.** $y=\left(\dfrac{x+1}{x+2}\right)^x$

**55.** Limits of the type

$$0/\infty,\quad \infty/0,\quad 0^\infty,\quad \infty\cdot\infty,\quad +\infty+(+\infty),$$
$$+\infty-(-\infty),\quad -\infty+(-\infty),\quad -\infty-(+\infty)$$

are *not* indeterminate forms. Find the following limits by inspection.

(a) $\lim_{x\to 0^+}\dfrac{x}{\ln x}$ (b) $\lim_{x\to +\infty}\dfrac{x^3}{e^{-x}}$

(c) $\lim_{x\to (\pi/2)^-}(\cos x)^{\tan x}$ (d) $\lim_{x\to 0^+}(\ln x)\cot x$

(e) $\lim_{x\to 0^+}\left(\dfrac{1}{x}-\ln x\right)$ (f) $\lim_{x\to -\infty}(x+x^3)$

$x_1 e^{-x_1} = x_2 e^{-x_2}$

$x_1 \neq 0$

$\Rightarrow \ln x_1 - x_1 = \ln x_2 - x_2$



56. There is a myth that circulates among beginning calculus students which states that all indeterminate forms of types $0^0$, $\infty^0$, and $1^\infty$ have value 1 because "anything to the zero power is 1" and "1 to any power is 1." The fallacy is that $0^0$, $\infty^0$, and $1^\infty$ are not powers of numbers, but rather descriptions of limits. The following examples, which were suggested by Prof. Jack Staib of Drexel University, show that such indeterminate forms can have any positive real value:
(a) $\lim_{x\to 0^+} [x^{(\ln a)/(1+\ln x)}] = a$ (form $0^0$)
(b) $\lim_{x\to +\infty} [x^{(\ln a)/(1+\ln x)}] = a$ (form $\infty^0$)
(c) $\lim_{x\to 0} [(x+1)^{(\ln a)/x}] = a$ (form $1^\infty$).
Verify these results.

57–60 Verify that L'Hôpital's rule is of no help in finding the limit; then find the limit, if it exists, by some other method.

7. $\lim_{x\to+\infty} \dfrac{x+\sin 2x}{x}$

58. $\lim_{x\to+\infty} \dfrac{2x-\sin x}{3x+\sin x}$

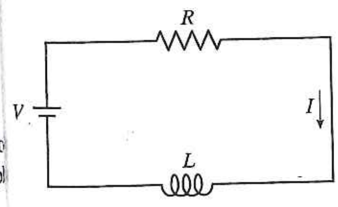
9. $\lim_{x\to+\infty} \dfrac{x(2+\sin 2x)}{x+1}$

60. $\lim_{x\to+\infty} \dfrac{x(2+\sin x)}{x^2+1}$

1. The accompanying schematic diagram represents an electrical circuit consisting of an electromotive force that produces a voltage $V$, a resistor with resistance $R$, and an inductor with inductance $L$. It is shown in electrical circuit theory that if the voltage is first applied at time $t=0$, then the current $I$ flowing through the circuit at time $t$ is given by

$$I = \frac{V}{R}(1 - e^{-Rt/L})$$

What is the effect on the current at a fixed time $t$ if the resistance approaches 0 (i.e., $R \to 0^+$)?




◀ Figure Ex-61

(a) Show that $\lim_{x\to\pi/2} (\pi/2 - x)\tan x = 1$.
(b) Show that

$$\lim_{x\to\pi/2}\left(\frac{1}{\pi/2 - x} - \tan x\right) = 0$$

(c) It follows from part (b) that the approximation

$$\tan x \approx \frac{1}{\pi/2 - x}$$

should be good for values of $x$ near $\pi/2$. Use a calculator to find $\tan x$ and $1/(\pi/2 - x)$ for $x = 1.57$; compare the results.

(a) Use a CAS to show that if $k$ is a positive constant, then

$$\lim_{x\to+\infty} x(k^{1/x} - 1) = \ln k$$

(b) Confirm this result using L'Hôpital's rule. [*Hint:* Express the limit in terms of $t = 1/x$.]
(c) If $n$ is a positive integer, then it follows from part (a) with $x = n$ that the approximation

$$n(\sqrt[n]{k} - 1) \approx \ln k$$

should be good when $n$ is large. Use this result and the square root key on a calculator to approximate the values of $\ln 0.3$ and $\ln 2$ with $n = 1024$, then compare the values obtained with values of the logarithms generated directly from the calculator. [*Hint:* The $n$th roots for which $n$ is a power of 2 can be obtained as successive square roots.]

64. Find all values of $k$ and $l$ such that

$$\lim_{x\to 0}\frac{k+\cos lx}{x^2} = -4$$

**FOCUS ON CONCEPTS**

65. Let $f(x) = x^2\sin(1/x)$.
(a) Are the limits $\lim_{x\to 0^+} f(x)$ and $\lim_{x\to 0^-} f(x)$ indeterminate forms?
(b) Use a graphing utility to generate the graph of $f$, and use the graph to make conjectures about the limits in part (a).
(c) Use the Squeezing Theorem (1.6.2) to confirm that your conjectures in part (b) are correct.

66. (a) Explain why L'Hôpital's rule does not apply to the problem

$$\lim_{x\to 0}\frac{x^2\sin(1/x)}{\sin x}$$

(b) Find the limit.

67. Find $\lim_{x\to 0^+} \dfrac{x\sin(1/x)}{\sin x}$ if it exists.

68. Suppose that functions $f$ and $g$ are differentiable at $x = a$ and that $f(a) = g(a) = 0$. If $g'(a) \neq 0$, show that

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$$

without using L'Hôpital's rule. [*Hint:* Divide the numerator and denominator of $f(x)/g(x)$ by $x - a$ and use the definitions for $f'(a)$ and $g'(a)$.]

69. **Writing** Were we to use L'Hôpital's rule to evaluate either

$$\lim_{x\to 0}\frac{\sin x}{x} \quad \text{or} \quad \lim_{x\to+\infty}\left(1+\frac{1}{x}\right)^x$$

we could be accused of circular reasoning. Explain why.

70. **Writing** Exercise 56 shows that the indeterminate forms $0^0$ and $\infty^0$ can assume any positive real value. However, it is often the case that these indeterminate forms have value 1. Read the article "Indeterminate Forms of Exponential Type" by John Baxley and Elmer Hayashi in the June–July 1978 issue of *The American Mathematical Monthly*, and write a short report on why this is the case.

## CHAPTER 6 REVIEW EXERCISES

Graphing Utility

1. In each part, find $f^{-1}(x)$ if the inverse exists.
   (a) $f(x) = (e^x)^2 + 1$
   (b) $f(x) = \sin\left(\dfrac{1-2x}{x}\right), \quad \dfrac{2}{4+\pi} \le x \le \dfrac{2}{4-\pi}$
   (c) $f(x) = \dfrac{1}{1 + 3\tan^{-1} x}$

2. (a) State the restrictions on the domains of $\sin x$, $\cos x$, $\tan x$, and $\sec x$ that are imposed to make those functions one-to-one in the definitions of $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$, and $\sec^{-1} x$.
   (b) Sketch the graphs of the restricted trigonometric functions in part (a) and their inverses.

3. In each part, find the exact numerical value of the given expression.
   (a) $\cos[\cos^{-1}(4/5) + \sin^{-1}(5/13)]$
   (b) $\sin[\sin^{-1}(4/5) + \cos^{-1}(5/13)]$

4. In each part, sketch the graph, and check your work with a graphing utility.
   (a) $f(x) = 3\sin^{-1}(x/2)$
   (b) $f(x) = \cos^{-1} x - \pi/2$
   (c) $f(x) = 2\tan^{-1}(-3x)$
   (d) $f(x) = \cos^{-1} x + \sin^{-1} x$

5. Suppose that the graph of $y = \log x$ is drawn with equal scales of 1 inch per unit in both the $x$- and $y$-directions. If a bug wants to walk along the graph until it reaches a height of 5 ft above the $x$-axis, how many miles to the right of the origin will it have to travel?

6. Find the largest value of $a$ such that the function $f(x) = xe^{-x}$ has an inverse on the interval $(-\infty, a]$.

7. Express the following function as a rational function of $x$:
$$3\ln\left(e^{2x}(e^x)^3\right) + 2\exp(\ln 1)$$

8. Suppose that $y = Ce^{kt}$, where $C$ and $k$ are constants, and let $Y = \ln y$. Show that the graph of $Y$ versus $t$ is a line, and state its slope and $Y$-intercept.

9. (a) Sketch the curves $y = \pm e^{-x/2}$ and $y = e^{-x/2}\sin 2x$ for $-\pi/2 \le x \le 3\pi/2$ in the same coordinate system, and check your work using a graphing utility.
   (b) Find all $x$-intercepts of the curve $y = e^{-x/2}\sin 2x$ in the stated interval, and find the $x$-coordinates of all points where this curve intersects the curves $y = \pm e^{-x/2}$.

10. Suppose that a package of medical supplies is dropped from a helicopter straight down by parachute into a remote area. The velocity $v$ (in feet per second) of the package $t$ seconds after it is released is given by $v = 24.61(1 - e^{-1.3t})$.
    (a) Graph $v$ versus $t$.
    (b) Show that the graph has a horizontal asymptote $v = c$.
    (c) The constant $c$ is called the ***terminal velocity***. Explain what the terminal velocity means in practical terms.
    (d) Can the package actually reach its terminal velocity? Explain.
    (e) How long does it take for the package to reach 98% of its terminal velocity?

11. A breeding group of 20 bighorn sheep is released in a protected area in Colorado. It is expected that with careful management the number of sheep, $N$, after $t$ years will be given by the formula
$$N = \frac{220}{1 + 10(0.83^t)}$$
and that the sheep population will be able to maintain itself without further supervision once the population reaches a size of 80.
    (a) Graph $N$ versus $t$.
    (b) How many years must the state of Colorado maintain a program to care for the sheep?
    (c) How many bighorn sheep can the environment in the protected area support? [*Hint:* Examine the graph of $N$ versus $t$ for large values of $t$.]

12. An oven is preheated and then remains at a constant temperature. A potato is placed in the oven to bake. Suppose that the temperature $T$ (in °F) of the potato $t$ minutes later is given by $T = 400 - 325(0.97^t)$. The potato will be considered done when its temperature is anywhere between 260°F and 280°F.
    (a) During what interval of time would the potato be considered done?
    (b) How long does it take for the difference between the potato and oven temperatures to be cut in half?

13. (a) Show that the graphs of $y = \ln x$ and $y = x^{0.2}$ intersect.
    (b) Approximate the solution(s) of the equation $\ln x = x^{0.2}$ to three decimal places.

14. (a) Show that for $x > 0$ and $k \neq 0$ the equations
$$x^k = e^x \quad \text{and} \quad \frac{\ln x}{x} = \frac{1}{k}$$
have the same solutions.
    (b) Use the graph of $y = (\ln x)/x$ to determine the values of $k$ for which the equation $x^k = e^x$ has two distinct positive solutions.
    (c) Estimate the positive solution(s) of $x^8 = e^x$.

**15–18** Find the limits.

15. $\displaystyle\lim_{t\to\pi/2^+} e^{\tan t}$

16. $\displaystyle\lim_{\theta\to 0^+} \ln(\sin 2\theta) - \ln(\tan\theta)$

17. $\displaystyle\lim_{x\to+\infty}\left(1 + \frac{3}{x}\right)^{-x}$

18. $\displaystyle\lim_{x\to+\infty}\left(1 + \frac{a}{x}\right)^{bx}, \quad a, b > 0$

**19–20** Find $dy/dx$ by first using algebraic properties of the natural logarithm function.

19. $y = \ln\left(\dfrac{(x+1)(x+2)^2}{(x+3)^3(x+4)^4}\right)$

20. $y = \ln\left(\dfrac{\sqrt{x}\sqrt[3]{x+1}}{\sin x \sec x}\right)$

**21–38** Find $dy/dx$. ■

21. $y = \ln 2x$
22. $y = (\ln x)^2$
23. $y = \sqrt[3]{\ln x + 1}$
24. $y = \ln(\sqrt[3]{x} + 1)$
25. $y = \log(\ln x)$
26. $y = \dfrac{1 + \log x}{1 - \log x}$
27. $y = \ln(x^{3/2}\sqrt{1 + x^4})$
28. $y = \ln\left(\dfrac{\sqrt{x}\cos x}{1 + x^2}\right)$
29. $y = e^{\ln(x^2+1)}$
30. $y = \ln\left(\dfrac{1 + e^x + e^{2x}}{1 - e^{3x}}\right)$
31. $y = 2xe^{\sqrt{x}}$
32. $y = \dfrac{a}{1 + be^{-x}}$
33. $y = \dfrac{1}{\pi}\tan^{-1} 2x$
34. $y = 2^{\sin^{-1} x}$
35. $y = x^{(e^x)}$
36. $y = (1 + x)^{1/x}$
37. $y = \sec^{-1}(2x + 1)$
38. $y = \sqrt{\cos^{-1} x^2}$

**39–40** Find $dy/dx$ using logarithmic differentiation. ■

39. $y = \dfrac{x^3}{\sqrt{x^2 + 1}}$
40. $y = \sqrt[3]{\dfrac{x^2 - 1}{x^2 + 1}}$

41. (a) Make a conjecture about the shape of the graph of $y = \frac{1}{2}x - \ln x$, and draw a rough sketch.
    (b) Check your conjecture by graphing the equation over the interval $0 < x < 5$ with a graphing utility.
    (c) Show that the slopes of the tangent lines to the curve at $x = 1$ and $x = e$ have opposite signs.
    (d) What does part (c) imply about the existence of a horizontal tangent line to the curve? Explain.
    (e) Find the exact $x$-coordinates of all horizontal tangent lines to the curve.

42. Recall from Section 6.1 that the loudness $\beta$ of a sound in decibels (dB) is given by $\beta = 10\log(I/I_0)$, where $I$ is the intensity of the sound in watts per square meter (W/m$^2$) and $I_0$ is a constant that is approximately the intensity of a sound at the threshold of human hearing. Find the rate of change of $\beta$ with respect to $I$ at the point where
    (a) $I/I_0 = 10$ (b) $I/I_0 = 100$ (c) $I/I_0 = 1000$.

43. A particle is moving along the curve $y = x\ln x$. Find all values of $x$ at which the rate of change of $y$ with respect to time is three times that of $x$. [Assume that $dx/dt$ is never zero.]

44. Find the equation of the tangent line to the graph of $y = \ln(5 - x^2)$ at $x = 2$.

45. Find the value of $b$ so that the line $y = x$ is tangent to the graph of $y = \log_b x$. Confirm your result by graphing both $y = x$ and $y = \log_b x$ in the same coordinate system.

46. In each part, find the value of $k$ for which the graphs of $y = f(x)$ and $y = \ln x$ share a common tangent line at their point of intersection. Confirm your result by graphing $y = f(x)$ and $y = \ln x$ in the same coordinate system.
    (a) $f(x) = \sqrt{x} + k$ (b) $f(x) = k\sqrt{x}$

47. If $f$ and $g$ are inverse functions and $f$ is differentiable on its domain, must $g$ be differentiable on its domain? Give a reasonable informal argument to support your answer.

48. In each part, find $(f^{-1})'(x)$ using Formula (2) of Section 6.3, and check your answer by differentiating $f^{-1}$ directly.
    (a) $f(x) = 3/(x + 1)$ (b) $f(x) = \sqrt{e^x}$

49. Find a point on the graph of $y = e^{3x}$ at which the tangent line passes through the origin.

50. Show that the rate of change of $y = 5000e^{1.07x}$ is proportional to $y$.

51. Show that the function $y = e^{ax}\sin bx$ satisfies
$$y'' - 2ay' + (a^2 + b^2)y = 0$$
for any real constants $a$ and $b$.

52. Show that the function $y = \tan^{-1} x$ satisfies
$$y'' = -2\sin y\cos^3 y$$

53. Suppose that the population of deer on an island is modeled by the equation
$$P(t) = \frac{95}{5 - 4e^{-t/4}}$$
where $P(t)$ is the number of deer $t$ weeks after an initial observation at time $t = 0$.
    (a) Use a graphing utility to graph the function $P(t)$.
    (b) In words, explain what happens to the population over time. Check your conclusion by finding $\lim_{t\to+\infty} P(t)$.
    (c) In words, what happens to the *rate* of population growth over time? Check your conclusion by graphing $P'(t)$.

54. The equilibrium constant $k$ of a balanced chemical reaction changes with the absolute temperature $T$ according to the law
$$k = k_0 \exp\left(-\frac{q(T - T_0)}{2T_0T}\right)$$
where $k_0$, $q$, and $T_0$ are constants. Find the rate of change of $k$ with respect to $T$.

**55–56** Find the limit by interpreting the expression as an appropriate derivative. ■

55. $\displaystyle\lim_{h\to 0}\frac{(1 + h)^\pi - 1}{h}$
56. $\displaystyle\lim_{x\to e}\frac{1 - \ln x}{(x - e)\ln x}$

57. Suppose that $\lim f(x) = \pm\infty$ and $\lim g(x) = \pm\infty$. In each of the four possible cases, state whether $\lim[f(x) - g(x)]$ is an indeterminate form, and give a reasonable informal argument to support your answer.

58. (a) Under what conditions will a limit of the form
$$\lim_{x\to a}[f(x)/g(x)]$$
be an indeterminate form?
    (b) If $\lim_{x\to a} g(x) = 0$, must $\lim_{x\to a}[f(x)/g(x)]$ be an indeterminate form? Give some examples to support your answer.

**59–62** Evaluate the given limit. ■