MA1123 Assignment6

[due Monday 10 November, 2014]

1. $\int sin^2 x cos^4 x dx$ Use the 1/2 angle formulae
2. $\int \frac{sin(\frac{5}{x})}{x^2} dx$
3. $\int x^2\sqrt{1+x}dx$
4. $\int \frac{x}{\sqrt{2x+1}} dx$

Page 287, 44,48. Note these are Initial Value Problems. The given Initial Value allows you to determine the constant of integration.

Page 307/308,18,24, 29-32

Page 320 27-30

Page 336, 24,28

These pages follow.

FOCUS ON CONCEPTS

41. (a) Evaluate the integral $\int \sin x \cos x\, dx$ by two methods: first by letting $u = \sin x$, and then by letting $u = \cos x$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

42. (a) Evaluate the integral $\int (5x-1)^2\, dx$ by two methods: first square and integrate, then let $u = 5x - 1$.
(b) Explain why the two apparently different answers obtained in part (a) are really equivalent.

43–44 Solve the initial-value problems.

43. $\dfrac{dy}{dx} = \sqrt{5x+1},\ y(3) = -2$

44. $\dfrac{dy}{dx} = 2 + \sin 3x,\ y(\pi/3) = 0$

45. (a) Evaluate $\int [x/\sqrt{x^2+1}]\, dx$.
(b) Use a graphing utility to generate some typical integral curves of $f(x) = x/\sqrt{x^2+1}$ over the interval $(-5, 5)$.

46. (a) Evaluate $\int 2x \sin(25 - x^2)\, dx$.
(b) Use a graphing utility to generate some typical integral curves of $f(x) = 2x\sin(25 - x^2)$ over the interval $(-5, 5)$.

47. Find a function f such that the slope of the tangent line at a point (x, y) on the curve $y = f(x)$ is $\sqrt{3x+1}$ and the curve passes through the point $(0, 1)$.

48. A population of minnows in a lake is estimated to be 100,000 at the beginning of the year 2005. Suppose that t years after the beginning of 2005 the rate of growth of the population $p(t)$ (in thousands) is given by $p'(t) = (3 + 0.12t)^{3/2}$. Estimate the projected population at the beginning of the year 2010.

49. **Writing** If you want to evaluate an integral by u-substitution, how do you decide what part of the integrand to choose for u?

50. **Writing** The evaluation of an integral can sometimes result in apparently different answers (Exercises 41 and 42). Explain why this occurs and give an example. How might you show that two apparently different answers are actually equivalent?

✔ QUICK CHECK ANSWERS 4.3

1. (a) $1 + x^3$; $3x^2\, dx$ (b) x^2; $2x\, dx$ (c) $1 + 9x^2$; $18x\, dx$ 2. (a) $u^{-1/3}$ (b) $-u$ (c) $2\sqrt[3]{u}$

4.4 THE DEFINITION OF AREA AS A LIMIT; SIGMA NOTATION

Our main goal in this section is to use the rectangle method to give a precise mathematical definition of the "area under a curve."

SIGMA NOTATION

To simplify our computations, we will begin by discussing a useful notation for expressing lengthy sums in a compact form. This notation is called ***sigma notation*** or ***summation notation*** because it uses the uppercase Greek letter Σ (sigma) to denote various kinds of sums. To illustrate how this notation works, consider the sum

$$1^2 + 2^2 + 3^2 + 4^2 + 5^2$$

in which each term is of the form k^2, where k is one of the integers from 1 to 5. In sigma notation this sum can be written as

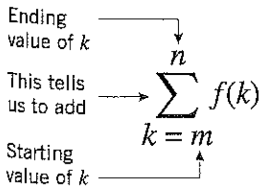
$$\sum_{k=1}^{5} k^2$$

which is read "the summation of k^2, where k runs from 1 to 5." The notation tells us to form the sum of the terms that result when we substitute successive integers for k in the expression k^2, starting with $k = 1$ and ending with $k = 5$.

More generally, if $f(k)$ is a function of k, and if m and n are integers such that $m \le n$, then

$$\sum_{k=m}^{n} f(k) \tag{1}$$

denotes the sum of the terms that result when we substitute successive integers for k, starting with $k = m$ and ending with $k = n$ (Figure 4.4.1).



▲ **Figure 4.4.1**

EXERCISE SET 4.5

1–4 Find the value of

(a) $\sum_{k=1}^{n} f(x_k^*)\,\Delta x_k$ (b) $\max \Delta x_k$.

1. $f(x) = x + 1;\ a = 0, b = 4;\ n = 3;$
$\Delta x_1 = 1, \Delta x_2 = 1, \Delta x_3 = 2;$
$x_1^* = \frac{1}{3}, x_2^* = \frac{3}{2}, x_3^* = 3$
2. $f(x) = \cos x;\ a = 0, b = 2\pi;\ n = 4;$
$\Delta x_1 = \pi/2, \Delta x_2 = 3\pi/4, \Delta x_3 = \pi/2, \Delta x_4 = \pi/4;$
$x_1^* = \pi/4, x_2^* = \pi, x_3^* = 3\pi/2, x_4^* = 7\pi/4$
3. $f(x) = 4 - x^2;\ a = -3, b = 4;\ n = 4;$
$\Delta x_1 = 1, \Delta x_2 = 2, \Delta x_3 = 1, \Delta x_4 = 3;$
$x_1^* = -\frac{5}{2}, x_2^* = -1, x_3^* = \frac{1}{4}, x_4^* = 3$
4. $f(x) = x^3;\ a = -3, b = 3;\ n = 4;$
$\Delta x_1 = 2, \Delta x_2 = 1, \Delta x_3 = 1, \Delta x_4 = 2;$
$x_1^* = -2, x_2^* = 0, x_3^* = 0, x_4^* = 2$

5–8 Use the given values of a and b to express the following limits as integrals. (Do not evaluate the integrals.)

5. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^2 \Delta x_k;\ a = -1, b = 2$

6. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (x_k^*)^3 \Delta x_k;\ a = 1, b = 2$

7. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} 4x_k^*(1 - 3x_k^*)\Delta x_k;\ a = -3, b = 3$

8. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (\sin^2 x_k^*)\Delta x_k;\ a = 0, b = \pi/2$

9–10 Use Definition 4.5.1 to express the integrals as limits of Riemann sums. (Do not evaluate the integrals.)

9. (a) $\int_1^2 2x\,dx$ (b) $\int_0^1 \frac{x}{x+1}\,dx$

10. (a) $\int_1^2 \sqrt{x}\,dx$ (b) $\int_{-\pi/2}^{\pi/2} (1 + \cos x)\,dx$

FOCUS ON CONCEPTS

11. Explain informally why Theorem 4.5.4(a) follows from Definition 4.5.1.

12. Explain informally why Theorem 4.5.6(a) follows from Definition 4.5.1.

13–16 Sketch the region whose signed area is represented by the definite integral, and evaluate the integral using an appropriate formula from geometry, where needed.

13. (a) $\int_0^3 x\,dx$ (b) $\int_{-2}^{-1} x\,dx$
(c) $\int_{-1}^{4} x\,dx$ (d) $\int_{-5}^{5} x\,dx$

14. (a) $\int_0^2 \left(1 - \frac{1}{2}x\right) dx$ (b) $\int_{-1}^{1} \left(1 - \frac{1}{2}x\right) dx$
(c) $\int_2^3 \left(1 - \frac{1}{2}x\right) dx$ (d) $\int_0^3 \left(1 - \frac{1}{2}x\right) dx$

15. (a) $\int_0^5 2\,dx$ (b) $\int_0^{\pi} \cos x\,dx$
(c) $\int_{-1}^{2} |2x - 3|\,dx$ (d) $\int_{-1}^{1} \sqrt{1 - x^2}\,dx$

16. (a) $\int_{-10}^{-5} 6\,dx$ (b) $\int_{-\pi/3}^{\pi/3} \sin x\,dx$
(c) $\int_0^3 |x - 2|\,dx$ (d) $\int_0^2 \sqrt{4 - x^2}\,dx$

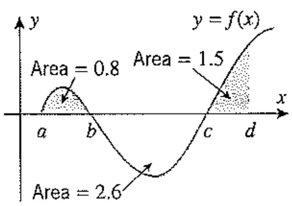
17. In each part, evaluate the integral, given that
$$f(x) = \begin{cases} |x - 2|, & x \geq 0 \\ x + 2, & x < 0 \end{cases}$$
(a) $\int_{-2}^{0} f(x)\,dx$ (b) $\int_{-2}^{2} f(x)\,dx$
(c) $\int_0^6 f(x)\,dx$ (d) $\int_{-4}^{6} f(x)\,dx$

18. In each part, evaluate the integral, given that
$$f(x) = \begin{cases} 2x, & x \leq 1 \\ 2, & x > 1 \end{cases}$$
(a) $\int_0^1 f(x)\,dx$ (b) $\int_{-1}^{1} f(x)\,dx$
(c) $\int_1^{10} f(x)\,dx$ (d) $\int_{1/2}^{5} f(x)\,dx$

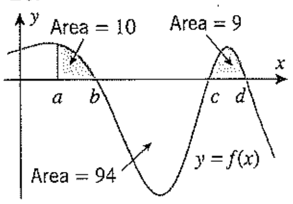
FOCUS ON CONCEPTS

19–20 Use the areas shown in the figure to find
(a) $\int_a^b f(x)\,dx$ (b) $\int_b^c f(x)\,dx$
(c) $\int_a^c f(x)\,dx$ (d) $\int_a^d f(x)\,dx$.

19.



20.



21. Find $\int_{-1}^{2} [f(x) + 2g(x)]\,dx$ if

$$\int_{-1}^{2} f(x)\,dx = 5 \quad \text{and} \quad \int_{-1}^{2} g(x)\,dx = -3$$

22. Find $\int_{1}^{4} [3f(x) - g(x)]\,dx$ if

$$\int_{1}^{4} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{4} g(x)\,dx = 10$$

23. Find $\int_{1}^{5} f(x)\,dx$ if

$$\int_{0}^{1} f(x)\,dx = -2 \quad \text{and} \quad \int_{0}^{5} f(x)\,dx = 1$$

24. Find $\int_{3}^{-2} f(x)\,dx$ if

$$\int_{-2}^{1} f(x)\,dx = 2 \quad \text{and} \quad \int_{1}^{3} f(x)\,dx = -6$$

25–28 Use Theorem 4.5.4 and appropriate formulas from geometry to evaluate the integrals.

25. $\int_{-1}^{3} (4 - 5x)\,dx$

26. $\int_{-2}^{2} (1 - 3|x|)\,dx$

27. $\int_{0}^{1} (x + 2\sqrt{1 - x^2})\,dx$

28. $\int_{-3}^{0} (2 + \sqrt{9 - x^2})\,dx$

29–32 True–False Determine whether the statement is true or false. Explain your answer.

29. If $f(x)$ is integrable on $[a, b]$, then $f(x)$ is continuous on $[a, b]$.

30. It is the case that

$$0 < \int_{-1}^{1} \frac{\cos x}{\sqrt{1 + x^2}}\,dx$$

31. If the integral of $f(x)$ over the interval $[a, b]$ is negative, then $f(x) \le 0$ for $a \le x \le b$.

32. The function

$$f(x) = \begin{cases} 0, & x \le 0 \\ x^2, & x > 0 \end{cases}$$

is integrable over every closed interval $[a, b]$.

33–34 Use Theorem 4.5.6 to determine whether the value of the integral is positive or negative.

33. (a) $\int_{2}^{3} \frac{\sqrt{x}}{1 - x}\,dx$ (b) $\int_{0}^{4} \frac{x^2}{3 - \cos x}\,dx$

34. (a) $\int_{-3}^{-1} \frac{x^4}{\sqrt{3 - x}}\,dx$ (b) $\int_{-2}^{2} \frac{x^3 - 9}{|x| + 1}\,dx$

35. Prove that if f is continuous and if $m \le f(x) \le M$ on $[a, b]$, then

$$m(b - a) \le \int_{a}^{b} f(x)\,dx \le M(b - a)$$

36. Find the maximum and minimum values of $\sqrt{x^3 + 2}$ for $0 \le x \le 3$. Use these values, and the inequalities in Exercise 35, to find bounds on the value of the integral

$$\int_{0}^{3} \sqrt{x^3 + 2}\,dx$$

37–38 Evaluate the integrals by completing the square and applying appropriate formulas from geometry.

37. $\int_{0}^{10} \sqrt{10x - x^2}\,dx$

38. $\int_{0}^{3} \sqrt{6x - x^2}\,dx$

39–40 Evaluate the limit by expressing it as a definite integral over the interval $[a, b]$ and applying appropriate formulas from geometry.

39. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} (3x_k^* + 1)\Delta x_k;\ a = 0, b = 1$

40. $\lim\limits_{\max \Delta x_k \to 0} \sum_{k=1}^{n} \sqrt{4 - (x_k^*)^2}\,\Delta x_k;\ a = -2, b = 2$

FOCUS ON CONCEPTS

41. Let $f(x) = C$ be a constant function.

(a) Use a formula from geometry to show that

$$\int_{a}^{b} f(x)\,dx = C(b - a)$$

(b) Show that any Riemann sum for $f(x)$ over $[a, b]$ evaluates to $C(b - a)$. Use Definition 4.5.1 to show that

$$\int_{a}^{b} f(x)\,dx = C(b - a)$$

42. Define a function f on $[0, 1]$ by

$$f(x) = \begin{cases} 1, & 0 < x \le 1 \\ 0, & x = 0 \end{cases}$$

Use Definition 4.5.1 to show that

$$\int_{0}^{1} f(x)\,dx = 1$$

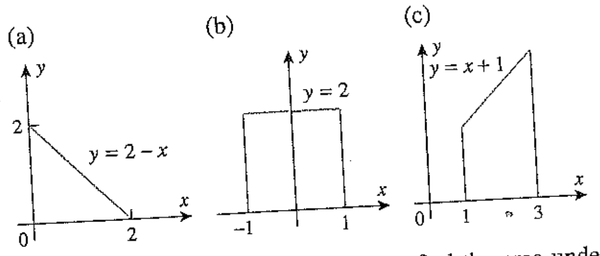
43. It can be shown that every interval contains both rational and irrational numbers. Accepting this to be so, do you believe that the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}$$

is integrable on a closed interval $[a, b]$? Explain your reasoning.

EXERCISE SET 4.6 Graphing Utility C CAS

1. In each part, use a definite integral to find the area of the region, and check your answer using an appropriate formula from geometry.



2. In each part, use a definite integral to find the area under the curve $y = f(x)$ over the stated interval, and check your answer using an appropriate formula from geometry.
   (a) $f(x) = x$; $[0, 5]$
   (b) $f(x) = 5$; $[3, 9]$
   (c) $f(x) = x + 3$; $[-1, 2]$

3. In each part, sketch the analogue of Figure 4.6.10 for the specified region. [Let $y = f(x)$ denote the upper boundary of the region. If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
   (a) The region in part (a) of Exercise 1.
   (b) The region in part (b) of Exercise 1.
   (c) The region in part (c) of Exercise 1.

4. In each part, sketch the analogue of Figure 4.6.10 for the function and interval specified. [If x^* is unique, label both it and $f(x^*)$ on your sketch. Otherwise, label $f(x^*)$ on your sketch, and determine all values of x^* that satisfy Equation (8).]
   (a) The function and interval in part (a) of Exercise 2.
   (b) The function and interval in part (b) of Exercise 2.
   (c) The function and interval in part (c) of Exercise 2.

5–8 Find the area under the curve $y = f(x)$ over the stated interval.

5. $f(x) = x^3$; $[2, 3]$
6. $f(x) = x^4$; $[-1, 1]$
7. $f(x) = 3\sqrt{x}$; $[1, 4]$
8. $f(x) = x^{-2/3}$; $[1, 27]$

9–10 Find all values of x^* in the stated interval that satisfy Equation (8) in the Mean-Value Theorem for Integrals (4.6.2), and explain what these numbers represent.

9. (a) $f(x) = \sqrt{x}$; $[0, 3]$
   (b) $f(x) = x^2 + x$; $[-12, 0]$
10. (a) $f(x) = \sin x$; $[-\pi, \pi]$ (b) $f(x) = 1/x^2$; $[1, 3]$

11–22 Evaluate the integrals using Part 1 of the Fundamental Theorem of Calculus.

11. $\displaystyle\int_{-2}^{1} (x^2 - 6x + 12)\, dx$
12. $\displaystyle\int_{-1}^{2} 4x(1 - x^2)\, dx$
13. $\displaystyle\int_{1}^{4} \frac{4}{x^2}\, dx$
14. $\displaystyle\int_{1}^{2} \frac{1}{x^6}\, dx$
15. $\displaystyle\int_{4}^{9} 2x\sqrt{x}\, dx$
16. $\displaystyle\int_{1}^{4} \frac{1}{x\sqrt{x}}\, dx$
17. $\displaystyle\int_{-\pi/2}^{\pi/2} \sin\theta\, d\theta$
18. $\displaystyle\int_{0}^{\pi/4} \sec^2\theta\, d\theta$
19. $\displaystyle\int_{-\pi/4}^{\pi/4} \cos x\, dx$
20. $\displaystyle\int_{0}^{\pi/3} (2x - \sec x \tan x)\, dx$
21. $\displaystyle\int_{1}^{4} \left(\frac{1}{\sqrt{t}} - 3\sqrt{t}\right) dt$
22. $\displaystyle\int_{\pi/6}^{\pi/2} \left(x + \frac{2}{\sin^2 x}\right) dx$

23–24 Use Theorem 4.5.5 to evaluate the given integrals.

23. (a) $\displaystyle\int_{-1}^{1} |2x - 1|\, dx$ (b) $\displaystyle\int_{0}^{3\pi/4} |\cos x|\, dx$
24. (a) $\displaystyle\int_{-1}^{2} \sqrt{2 + |x|}\, dx$ (b) $\displaystyle\int_{0}^{\pi/2} \left|\tfrac{1}{2} - \cos x\right| dx$

25–26 A function $f(x)$ is defined piecewise on an interval. In these exercises: (a) Use Theorem 4.5.5 to find the integral of $f(x)$ over the interval. (b) Find an antiderivative of $f(x)$ on the interval. (c) Use parts (a) and (b) to verify Part 1 of the Fundamental Theorem of Calculus.

25. $f(x) = \begin{cases} x, & 0 \le x \le 1 \\ x^2, & 1 < x \le 2 \end{cases}$
26. $f(x) = \begin{cases} \sqrt{x}, & 0 \le x < 1 \\ 1/x^2, & 1 \le x \le 4 \end{cases}$

27–30 True–False Determine whether the statement is true or false. Explain your answer.

27. There does not exist a differentiable function $F(x)$ such that $F'(x) = |x|$.
28. If $f(x)$ is continuous on the interval $[a, b]$, and if the definite integral of $f(x)$ over this interval has value 0, then the equation $f(x) = 0$ has at least one solution in the interval $[a, b]$.
29. If $F(x)$ is an antiderivative of $f(x)$ and $G(x)$ is an antiderivative of $g(x)$, then
$$\int_a^b f(x)\, dx = \int_a^b g(x)\, dx$$
if and only if
$$G(a) + F(b) = F(a) + G(b)$$
30. If $f(x)$ is continuous everywhere and
$$F(x) = \int_0^x f(t)\, dt$$
then the equation $F(x) = 0$ has at least one solution.

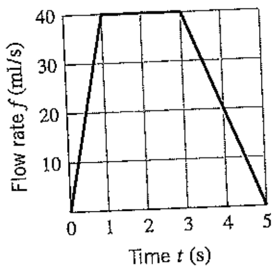
21. Water is run at a constant rate of 1 ft^3/min to fill a cylindrical tank of radius 3 ft and height 5 ft. Assuming that the tank is initially empty, make a conjecture about the average weight of the water in the tank over the time period required to fill it, and then check your conjecture by integrating. [Take the weight density of water to be 62.4 lb/ft^3.]

22. (a) The temperature of a 10 m long metal bar is 15°C at one end and 30°C at the other end. Assuming that the temperature increases linearly from the cooler end to the hotter end, what is the average temperature of the bar?
(b) Explain why there must be a point on the bar where the temperature is the same as the average, and find it.

23. A traffic engineer monitors the rate at which cars enter the main highway during the afternoon rush hour. From her data she estimates that between 4:30 P.M. and 5:30 P.M. the rate $R(t)$ at which cars enter the highway is given by the formula $R(t) = 100(1 - 0.0001t^2)$ cars per minute, where t is the time (in minutes) since 4:30 P.M. Find the average rate, in cars per minute, at which cars enter the highway during the first half-hour of rush hour.

24. Suppose that the value of a yacht in dollars after t years of use is $V(t) = 275{,}000\sqrt{\frac{20}{t+20}}$. What is the average value of the yacht over its first 10 years of use?

25. A large juice glass containing 60 ml of orange juice is replenished by a server. The accompanying figure shows the rate at which orange juice is poured into the glass in milliliters per second (ml/s). Show that the average rate of change of the volume of juice in the glass during these 5 s is equal to the average value of the rate of flow of juice into the glass.



◂ **Figure Ex-25**

C 26. The function J_0 defined by

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x \sin t)\,dt$$

is called the ***Bessel function of order zero***.
(a) Find a function f and an interval $[a, b]$ for which $J_0(1)$ is the average value of f over $[a, b]$.
(b) Estimate $J_0(1)$.
(c) Use a CAS to graph the equation $y = J_0(x)$ over the interval $0 \le x \le 8$.
(d) Estimate the smallest positive zero of J_0.

27. Find a positive value of k such that the average value of $f(x) = \sqrt{3x}$ over the interval $[0, k]$ is 6.

28. Suppose that a tumor grows at the rate of $r(t) = kt$ grams per week for some positive constant k, where t is the number of weeks since the tumor appeared. When, during the second 26 weeks of growth, is the mass of the tumor the same as its average mass during that period?

29. **Writing** Consider the following statement: *The average value of the rate of change of a function over an interval is equal to the average rate of change of the function over that interval.* Write a short paragraph that explains why this statement may be interpreted as a rewording of Part 1 of the Fundamental Theorem of Calculus.

30. **Writing** If an automobile gets an average of 25 miles per gallon of gasoline, then it is also the case that on average the automobile expends 1/25 gallon of gasoline per mile. Interpret this statement using the concept of the average value of a function over an interval.

✔ QUICK CHECK ANSWERS 4.8

1. $\frac{1}{n}\sum_{k=1}^{n} a_k$ 2. $\frac{1}{b-a}\int_a^b f(x)\,dx$ 3. $f(x^*)$ 4. 40