MA1123 Assignment6

[due Monday 10 November, 2014]

Page 251, 23,24
Page 257/8, 11-14,22,32
Page 260, 16,18
Page 263, 70
Page 264, 6
These pages follow.

23. The ***mechanic's rule*** for approximating square roots states that $\sqrt{a} \approx x_{n+1}$, where

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right), \quad n = 1, 2, 3, \ldots$$

and x_1 is any positive approximation to $\sqrt{a}$.

(a) Apply Newton's Method to

$$f(x) = x^2 - a$$

to derive the mechanic's rule.

(b) Use the mechanic's rule to approximate $\sqrt{10}$.

24. Many calculators compute reciprocals using the approximation $1/a \approx x_{n+1}$, where

$$x_{n+1} = x_n(2 - ax_n), \quad n = 1, 2, 3, \ldots$$

and x_1 is an initial approximation to $1/a$. This formula makes it possible to perform divisions using multiplications and subtractions, which is a faster procedure than dividing directly.

(a) Apply Newton's Method to

$$f(x) = \frac{1}{x} - a$$

to derive this approximation.

(b) Use the formula to approximate $\frac{1}{17}$.

25. Use Newton's Method to approximate the absolute minimum of $f(x) = \frac{1}{4}x^4 + x^2 - 5x$.

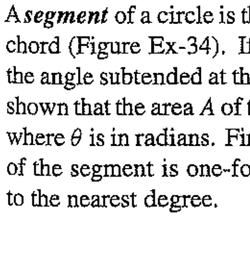
26. Use Newton's Method to approximate the absolute maximum of $f(x) = x \sin x$ on the interval $[0, \pi]$.

27. Use Newton's Method to approximate the coordinates of the the point on the parabola $y = x^2$ that is closest to the point $(1, 0)$.

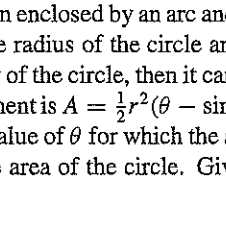
28. Use Newton's Method to approximate the dimensions of the rectangle of largest area that can be inscribed under the curve $y = \cos x$ for $0 \le x \le \pi/2$ (Figure Ex-28).

29. (a) Show that on a circle of radius r, the central angle θ that subtends an arc whose length is 1.5 times the length L of its chord satisfies the equation $\theta = 3\sin(\theta/2)$ (Figure Ex-29).

(b) Use Newton's Method to approximate θ.

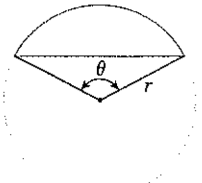


▲ Figure Ex-28



▲ Figure Ex-29

30. A ***segment*** of a circle is the region enclosed by an arc and its chord (Figure Ex-34). If r is the radius of the circle and θ the angle subtended at the center of the circle, then it can be shown that the area A of the segment is $A = \frac{1}{2}r^2(\theta - \sin\theta)$, where θ is in radians. Find the value of θ for which the area of the segment is one-fourth the area of the circle. Give θ to the nearest degree.



◀ Figure Ex-30

31–32 Use Newton's Method to approximate all real values of y satisfying the given equation for the indicated value of x.

31. $xy^4 + x^3y = 1;\ x = 1$

32. $xy - \cos\left(\frac{1}{2}xy\right) = 0;\ x = 2$

33. An ***annuity*** is a sequence of equal payments that are paid or received at regular time intervals. For example, you may want to deposit equal amounts at the end of each year into an interest-bearing account for the purpose of accumulating a lump sum at some future time. If, at the end of each year, interest of $i \times 100\%$ on the account balance for that year is added to the account, then the account is said to pay $i \times 100\%$ interest, ***compounded annually***. It can be shown that if payments of Q dollars are deposited at the end of each year into an account that pays $i \times 100\%$ compounded annually, then at the time when the nth payment and the accrued interest for the past year are deposited, the amount $S(n)$ in the account is given by the formula

$$S(n) = \frac{Q}{i}[(1+i)^n - 1]$$

Suppose that you can invest \$5000 in an interest-bearing account at the end of each year, and your objective is to have \$250,000 on the 25th payment. Approximately what annual compound interest rate must the account pay for you to achieve your goal? [*Hint:* Show that the interest rate i satisfies the equation $50i = (1+i)^{25} - 1$, and solve it using Newton's Method.]

FOCUS ON CONCEPTS

34. (a) Use a graphing utility to generate the graph of

$$f(x) = \frac{x}{x^2 + 1}$$

and use it to explain what happens if you apply Newton's Method with a starting value of $x_1 = 2$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

(b) Use the graph generated in part (a) to explain what happens if you apply Newton's Method with a starting value of $x_1 = 0.5$. Check your conclusion by computing x_2, x_3, x_4, and x_5.

35. (a) Apply Newton's Method to $f(x) = x^2 + 1$ with a starting value of $x_1 = 0.5$, and determine if the values of $x_2, \ldots, x_{10}$ appear to converge.

(b) Explain what is happening.

36. In each part, explain what happens if you apply Newton's Method to a function f when the given condition is satisfied for some value of n.

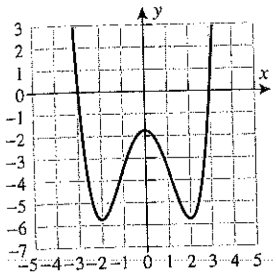
(a) $f(x_n) = 0$ (b) $x_{n+1} = x_n$

(c) $x_{n+2} = x_n \neq x_{n+1}$

✔ QUICK CHECK EXERCISES 3.8 *(See page 259 for answers.)*

1. Let $f(x) = x^2 - x$.
   (a) An interval on which f satisfies the hypotheses of Rolle's Theorem is ______.
   (b) Find all values of c that satisfy the conclusion of Rolle's Theorem for the function f on the interval in part (a).

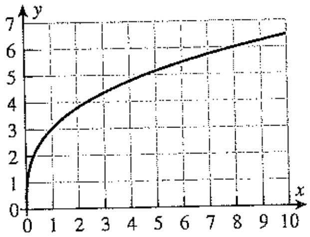
2. Use the accompanying graph of f to find an interval $[a, b]$ on which Rolle's Theorem applies, and find all values of c in that interval that satisfy the conclusion of the theorem.



◀ **Figure Ex-2**

3. Let $f(x) = x^2 - x$.
   (a) Find a point b such that the slope of the secant line through $(0, 0)$ and $(b, f(b))$ is 1.
   (b) Find all values of c that satisfy the conclusion of the Mean-Value Theorem for the function f on the interval $[0, b]$, where b is the point found in part (a).

4. Use the graph of f in the accompanying figure to estimate all values of c that satisfy the conclusion of the Mean-Value Theorem on the interval
   (a) $[0, 8]$ (b) $[0, 4]$.



◀ **Figure Ex-4**

5. Find a function f such that the graph of f contains the point $(1, 5)$ and such that for every value of x_0 the tangent line to the graph of f at x_0 is parallel to the tangent line to the graph of $y = x^2$ at x_0.

EXERCISE SET 3.8 ⊡ Graphing Utility

1–4 Verify that the hypotheses of Rolle's Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

1. $f(x) = x^2 - 8x + 15$; $[3, 5]$
2. $f(x) = x^3 - 3x^2 + 2x$; $[0, 2]$
3. $f(x) = \cos x$; $[\pi/2, 3\pi/2]$
4. $f(x) = (x^2 - 1)/(x - 2)$; $[-1, 1]$

5–8 Verify that the hypotheses of the Mean-Value Theorem are satisfied on the given interval, and find all values of c in that interval that satisfy the conclusion of the theorem.

5. $f(x) = x^2 - x$; $[-3, 5]$
6. $f(x) = x^3 + x - 4$; $[-1, 2]$
7. $f(x) = \sqrt{x + 1}$; $[0, 3]$
8. $f(x) = x - \dfrac{1}{x}$; $[3, 4]$

⊡ 9. (a) Find an interval $[a, b]$ on which
$$f(x) = x^4 + x^3 - x^2 + x - 2$$
satisfies the hypotheses of Rolle's Theorem.
   (b) Generate the graph of $f'(x)$, and use it to make rough estimates of all values of c in the interval obtained in part (a) that satisfy the conclusion of Rolle's Theorem.
   (c) Use Newton's Method to improve on the rough estimates obtained in part (b).

⊡ 10. Let $f(x) = x^3 - 4x$.
   (a) Find the equation of the secant line through the points $(-2, f(-2))$ and $(1, f(1))$.
   (b) Show that there is only one point c in the interval $(-2, 1)$ that satisfies the conclusion of the Mean-Value Theorem for the secant line in part (a).
   (c) Find the equation of the tangent line to the graph of f at the point $(c, f(c))$.
   (d) Use a graphing utility to generate the secant line in part (a) and the tangent line in part (c) in the same coordinate system, and confirm visually that the two lines seem parallel.

11–14 True–False Determine whether the statement is true or false. Explain your answer.

11. Rolle's Theorem says that if f is a continuous function on $[a, b]$ and $f(a) = f(b)$, then there is a point between a and b at which the curve $y = f(x)$ has a horizontal tangent line.

12. If f is continuous on a closed interval $[a, b]$ and differentiable on (a, b), then there is a point between a and b at which the instantaneous rate of change of f matches the average rate of change of f over $[a, b]$.

13. The Constant Difference Theorem says that if two functions have derivatives that differ by a constant on an interval, then the functions are equal on the interval.

14. One application of the Mean-Value Theorem is to prove that a function with positive derivative on an interval must be increasing on that interval.

FOCUS ON CONCEPTS

15. Let $f(x) = \tan x$.
   (a) Show that there is no point c in the interval $(0, \pi)$ such that $f'(c) = 0$, even though $f(0) = f(\pi) = 0$.
   (b) Explain why the result in part (a) does not contradict Rolle's Theorem.

16. Let $f(x) = x^{2/3}$, $a = -1$, and $b = 8$.
   (a) Show that there is no point c in (a, b) such that
   $$f'(c) = \frac{f(b) - f(a)}{b - a}$$
   (b) Explain why the result in part (a) does not contradict the Mean-Value Theorem.

17. (a) Show that if f is differentiable on $(-\infty, +\infty)$, and if $y = f(x)$ and $y = f'(x)$ are graphed in the same coordinate system, then between any two x-intercepts of f there is at least one x-intercept of f'.
   (b) Give some examples that illustrate this.

18. Review Formulas (8) and (9) in Section 2.1 and use the Mean-Value Theorem to show that if f is differentiable on $(-\infty, +\infty)$, then for any interval $[x_0, x_1]$ there is at least one point in (x_0, x_1) where the instantaneous rate of change of y with respect to x is equal to the average rate of change over the interval.

19–21 Use the result of Exercise 18 in these exercises.

19. An automobile travels 4 mi along a straight road in 5 min. Show that the speedometer reads exactly 48 mi/h at least once during the trip.

20. At 11 A.M. on a certain morning the outside temperature was 76°F. At 11 P.M. that evening it had dropped to 52°F.
   (a) Show that at some instant during this period the temperature was decreasing at the rate of 2°F/h.
   (b) Suppose that you know the temperature reached a high of 88°F sometime between 11 A.M. and 11 P.M. Show that at some instant during this period the temperature was decreasing at a rate greater than 3°F/h.

21. Suppose that two runners in a 100 m dash finish in a tie. Show that they had the same velocity at least once during the race.

22. Use the fact that
$$\frac{d}{dx}(3x^4 + x^2 - 4x) = 12x^3 + 2x - 4$$
to show that the equation $12x^3 + 2x - 4 = 0$ has at least one solution in the interval $(0, 1)$.

23. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in the interval $(-\infty, +\infty)$, and if f and g have the same value at some point x_0, then $f(x) = g(x)$ for all x in $(-\infty, +\infty)$.
   (b) Use the result in part (a) to confirm the trigonometric identity $\sin^2 x + \cos^2 x = 1$.

24. (a) Use the Constant Difference Theorem (3.8.3) to show that if $f'(x) = g'(x)$ for all x in $(-\infty, +\infty)$, and if $f(x_0) - g(x_0) = c$ at some point x_0, then
   $$f(x) - g(x) = c$$
   for all x in $(-\infty, +\infty)$.
   (b) Use the result in part (a) to show that the function
   $$h(x) = (x - 1)^3 - (x^2 + 3)(x - 3)$$
   is constant for all x in $(-\infty, +\infty)$, and find the constant.
   (c) Check the result in part (b) by multiplying out and simplifying the formula for $h(x)$.

FOCUS ON CONCEPTS

25. (a) Use the Mean-Value Theorem to show that if f is differentiable on an interval, and if $|f'(x)| \le M$ for all values of x in the interval, then
   $$|f(x) - f(y)| \le M|x - y|$$
   for all values of x and y in the interval.
   (b) Use the result in part (a) to show that
   $$|\sin x - \sin y| \le |x - y|$$
   for all real values of x and y.

26. (a) Use the Mean-Value Theorem to show that if f is differentiable on an open interval, and if $|f'(x)| \ge M$ for all values of x in the interval, then
   $$|f(x) - f(y)| \ge M|x - y|$$
   for all values of x and y in the interval.
   (b) Use the result in part (a) to show that
   $$|\tan x - \tan y| \ge |x - y|$$
   for all values of x and y in the interval $(-\pi/2, \pi/2)$.
   (c) Use the result in part (b) to show that
   $$|\tan x + \tan y| \ge |x + y|$$
   for all values of x and y in the interval $(-\pi/2, \pi/2)$.

27. (a) Use the Mean-Value Theorem to show that
   $$\sqrt{y} - \sqrt{x} < \frac{y - x}{2\sqrt{x}}$$
   if $0 < x < y$.
   (b) Use the result in part (a) to show that if $0 < x < y$, then $\sqrt{xy} < \frac{1}{2}(x + y)$.

28. Show that if f is differentiable on an open interval and $f'(x) \ne 0$ on the interval, the equation $f(x) = 0$ can have at most one real root in the interval.

29. Use the result in Exercise 28 to show the following:
   (a) The equation $x^3 + 4x - 1 = 0$ has exactly one real root.
   (b) If $b^2 - 3ac < 0$ and if $a \ne 0$, then the equation
   $$ax^3 + bx^2 + cx + d = 0$$
   has exactly one real root.

9–12 Analyze the trigonometric function f over the specified interval, stating where f is increasing, decreasing, concave up, and concave down, and stating the x-coordinates of all inflection points. Confirm that your results are consistent with the graph of f generated with a graphing utility.

9. $f(x) = \cos x;\ [0, 2\pi]$

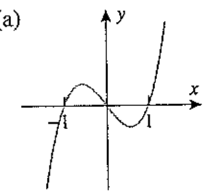
10. $f(x) = \tan x;\ (-\pi/2, \pi/2)$

11. $f(x) = \sin x \cos x;\ [0, \pi]$

12. $f(x) = \cos^2 x - 2\sin x;\ [0, 2\pi]$

13. In each part, sketch a continuous curve $y = f(x)$ with the stated properties.
(a) $f(2) = 4,\ f'(2) = 1,\ f''(x) < 0$ for $x < 2$, $f''(x) > 0$ for $x > 2$
(b) $f(2) = 4,\ f''(x) > 0$ for $x < 2,\ f''(x) < 0$ for $x > 2$, $\lim_{x\to 2^-} f'(x) = +\infty,\ \lim_{x\to 2^+} f'(x) = +\infty$
(c) $f(2) = 4,\ f''(x) < 0$ for $x \neq 2,\ \lim_{x\to 2^-} f'(x) = 1$, $\lim_{x\to 2^+} f'(x) = -1$

14. In parts (a)–(d), the graph of a polynomial with degree at most 6 is given. Find equations for polynomials that produce graphs with these shapes, and check your answers with a graphing utility.

(a)


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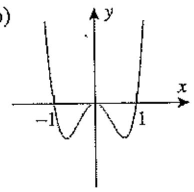
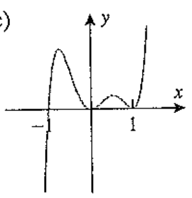
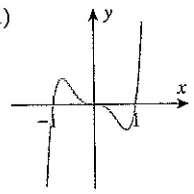


(c)


(d)


15. For a general quadratic polynomial
$$f(x) = ax^2 + bx + c \quad (a \neq 0)$$
find conditions on a, b, and c to ensure that f is always increasing or always decreasing on $[0, +\infty)$.

16. For the general cubic polynomial
$$f(x) = ax^3 + bx^2 + cx + d \quad (a \neq 0)$$
find conditions on a, b, c, and d to ensure that f is always increasing or always decreasing on $(-\infty, +\infty)$.

17. (a) Where on the graph of $y = f(x)$ would you expect y to be increasing or decreasing most rapidly with respect to x?
(b) In words, what is a relative extremum?
(c) State a procedure for determining where the relative extrema of f occur.

18. Determine whether the statement is true or false. If it is false, give an example for which the statement fails.
(a) If f has a relative maximum at x_0, then $f(x_0)$ is the largest value that $f(x)$ can have.
(b) If the largest value for f on the interval (a, b) is at x_0, then f has a relative maximum at x_0.
(c) A function f has a relative extremum at each of its critical points.

19. (a) According to the first derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?
(b) According to the second derivative test, what conditions ensure that f has a relative maximum at x_0? A relative minimum?

20–22 Locate the critical points and identify which critical points correspond to stationary points.

20. (a) $f(x) = x^3 + 3x^2 - 9x + 1$
(b) $f(x) = x^4 - 6x^2 - 3$

21. (a) $f(x) = \dfrac{x}{x^2 + 2}$ (b) $f(x) = \dfrac{x^2 - 3}{x^2 + 1}$

22. (a) $f(x) = x^{1/3}(x - 4)$ (b) $f(x) = x^{4/3} - 6x^{1/3}$

23. In each part, find all critical points, and use the first derivative test to classify them as relative maxima, relative minima, or neither.
(a) $f(x) = x^{1/3}(x - 7)^2$
(b) $f(x) = 2\sin x - \cos 2x,\quad 0 \leq x \leq 2\pi$
(c) $f(x) = 3x - (x - 1)^{3/2}$

24. In each part, find all critical points, and use the second derivative test (where possible) to classify them as relative maxima, relative minima, or neither.
(a) $f(x) = x^{-1/2} + \frac{1}{9}x^{1/2}$
(b) $f(x) = x^2 + 8/x$
(c) $f(x) = \sin^2 x - \cos x,\quad 0 \leq x \leq 2\pi$

25–32 Give a graph of the function f, and identify the limits as $x \to \pm\infty$, as well as locations of all relative extrema, inflection points, and asymptotes (as appropriate).

25. $f(x) = x^4 - 3x^3 + 3x^2 + 1$

26. $f(x) = x^5 - 4x^4 + 4x^3$

27. $f(x) = \tan(x^2 + 1)$ **28.** $f(x) = x - \cos x$

29. $f(x) = \dfrac{x^2}{x^2 + 2x + 5}$ **30.** $f(x) = \dfrac{25 - 9x^2}{x^3}$

31. $f(x) = \begin{cases} \frac{1}{2}x^2, & x \leq 0 \\ -x^2, & x > 0 \end{cases}$

32. $f(x) = (1 + x)^{2/3}(3 - x)^{1/3}$

33–38 Use any method to find the relative extrema of the function f.

33. $f(x) = x^3 + 5x - 2$ **34.** $f(x) = x^4 - 2x^2 + 7$

35. $f(x) = x^{4/5}$ **36.** $f(x) = 2x + x^{2/3}$

when $t = 90$ days. [First find ϕ from Kepler's equation, and then use this value of ϕ to find the distance. Use $a = 150 \times 10^6$ km, $e = 0.0167$, and $T = 365$ days.]

67. Using the formulas in Exercise 66, find the distance from the planet Mars to the Sun when $t = 1$ year. For Mars use $a = 228 \times 10^6$ km, $e = 0.0934$, and $T = 1.88$ years.

68. Suppose that f is continuous on the closed interval $[a, b]$ and differentiable on the open interval (a, b), and suppose that $f(a) = f(b)$. Is it true or false that f must have at least one stationary point in (a, b)? Justify your answer.

69. In each part, determine whether all of the hypotheses of Rolle's Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of c guaranteed in the conclusion of the theorem.
(a) $f(x) = \sqrt{4 - x^2}$ on $[-2, 2]$
(b) $f(x) = x^{2/3} - 1$ on $[-1, 1]$
(c) $f(x) = \sin(x^2)$ on $[0, \sqrt{\pi}]$

70. In each part, determine whether all of the hypotheses of the Mean-Value Theorem are satisfied on the stated interval. If not, state which hypotheses fail; if so, find all values of c guaranteed in the conclusion of the theorem.
(a) $f(x) = |x - 1|$ on $[-2, 2]$
(b) $f(x) = \dfrac{x+1}{x-1}$ on $[2, 3]$
(c) $f(x) = \begin{cases} 3 - x^2 & \text{if } x \le 1 \\ 2/x & \text{if } x > 1 \end{cases}$ on $[0, 2]$

71. Use the fact that

$$\frac{d}{dx}(x^6 - 2x^2 + x) = 6x^5 - 4x + 1$$

to show that the equation $6x^5 - 4x + 1 = 0$ has at least one solution in the interval $(0, 1)$.

72. Let $g(x) = x^3 - 4x + 6$. Find $f(x)$ so that $f'(x) = g'(x)$ and $f(1) = 2$.

CHAPTER 3 MAKING CONNECTIONS

1. Suppose that $g(x)$ is a function that is defined and differentiable for all real numbers x and that $g(x)$ has the following properties:
(i) $g(0) = 2$ and $g'(0) = -\frac{2}{3}$.
(ii) $g(4) = 3$ and $g'(4) = 3$.
(iii) $g(x)$ is concave up for $x < 4$ and concave down for $x > 4$.
(iv) $g(x) \ge -10$ for all x.

Use these properties to answer the following questions.
(a) How many zeros does g have?
(b) How many zeros does g' have?
(c) Exactly one of the following limits is possible:

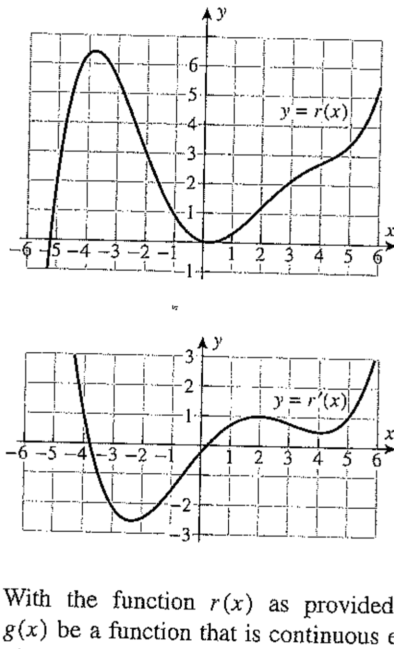
$$\lim_{x\to+\infty} g'(x) = -5, \quad \lim_{x\to+\infty} g'(x) = 0, \quad \lim_{x\to+\infty} g'(x) = 5$$

Identify which of these results is possible and draw a rough sketch of the graph of such a function $g(x)$. Explain why the other two results are impossible.

2. The two graphs in the accompanying figure depict a function $r(x)$ and its derivative $r'(x)$.
(a) Approximate the coordinates of each inflection point on the graph of $y = r(x)$.
(b) Suppose that $f(x)$ is a function that is continuous everywhere and whose *derivative* satisfies

$$f'(x) = (x^2 - 4) \cdot r(x)$$

What are the critical points for $f(x)$? At each critical point, identify whether $f(x)$ has a (relative) maximum, minimum, or neither a maximum or minimum. Approximate $f''(1)$.



◀ **Figure Ex-2**

3. With the function $r(x)$ as provided in Exercise 2, let $g(x)$ be a function that is continuous everywhere such that $g'(x) = x - r(x)$. For which values of x does $g(x)$ have an inflection point?

4. Suppose that f is a function whose derivative is continuous everywhere. Assume that there exists a real number c such that when Newton's Method is applied to f, the inequality
$$|x_n - c| < \frac{1}{n}$$
is satisfied for all values of $n = 1, 2, 3, \ldots$.
(a) Explain why
$$|x_{n+1} - x_n| < \frac{2}{n}$$
for all values of $n = 1, 2, 3, \ldots$.
(b) Show that there exists a positive constant M such that
$$|f(x_n)| \le M|x_{n+1} - x_n| < \frac{2M}{n}$$
for all values of $n = 1, 2, 3, \ldots$.
(c) Prove that if $f(c) \neq 0$, then there exists a positive integer N such that
$$\frac{|f(c)|}{2} < |f(x_n)|$$
if $n > N$. [*Hint:* Argue that $f(x) \to f(c)$ as $x \to c$ and then apply Definition 1.4.1 with $\epsilon = \frac{1}{2}|f(c)|$.]
(d) What can you conclude from parts (b) and (c)?

5. What are the important elements in the argument suggested by Exercise 4? Can you extend this argument to a wider collection of functions?

6. A bug crawling on a linoleum floor along the edge of a plush carpet encounters an irregularity in the form of a 2 in by 3 in rectangular section of carpet that juts out into the linoleum as illustrated in Figure Ex-6a. The bug crawls at 0.7 in/s on the linoleum, but only at 0.3 in/s through the carpet, and its goal is to travel from point A to point B. Four possible routes from A to B are as follows: (i) crawl on linoleum along the edge of the carpet; (ii) crawl through the carpet to a point on the wider side of the rectangle, and finish the journey on linoleum along the edge of the carpet; (iii) crawl through the carpet to a point on the shorter side of the rectangle, and finish the journey on linoleum along the edge of the carpet; or (iv) crawl through the carpet directly to point B. (See Figure Ex-6b.)

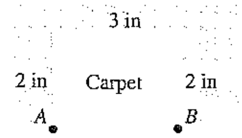


Figure Ex-6a

(a) Calculate the times it would take the bug to crawl from A to B via routes (i) and (iv).
(b) Suppose the bug follows route (ii) and use x to represent the total distance the bug crawls on linoleum. Identify the appropriate interval for x in this case, and determine the shortest time for the bug to complete the journey using route (ii).
(c) Suppose the bug follows route (iii) and again use x to represent the total distance the bug crawls on linoleum. Identify the appropriate interval for x in this case, and determine the shortest time for the bug to complete the journey using route (iii).
(d) Which of routes (i), (ii), (iii), or (iv) is quickest? What is the shortest time for the bug to complete the journey?

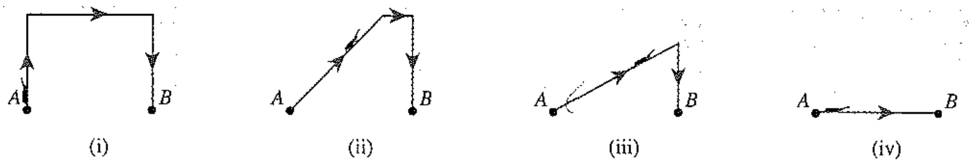


Figure Ex-6b