1 Forier Series

1.1 Real Fourier Series

The Fourier Series is an expansion of a function much like the Taylor Series. In the mid-19th century mathematicians were interested in solving several partial differential equations originating from physics such as the heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{1}$$

where $u(x,t)$ is the temperature at a point x at time t and k is a constant indicating the thermal conductivity of a substance.

These kind of equations could initially only be solved under certain simplifying assumptions, for example that the spatial profile of the temperature was a sine or cosine wave:

$$u(x,t) = a(t)\cos(\lambda x) \tag{2}$$

Due to $\cos(\lambda x)$ being an eigenfunction of the second derivative, the heat equation, under these assumptions, could be reduced to a first order ordinary differential equation for $a(t)$:

$$\frac{da}{dt} = -\lambda^2 a \tag{3}$$

the solution of which is:

$$a(t) = e^{-\lambda^2 t} \tag{4}$$

Fourier's idea was to solve the heat equation with no assumptions about the spatial profile of the temperature, under the assumption that any function can be expanded as a sum of sine and cosine waves. This would allow one to use the eigenfunction properties of $\cos(x)$ and $\sin(x)$ to solve the heat equation in general.

Fourier initially considered the case of **periodic functions**. A periodic function is a function that repeats its values after some set interval. Specifically a function is periodic if there exists some C such that:

$$f(x+C) = f(x) \tag{5}$$

for all x.
In general there will be many such values. For instance, $\cos(x)$ repeats itself over intervals of $2\pi, 4\pi, ...$ and hence $C = 2\pi n$. However all these values are simply multiples of $C = 2\pi$, the smallest interval over which $\cos(x)$ repeats itself. This smallest interval will be known as **the period** in the course and denoted L. So,

for $\cos(x)$ we have the period $L = 2\pi$.[1]
Periodic functions can be thought of as functions on a circle of circumfrence L. Points on the unit circle for example are labelled by a coordinate $\theta \in [0, 2\pi)$. Naturally we have the condition $f(\theta) = f(\theta + 2\pi)$, for any function $f(\theta)$ on the circle, as θ and $\theta + 2\pi$ are simply two different labels for the same point. Similarly a function on a circle of cirumference L will obey:

$$f(x) = f(x + L) \tag{6}$$

Fourier's proposal was that all periodic functions could be expanded as a combination of sine and cosine waves, that one could find a series of the following form:

$$f(x) = \sum_{n=0}^{\infty} a_n \cos(\lambda_n x) + \sum_{n=0}^{\infty} b_n \sin(\lambda_n x) \tag{7}$$

with $f(x)$ expanded in terms of trigonometric functions just as the Taylor expansion expresses a function as a sum of monomials x^n.

The first question is the approriate form of the cosine and sine waves, $\cos(\lambda_n x)$ and $\sin(\lambda_n x)$. To be used in the expansion of a function of period L, they must also repeat in value over that interval. As $\cos(x)$ and $\sin(x)$ have period 2π, we can modify their arguement so that they have period L:

$$\cos\left(\frac{2\pi}{L}x\right) \tag{8}$$

$$\sin\left(\frac{2\pi}{L}x\right) \tag{9}$$

Functions with integer multiples of this arguement:

$$\cos\left(\frac{2\pi n}{L}x\right) \tag{10}$$

$$\sin\left(\frac{2\pi n}{L}x\right) \tag{11}$$

have period $\frac{L}{n}$ and hence also repeat over an interval of size L and can be included in the expansion. Hence we propose that for a function $f(x)$ of period L:

$$f(x) = \sum_{n=0}^{\infty} \cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=0}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right) \tag{12}$$

We will simplfy this a bit. First of all looking at the case $n = 0$, we find that the trigonometric functions are:

$$\cos(0) = 1 \tag{13}$$

$$\sin(0) = 0 \tag{14}$$

[1]Other texts will use "period" for any interval over which the function repeats itself and fundamental/prime period for the smallest interval.

hence we can remove the $n = 0$ case from the sine wave series (as it vanishes) and seperate the $n = 0$ case of the cosine series (as it is simply a constant). This gives:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right) \tag{15}$$

Although this is commonly used as a definition of the Fourier Series, it produces unwanted extra factors of 2 in certain expressions that we will derive later. For this reason the Fourier series is commonly defined as:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right) \tag{16}$$

This expansion will be known as **the Real Fourier Series** in this course.

In terms of interpretation, if the function $f(x)$ describes a sound wave for example, the each sine and cosine wave would be an individual frequency of sound. Low values of n would correspond to low pitched sounds and high n to high pitched sounds. The coefficients a_n, b_n would then measure the amplitudes of each pitch, or how much that pitch contributes to the overall sounds.
If $f(x)$ describes a beam of light, the sine and cosine waves for different values of n describe different frequencies of light (i.e. different colours). Low values of n being radio waves, mid-range values being visible light and high n values being ultraviolet. a_n, b_n then measure the intensity of each frequency in the beam of light.
Another interpretation is that the sine and cosine waves capture details of the function on certain length scales. The waves indexed by n capturing details on the scale L/n. So cutting off the Fourier at some fixed value N, as an approximation of the function, would produce a function which is similar to $f(x)$ when looked at on scales larger than L/N.
The main advantages of the Fourier Series over the Taylor series are:

1. It preserves the periodicity of a function. In the Taylor series the monomials x^n are not periodic.

2. $\cos(x)$ and $\sin(x)$ are eigenfunctions of the second derivative and so the expansion simplifies many second-order partial differential equations.

3. The Fourier Series seperates information on different length scales. This can be useful if one wishes to remove information on certain scales. For example humans can not hear certain frequencies of sound, which are removed using the Fourier Series (or more correctly the Fourier Transform, to be introduced later) as it can seperate out these high frequency sounds.

For the Taylor series about a point $x = a$, given by:

$$f(x) = \sum_{n=0}^{\infty} d_n (x-a)^n \tag{17}$$

We have a method to compute the coefficients in the expansion, namely:

$$d_n = \frac{f^{(n)}(a)}{n!} \tag{18}$$

We need similar methods of computing the coefficients a_0, a_n, b_n for the Fourier Series to be useful.

In order to motivate the formulae for a_0, a_n, b_n we use an analogy. The Fourier Series:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right) \tag{19}$$

expands a function much the same as how in linear algebra we expand a vector:

$$\underline{V} = \sum_{n=1}^{N} v_n \underline{e}_n \tag{20}$$

Here we expand a general vector as a combination of basis vectors $\underline{e}_n$ with coefficients v_n. Similarly, one could see the Fourier series as a expansion of a general periodic function as a combination of "basis" functions:

$$\cos\left(\frac{2\pi n}{L}x\right) \tag{21}$$

$$\sin\left(\frac{2\pi n}{L}x\right) \tag{22}$$

with coefficients a_0, a_n, b_n. Note that this allows one to view a periodic function as an "infinite-dimensional" vector, this insight is one of the key components of the branch of mathematics known as Functional Analysis. To compute the coefficients in the linear algebra case one uses the dot-product:

$$v_n = \underline{v} \cdot \underline{e}_n \tag{23}$$

In the case of functions we might guess that the coefficients are given by some version of this formula. For functions the analogue of the dot-product will be the integral over their period. The analogue of this formula then would be:

$$a_n = D \int_{-L/2}^{L/2} f(x) \cos\left(\frac{2\pi n}{L}x\right) dx \tag{24}$$

$$b_n = E \int_{-L/2}^{L/2} f(x) \sin\left(\frac{2\pi n}{L}x\right) dx \tag{25}$$

with contants D, E as we don't know if the coefficients are only proportional to these integrals.

Let us know prove that this proposal for the coefficients is correct. We will evaluate the integral:

$$\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi n}{L}x\right)dx \tag{26}$$

First of all, we replace $f(x)$ with its Fourier Series:

$$\int_{-L/2}^{L/2}\left(\frac{a_0}{2}+\sum_{m=1}^{\infty} a_m\cos\left(\frac{2\pi m}{L}x\right)+\sum_{m=1}^{\infty} b_n\sin\left(\frac{2\pi m}{L}x\right)\right)\cos\left(\frac{2\pi n}{L}x\right)dx \tag{27}$$

We use m for the sine and cosine functions from the Fourier series of $f(x)$ and n for the cosine function we are integrating against.
we make the unjustified assumption[2] that the infinite sums can be pulled outside the integral and we find:

$$\begin{aligned}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi n}{L}x\right)dx &= \frac{a_0}{2}\int_{-L/2}^{L/2}\cos\left(\frac{2\pi n}{L}x\right)dx &(28)\\ &+\sum_{m=1}^{\infty} a_m\int_{-L/2}^{L/2}\cos\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)dx &(29)\\ &+\sum_{m=1}^{\infty} b_m\int_{-L/2}^{L/2}\sin\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)dx &(30)\end{aligned}$$

We work out each of the three integrals in turn. First of all:

$$\begin{aligned}\int_{-L/2}^{L/2}\cos\left(\frac{2\pi n}{L}x\right)dx &= \left[\frac{L}{2\pi n}\sin\left(\frac{2\pi n}{L}x\right)\right]_{-L/2}^{L/2} &(31)\\ &= \frac{L}{2\pi n}(\sin(\pi n)-\sin(-\pi n)) = 0 &(32)\end{aligned}$$

As $\sin(\pi n) = 0$. This is an integral of a cosine function over a multiple of its period, which we have just found to vanish. The same is also true of a sine function. This will be useful below.

The second integral:

$$\int_{-L/2}^{L/2}\cos\left(\frac{2\pi m}{L}x\right)\cos\left(\frac{2\pi n}{L}x\right)dx \tag{33}$$

can be integrated using the trigonometric identity $\cos(a)\cos(b) = \frac{1}{2}\left[\cos(a-b)+\cos(a+b)\right]$. Hence this integral seperates into:

$$\frac{1}{2}\int_{-L/2}^{L/2}\cos\left(\frac{2\pi(m-n)}{L}x\right)dx+\frac{1}{2}\int_{-L/2}^{L/2}\cos\left(\frac{2\pi(m+n)}{L}x\right)dx \tag{34}$$

[2]This assumption does in fact turn out to be valid, but the proof of this requires the Lesbesgue theory of integration.

The second of these two integrals always vanishes as $m + n$ is just some integer p, producing the integral:

$$\frac{1}{2}\int_{-L/2}^{L/2} \cos\left(\frac{2\pi(p)}{L}x\right) dx \tag{35}$$

which is the integral of a cosine function over a multiple of its period, which vanishes. The first of the integrals also produces an integral of a cosine function, unless $m = n$ in which case we have:

$$\int_{-L/2}^{L/2} \frac{1}{2} dx = \frac{L}{2} \tag{36}$$

So we see that the result from this second integral is $\frac{L}{2}\delta_{nm}$. It vanishes unless $n = m$, in which case it is $\frac{L}{2}$. δ_{nm} known as the Kronekcer Delta is defined by the properties:

$$\delta_{nm} = 1, \quad n = m \tag{37}$$

$$= 0, \quad n \neq m. \tag{38}$$

The final integral:

$$\int_{-L/2}^{L/2} \sin\left(\frac{2\pi m}{L}x\right) \cos\left(\frac{2\pi n}{L}x\right) dx \tag{39}$$

can be integrated using the identity $\sin(a)\cos(b) = \frac{1}{2}\left[\sin(a - b) + \sin(a + b)\right]$ and so this integral becomes:

$$\frac{1}{2}\int_{-L/2}^{L/2} \sin\left(\frac{2\pi(m-n)}{L}x\right) dx + \frac{1}{2}\int_{-L/2}^{L/2} \sin\left(\frac{2\pi(m+n)}{L}x\right) dx \tag{40}$$

Just like the previous case we have the integral of a trigonometric function over a multiple of its period, which vanishes. In the previous integral the $n = m$ produced a constant function $\frac{1}{2}$ rather than a trigonometric function and so the integral was nonzero in that case. However in this integral the $n = m$ case produces $\sin(0) = 0$, so even this is zero. Hence this third integral vanishes.

Over all then we have:

$$\int_{-L/2}^{L/2} f(x) \cos\left(\frac{2\pi n}{L}x\right) dx = \sum_{m=1}^{\infty} a_m \frac{L}{2}\delta_{nm} \tag{41}$$

as the first and third integrals vanish and the second is $\frac{L}{2}\delta_{nm}$. In the sum on the right hand side, all terms vanish except for the term with $n = m$, as the Kronecker delta is zero for the other terms. Hence:

$$\int_{-L/2}^{L/2} f(x) \cos\left(\frac{2\pi n}{L}x\right) dx = a_n \frac{L}{2} \tag{42}$$

or:

$$a_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \cos\left(\frac{2\pi n}{L} x\right) dx \tag{43}$$

and so we have found the formula for a_n. The formula for b_n is worked out in exactly the same way. One simply replaces $\cos\left(\frac{2\pi n}{L}x\right)$ with $\sin\left(\frac{2\pi n}{L}x\right)$. After that the integrals are the same except one needs the identity $\sin(a)\cos(b) = \frac{1}{2}\left[\cos(a-b) - \cos(a+b)\right]$ for the third integral. This gives:

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin\left(\frac{2\pi n}{L} x\right) dx \tag{44}$$

The formula for b_n.

Finally we deal with a_0. This coefficient comes from the $n = 0$ case of the cosine waves. Hence with might expect it to be related to integrating $f(x)$ against the $n = 0$ cosine wave, which is $\cos(0) = 1$. That is:

$$a_0 = F \int_{-L/2}^{L/2} f(x) dx \tag{45}$$

again, we do not know the constant of proportionality. We evaluate the integral:

$$\int_{-L/2}^{L/2} f(x) dx \tag{46}$$

by replacing f(x) with its Fourier Series:

$$\int_{-L/2}^{L/2} f(x) dx = \frac{a_0}{2} \int_{-L/2}^{L/2} 1 dx \tag{47}$$

$$+ \sum_{m=1}^{\infty} a_m \int_{-L/2}^{L/2} \cos\left(\frac{2\pi m}{L} x\right) dx \tag{48}$$

$$+ \sum_{m=1}^{\infty} b_m \int_{-L/2}^{L/2} \sin\left(\frac{2\pi m}{L} x\right) dx \tag{49}$$

The second two integrals are integrals of trigonometric functions over their periods and hence they vanish, leaving us with:

$$\int_{-L/2}^{L/2} f(x) dx = \frac{a_0}{2} \int_{-L/2}^{L/2} 1 dx \tag{50}$$

The integral on the right-hand side is easy to perform, leaving us with:

$$\int_{-L/2}^{L/2} f(x) dx = \frac{a_0}{2} L \tag{51}$$

or:

$$a_0 = \frac{2}{L} \int_{-L/2}^{L/2} f(x) dx \tag{52}$$

It can be seen now why the factor of $\frac{1}{2}$ was introduced into the definition of a_0, since it makes its formula have the constant of proportionality as the formulae for a_n, b_n.

We now have all the information needed to compute the Fourier Series. The Series itself is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi n}{L}x\right) \tag{53}$$

with the coefficients given by:

$$a_0 = \frac{2}{L}\int_{-L/2}^{L/2} f(x)dx \tag{54}$$

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi n}{L}x\right)dx \tag{55}$$

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \tag{56}$$

Now that we have found the formulae for the coefficients we can work out some examples of the Fourier Series. Before we do so, we shall comment on a property of functions which can cause either the a_n or b_n coefficients to vanish. We mention the definitions first. A function is even, if it obeys the following condition:

$$f(x) = f(-x) \tag{57}$$

and odd if it obeys:

$$f(x) = -f(-x) \tag{58}$$

An even function is symmetric across $x = 0$ and an odd function is anti-symmetric across $x = 0$. All even monomials are even, e.g. x^2, x^4 and all odd monomials are odd, e.g. x, x^3, x^5. $\sin(\lambda x)$ is odd, for any constant λ and similarly $\cos(\lambda x)$ is even.

For the Fourier Series a function being odd or even as an effect on its coefficients. An odd function has $a_0 = a_n = 0$ and an even function has $b_n = 0$. The proof is not too difficult and we will only show it in the case of an even function. First of all, taking the formula for b_n:

$$b_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \tag{59}$$

with $f(x)$ an even function. We can split this integral into an integral on the positive axis and an integral on the negative axis:

$$b_n = \frac{2}{L}\int_{-L/2}^{0} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx + \frac{2}{L}\int_{0}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \tag{60}$$

We use the substitution $x \to -x$ in the first integral to obtain:

$$b_n = -\frac{2}{L}\int_{L/2}^{0} f(-x)\sin\left(-\frac{2\pi n}{L}x\right)dx + \frac{2}{L}\int_{0}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \tag{61}$$

The overall minus originates from the change in the measure: $dx \to -dx$.
The function being even means $f(x) = f(-x)$, so

$$b_n = -\frac{2}{L}\int_{L/2}^{0} f(x)\sin\left(-\frac{2\pi n}{L}x\right)dx + \frac{2}{L}\int_{0}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \tag{62}$$

$\sin(\lambda x)$ is odd, $\sin(-\lambda x) = -\sin(\lambda x)$, hence we have:

$$b_n = \frac{2}{L}\int_{L/2}^{0} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx + \frac{2}{L}\int_{0}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \tag{63}$$

We can then reverse the order of integration on the first integral to obtain:

$$b_n = -\frac{2}{L}\int_{0}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx + \frac{2}{L}\int_{0}^{L/2} f(x)\sin\left(\frac{2\pi n}{L}x\right)dx \tag{64}$$

both integrals now being the same except with opposite signs and so:

$$b_n = 0 \tag{65}$$

So we see that an even function as $b_n = 0$, for all n.

Let us then compute the Fourier Series of x and x^2.

We take the function:

$$f(x) = x, \quad -\pi < x \leq \pi \tag{66}$$
$$f(x) = f(x + 2\pi) \tag{67}$$

From this we can read off the period as $L = 2\pi$ and $f(x) = x$. Hence the formula for the Fourier Series becomes:

$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx) \tag{68}$$

Secondly x is an odd function, hence $a_0 = a_n = 0$ and so:

$$x = \sum_{n=1}^{\infty} b_n \sin(nx) \tag{69}$$

we only need to work out the b_n coefficients.
The general formula is:

$$b_n = \frac{2}{L} \int_{-L/2}^{L/2} f(x) \sin\left(\frac{2\pi n}{L} x\right) dx \tag{70}$$

with period $L = 2\pi$ and $f(x) = x$ this becomes:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx \tag{71}$$

This can be evaluated using integration by parts, choosing $u = x$ and $dv = \sin(nx)\, dx$ we find:

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx)\, dx \tag{72}$$
$$= \frac{1}{\pi} \left[\frac{-x\cos(nx)}{n} \right]_{-\pi}^{\pi} + \frac{1}{\pi} \int_{-\pi}^{\pi} \frac{\cos(nx)}{n} dx \tag{73}$$

Evaluating the second integral:

$$b_n = \frac{1}{\pi} \left[\frac{-x\cos(nx)}{n} \right]_{-\pi}^{\pi} + \frac{1}{\pi} \left[\frac{\sin(nx)}{n^2} \right]_{-\pi}^{\pi} \tag{74}$$

Substituting the limits we obtain and remembering:

1. $\cos(\lambda x) = \cos(-\lambda x)$
2. $\sin(n\pi) = \sin(-n\pi) = 0$

we find:

$$b_n = \frac{-2\cos(\pi n)}{n} \tag{75}$$

Often the expression for the Fourier series of a function can be simplified using the property of the cosine function: $\cos(\pi n) = (-1)^n$. So:

$$b_n = \frac{-2(-1)^n}{n} \tag{76}$$

We can then substitute the b_n coefficients into the Fourier series above in Eq.69 to find:

$$x = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx) \tag{77}$$

and so we have found the Fourier series of x. Technically speaking we have found the Fourier series of a periodic version of $f(x)$, so some texts will instead use:

$$x^{\circlearrowright} = \sum_{n=1}^{\infty} \frac{-2(-1)^n}{n} \sin(nx) \tag{78}$$

With $x^{\circlearrowright}$ denoting a periodic version of the function x.

We can see from the coefficients, which decrease as $\frac{1}{n}$, that it is the sine waves with a low value of n which contribute most to x.

Now for x^2, again we take its periodic version:

$$f(x) = x^2, \quad -\pi < x \le \pi \tag{79}$$

$$f(x) = f(x + 2\pi) \tag{80}$$

As x^2 is even, $b_n = 0$ for all n. The period is $L = 2\pi$ and the function is $f(x) = x^2$ and so the Fourier series is:

$$x^2 = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx) \tag{81}$$

To work out the a_n coefficients we use the general formula:

$$a_n = \frac{2}{L}\int_{-L/2}^{L/2} f(x)\cos\left(\frac{2\pi n}{L}x\right)dx \tag{82}$$

in this case $L = 2\pi$ and $f(x) = x^2$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)dx \tag{83}$$

We evaluate this using integration by parts with $u = x^2$, $dv = \cos(nx)dx$:

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2\cos(nx)dx \qquad = \frac{1}{\pi}\left(\left[\frac{x^2\sin(nx)}{n}\right]_{-\pi}^{\pi} - \int\frac{\sin(nx)}{n}2xdx\right) \tag{84}$$

The second integral itself must be evaluated using integration by parts with $u = 2x$ and $dv = \frac{\sin(nx)}{x}dx$. This gives:

$$a_n = \frac{1}{\pi}\left(\left[\frac{x^2\sin(nx)}{n}\right]_{-\pi}^{\pi} - \int\frac{\sin(nx)}{n}2xdx\right) \tag{85}$$

$$= \frac{1}{\pi}\left(\left[\frac{x^2\sin(nx)}{n}\right]_{-\pi}^{\pi} + \left[2x\frac{\cos(nx)}{n^2}\right]_{-\pi}^{\pi} - \int\frac{2\cos(nx)}{n^2}dx\right) \tag{86}$$

evaluating the last integral we have:

$$a_n = \frac{1}{\pi}\left(\left[\frac{x^2\sin(nx)}{n}\right]_{-\pi}^{\pi} + \left[2x\frac{\cos(nx)}{n^2}\right]_{-\pi}^{\pi} - \left[\frac{2\sin(nx)}{n^3}\right]_{-\pi}^{\pi}\right) \tag{87}$$

Substituting the limits and remembering:

1. $\sin(n\pi) = \sin(-n\pi) = 0$. This sets the first and third terms to zero.
2. $\cos(-n\pi) = \cos(n\pi) = (-1)^n$. This simplifies the second term.

we find:

$$a_n = \frac{4(-1)^n}{n^2} \tag{88}$$

This formula is not well defined when $n = 0$ and hence we can not use it to infer the value of a_0 which we must calculate seperately.
We use the formula:

$$a_0 = \frac{2}{L}\int_{-\pi}^{\pi} f(x)dx \tag{89}$$

in this case:

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x^2dx \tag{90}$$

This can be evaluated directly and we have:

$$a_0 = \frac{2\pi^2}{3} \tag{91}$$

Hence the Fourier series is:

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^n}{n^2}\cos(nx) \tag{92}$$

Again we can see that the cosine waves with low n contribute the most to the function, the contribution decreasing quite rapidly with n.

Although we have computed two examples of the Fourier series, we have said nothing at this point about the convergence of the series. For example in the case of $f(x) = x, x^2$, although we have found the coefficients it is possible that the Fourier series, when summed:

1. Diverges for all x.
2. Diverges over certain intervals: $x \in [a, b]$.
3. Diverges at isolated points.

unfortunately, we typically cannot use standard test like the ration or root test to find if the series converges or not, as they will generally give inconclusive answers.
In answer to this question we state (although we do not prove), **Dirichlet's theorem**. This states that if:

1. There are a finite number of extrema over the functions period.
2. There are a finite number of discontinuities over the period
3. $$\int |f(x)|^2 dx < \infty \tag{93}$$

Then:
The Fourier series converges to:

$$\frac{1}{2}\left[\lim_{x\to a^+} f(x) + \lim_{x\to a^-} f(x)\right] \tag{94}$$

which at points x where the function is continuous is simply $f(x)$.

Later we will see that integral:

$$\int |f(x)|^2 dx < \infty \tag{95}$$

is related to the sum of the Fourier coefficients and essentially gives the "length" (properly known as the norm) of the function when viewed as an infinite dimensional vector. The third condition essentially states that this "length" must be finite.
The first two conditions essentially rule out unusual functions like $\sin(1/x)$ which oscillate infinitely often, or like $e^{-\lceil \frac{1}{x} \rceil}$ which have an infinite number of discontinuities.[3]

[3] $\lceil x \rceil$ denotes the smallest integer larger than x, i.e. the integer produced when one rounds x up.

If we look at the truncated Fourier series:

$$f_N(x) = \frac{a_0}{2} + \sum_{n=1}^{N} a_n \cos\left(\frac{2\pi n}{L}x\right) + \sum_{n=1}^{N} b_n \sin\left(\frac{2\pi n}{L}x\right) \tag{96}$$

we often see the phenomena that this approximate series deviates from $f(x)$ more severely at discontinuities than at continuous points. In fact Gibbs and Wilbraham observed that the truncated Fourier series tends to disagree with $f(x)$ by a factor:

$$f_N(x) = f(x) + G \cdot a + g_N(x) \tag{97}$$

at discontinuties and

$$f_N(x) = f(x) + g_N(x) \tag{98}$$

at continuous points. With G a constant, known as the Wilbraham-Gibbs constant, a the size of the discontinuity and $g_N(x)$ a function which vanishes as $N \to \infty$. Hence, near discontinuties there is a finite error that never disappears even when we sum the entire series. This is known as Gibbs phenomena.
However as the series is summed this oscillation away from the correct value of $f(x)$ is concentrated more and more near the dicontinuity itself. In the limit where we sum the entire Fourier series, it essentially only occurs at the discontinuity and cancels against a similar finite error on the other side of the disconinuity.
It should be noted that Gibbs phenomena is related to the second condition in Dirichelt's theorem, as in that case there would be infinitely many Gibbs' phenomena oscillations which would prevent the Fourier series from converging.

1.2 Complex Fourier Series

We originally motivated the Fourier Series as a way of "transfering" the simple behaviour of the sine and cosine functions under the second derivative to any periodic function.
However this does not provide us with a way of dealing with partial differential equations involving the first derivative. In that case we must try to expand $f(x)$ in terms of an eigenfunction of the first derivative.
The obvious choice would be e^x, or in general e^{nx}. However, first of all, this function is not periodic and secondly it increases as $x \to \infty$ and so a series involving e^{nx} would probably not converge for large x. Contrast this behaviour with $\cos(x)$ and $\sin(x)$ which are bounded between 1 and -1.

For this reason we take e^{ix}, which is bounded, obeying $|e^{ix}| < 1$. Just as was the case for $\cos(x)$ and $\sin(x)$, this function has a period of 2π so we adjust it to obtain a function with period L:

$$e^{i\frac{2\pi}{L}x} \tag{99}$$

and then allow any variant with period L/n:

$$e^{i\frac{2\pi n}{L}x} \tag{100}$$

The series will also contain coefficients, however since there is only one type of function in this case there will only be a single set c_n. This **Complex Fourier Series** then takes the form:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x} \tag{101}$$

Notice that series runs from $n = -\infty$, so we will first explain the origin of these additional terms in the complex Fourier series.
The real Fourier Series is given by:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{L}\right) \tag{102}$$

The individual term for a fixed value of n is:

$$a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right) \tag{103}$$

Using the following identity:

$$\cos(\lambda x) = \frac{e^{i\lambda x} + e^{-i\lambda x}}{2} \tag{104}$$

$$\sin(\lambda x) = \frac{e^{i\lambda x} - e^{-i\lambda x}}{2i} \tag{105}$$

which expresses Sine and Cosine as sums of complex exponentials, the n-th term in the Fourier Series can be rewritten as:

$$\left(\frac{a_n - ib_n}{2}\right)e^{i\frac{2\pi nx}{L}} + \left(\frac{a_n + ib_n}{2}\right)e^{-i\frac{2\pi nx}{L}} \tag{106}$$

or simply:

$$\left(\frac{a_n - ib_n}{2}\right)e^{i\frac{2\pi (n)x}{L}} + \left(\frac{a_n + ib_n}{2}\right)e^{i\frac{2\pi (-n)x}{L}} \tag{107}$$

So we see that for the n-th term in the real Fourier Series we get a n and $-n$ term in the complex Fourier Series.
The coefficients of the n and $-n$ terms are also complex conjugates of eachother.

This means that our complex Fourier series is nothing more than a way of rewriting the real Fourier Series and hence all the same theorems about convergence of the series still apply. We also get the identity:

$$c_n = \frac{a_n - ib_n}{2} \tag{108}$$

Next we need a formula for the complex coefficents, c_n, just as we had for the real coefficients a_n and b_n.
Using the analogy with vectors, in this case complex vectors, we would guess at:

$$c_n = C\int_{-L/2}^{L/2} f(x)e^{-i(\frac{2\pi}{L})nx}dx \tag{109}$$

Where C is some constant we will work out in the course of proving the formula's validity. We use the function:

$$e^{-i(\frac{2\pi}{L})nx} \tag{110}$$

that is with $-i$ rather than i, again in analogy with complex vectors, where the dot product is computed using the transpose conjugate of one of the vectors.

We will now work out the integral guessed at in Eq.109.
So we have:

$$\int_{-L/2}^{L/2} f(x)e^{-i(\frac{2\pi}{L})nx}dx \tag{111}$$

as a first step we replace $f(x)$ by its complex Fourier Series:

$$\int_{-L/2}^{L/2}\left(\sum_{m=-\infty}^{\infty} c_m e^{i(\frac{2\pi}{L})mx}\right)e^{-i(\frac{2\pi}{L})nx}dx \tag{112}$$

bringing the sum and the coefficients outside the integral:

$$\sum_{m=-\infty}^{\infty} c_m \int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})mx} e^{-i(\frac{2\pi}{L})nx} dx \tag{113}$$

Multiplying the exponentials together:

$$\sum_{m=-\infty}^{\infty} c_m \int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})(m-n)x} dx \tag{114}$$

Looking at just the integral we have:

$$\int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})(m-n)x} dx \tag{115}$$

In the case where $m \neq n$ we have:

$$\int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})px} dx \tag{116}$$

with $p = m - n$.
This gives:

$$\int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})px} dx = \frac{L}{2\pi ip} \left[e^{i(\frac{2\pi}{L})px} \right]_{-L/2}^{L/2} \tag{117}$$

$$= \frac{L}{2\pi ip} \left(e^{ip\pi} - e^{-ip\pi} \right) = 0 \tag{118}$$

When $m = n$ we instead get the integral:

$$\int_{-L/2}^{L/2} 1 dx = L \tag{119}$$

These two results can be combined using the Kronecker delta:

$$\int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})(m-n)x} dx = \delta_{mn} L \tag{120}$$

and so we have:

$$\sum_{m=-\infty}^{\infty} c_m \int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})mx} e^{-i(\frac{2\pi}{L})nx} dx = \sum_{m=-\infty}^{\infty} c_m \delta_{mn} L \tag{121}$$

The Kronecker delta then sets $m = n$ and we have:

$$\sum_{m=-\infty}^{\infty} c_m \int_{-L/2}^{L/2} e^{i(\frac{2\pi}{L})mx} e^{-i(\frac{2\pi}{L})nx} dx = c_n L \tag{122}$$

Or in terms of the original form of the integral:

$$\int_{-L/2}^{L/2} f(x) e^{-i(\frac{2\pi}{L})nx} dx = c_n L \tag{123}$$

That is:

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i(\frac{2\pi}{L})nx} dx \tag{124}$$

Using this we can now work out the complex Fourier series of a function.

We begin by finding the complex Fourier series for a periodic version of x^2.

$$f(x) = x^2 \qquad -\pi < x < \pi; \tag{125}$$
$$f(x) = f(x + 2\pi) \tag{126}$$

In this case $L = 2\pi$ and so the formula for the coefficients reduces to:

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 e^{-inx} dx \tag{127}$$

This can be done using integration be parts taking $u = x^2$ and $dv = e^{-inx}dx$. This gives us:

$$c_n = \frac{1}{2\pi}\left(\left[\frac{x^2 e^{-inx}}{-in}\right]_{-\pi}^{\pi} - \int_{\pi}^{-\pi} \frac{2xe^{-inx}}{-in}\right) \tag{128}$$
$$= \frac{1}{2\pi}\left(\int_{\pi}^{-\pi} \frac{2xe^{-inx}}{in}\right) \tag{129}$$

as the boundary term cancels.
Performing integration be parts on this second integral, with $u = 2x$ $dv = \frac{e^{-inx}}{in}dx$ we find:

$$c_n = \frac{1}{2\pi}\left(\left[\frac{2xe^{-inx}}{n^2}\right]_{-\pi}^{\pi} - \int_{\pi}^{-\pi} \frac{2e^{-inx}}{n^2}\right) \tag{130}$$
$$= \frac{1}{2\pi}\left(\left[\frac{2xe^{-inx}}{n^2}\right]_{-\pi}^{\pi} - \left[\frac{e^{-inx}}{-in^3}\right]_{-\pi}^{\pi}\right) \tag{131}$$
$$= \frac{1}{2\pi}\left(\frac{4\pi(-1)^n}{n^2}\right) = \frac{2(-1)^n}{n^2} \tag{132}$$

It should be noted that this integral can also be evaluated using a trick known as "differentiation under the integral" sign.
Letting n be a continuous variable we get:

$$\frac{1}{2\pi}\int x^2 e^{-inx} dx = \frac{1}{2\pi}\int \left(\frac{-d^2}{dn^2}\right) e^{-inx} dx \tag{133}$$

We can take the n derivative out of the integral, as the integral is taken with respect to x:

$$\frac{1}{2\pi}\int x^2 e^{-inx} dx = \frac{1}{2\pi}\left(\frac{-d^2}{dn^2}\right)\int e^{-inx} dx \tag{134}$$

Now we evaluate the integral:

$$\frac{1}{2\pi}\left(\frac{-d^2}{dn^2}\right)\int e^{-inx} dx = \frac{1}{2\pi}\left(\frac{-d^2}{dn^2}\right)\frac{e^{-inx}}{-in} \tag{135}$$

Evaluating the derivative:

$$\frac{1}{2\pi}\left(\frac{-d^2}{dn^2}\right)\frac{e^{-inx}}{-in} = \frac{1}{2\pi}\left(\frac{-d}{dn}\right)\left(\frac{xe^{-inx}}{n} + \frac{e^{-inx}}{in^2}\right) \tag{136}$$

$$= \frac{1}{2\pi}\left(\frac{ix^2e^{-inx}}{n} + \frac{2xe^{-inx}}{n^2} + \frac{2e^{-inx}}{in^3}\right) \tag{137}$$

It can be seen that this has given the integration-by-parts chain, so substituting the limits π and $-\pi$ will produce the same results as the previous case.

We will now return to the original motivation for the Fourier series, the heat equation and demonstrate its solution.
The heat equation is given by:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \tag{138}$$

The function $u(x,t)$ is the temperature as a function of position and time and k is the thermal conductivity of the substance.
We know that any function of x can be expanded in terms of its Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{-i\lambda_n x} \tag{139}$$

$$\lambda_n = \frac{2\pi n}{L} \tag{140}$$

For the function $u(x,t)$ this means that u at any fixed time can be expanded like this. The time dependence is then to be found in the evolution of the coefficients c_n:

$$u(x,t) = \sum_{n=-\infty}^{\infty} c_n(t) e^{-i\lambda_n x} \tag{141}$$

$$\tag{142}$$

If we substitute this expansion into both sides of the heat equation we find:

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}\sum_{n=-\infty}^{\infty} c_n(t)e^{-i\lambda_n x} = \sum_{n=-\infty}^{\infty}\left(\frac{d}{dt}c_n(t)\right)e^{-i\lambda_n x} \tag{143}$$

$$k\frac{\partial^2 u}{\partial x^2} = \sum_{n=-\infty}^{\infty} c_n(t)k\frac{\partial^2 u}{\partial x^2}e^{-i\lambda_n x} = \sum_{n=-\infty}^{\infty}\left(-k\lambda_n^2 c_n(t)\right)e^{-i\lambda_n x} \tag{144}$$

Equating the two sides (the heat equation) we find:

$$\sum_{n=-\infty}^{\infty}\left(\frac{d}{dt}c_n(t)\right)e^{-i\lambda_n x} = \sum_{n=-\infty}^{\infty}\left(-k\lambda_n^2 c_n(t)\right)e^{-i\lambda_n x} \tag{145}$$

We can see from this that the heat equation essentially reduces to an ODE for the coefficients:

$$\frac{d}{dt}c_n(t) = -k\lambda_n^2 c_n(t) \tag{146}$$

Which as the solution:

$$c_n(t) = A_n e^{-k\lambda_n^2 t} \tag{147}$$

With A_n being the initial conditions $c_n(0) = A_n$. Placing this back in the Fourier series, we obtain as a solution to the heat equation:

$$u(x,t) = \sum_{n=-\infty}^{\infty} A_n e^{-k\lambda_n^2 t} e^{-i\lambda_n x} \tag{148}$$

From this expression for the temperature function, one can see the physical process of thermalisation. As $t \to \infty$ the functions $e^{-k\lambda_n^2 t} \to 0$, except for the case of $n = 0$ where the function reduces to 1 and hence has no time dependence. For that reason we find:

$$u(x,t) \to A_0, \quad t \to \infty \tag{149}$$

The temperature, for large times, becomes a constant distribution. This is thermalisation, as the whole body obtains a single constant temperature.

We begin with a theorem connecting the integral of a function with the sum of its coefficients, known as Parseval's theorem.
Parseval's theorem states:

$$\frac{1}{L}\int_{-L/2}^{L/2}|f(x)|^2 dx = \sum_{n=-\infty}^{\infty}|c_n|^2 \tag{150}$$

This is essentially just an analogue of the dot-product of a vector with itself:

$$\underline{V}\cdot\underline{V} = \sum_{i=1}^{N}|V_i|^2 \tag{151}$$

Before giving a proof we obtain the equivalent expression for the real Fourier series.
We have the relation:

$$c_n = \frac{a_n - ib_n}{2} \tag{152}$$

$$c_0 = \frac{a_0}{2} \tag{153}$$

and hence:

$$|c_n|^2 = \frac{a_n^2 + b_n^2}{4} \tag{154}$$

$$|c_0|^2 = \frac{a_0^2}{4} \tag{155}$$

and so:

$$\sum_{-\infty}^{\infty}|c_n|^2 = \frac{a_0^2}{4} + \sum_{n=-\infty, n\neq 0}^{\infty}\frac{a_n^2 + b_n^2}{4} \tag{156}$$

However the values of a_n and b_n appear in the expressions for c_n and c_{-n}, hence we actually copies of the sum over positive n:

$$\sum_{-\infty}^{\infty}|c_n|^2 = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\frac{a_n^2 + b_n^2}{2} \tag{157}$$

Giving the real Fourier series version of Parseval's theorem:

$$\frac{1}{L}\int_{-L/2}^{L/2}|f(x)|^2 dx = \frac{a_0^2}{4} + \sum_{n=1}^{\infty}\frac{a_n^2 + b_n^2}{2} \tag{158}$$

The Proof is a simple matter of integration. Firstly we take the integral:

$$\frac{1}{L}\int_{-L/2}^{L/2}|f(x)|^2 dx = \frac{1}{L}\int_{-L/2}^{L/2} f^*(x)f(x)dx \tag{159}$$

and replace both functions by their Fourier expansions, the expansion for $f^*(x)$ simply being the conjugate of that for $f(x)$:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{2\pi n}{L}x} \tag{160}$$

$$f^*(x) = \sum_{n=-\infty}^{\infty} c_n^* e^{-i\frac{2\pi n}{L}x} \tag{161}$$

so inserting this into the integral gives:

$$\frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2 dx = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n \frac{1}{L}\int_{-L/2}^{L/2} e^{i\frac{2\pi n}{L}x} e^{-i\frac{2\pi m}{L}x} dx \tag{162}$$

m being used to denote the terms in the expansion of $f^*(x)$. Also we have taken both sums outside the integral. Whether this is valid or not is a subtle mathematical issue that we ignore here.
Combining the exponentials we get:

$$\frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2 dx = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n \frac{1}{L}\int_{-L/2}^{L/2} e^{i\frac{2\pi(n-m)}{L}x} dx \tag{163}$$

The integral is now one we have performed before, which has the value $L\delta_{nm}$:

$$\frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2 dx = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} c_m^* c_n \delta_{nm} \tag{164}$$

This sets to zero all terms with $n \neq m$, reducing the sum to:

$$\frac{1}{L}\int_{-L/2}^{L/2} |f(x)|^2 dx = \sum_{m=-\infty}^{\infty} c_n^* c_n = \sum_{-\infty}^{\infty} |c_n|^2 \tag{165}$$

completing the proof.

Parseval's theorem can be used to calculate the values of various infinite sums, such as those involved in the Riemann-Zeta function. The Riemann-Zeta function, $\zeta(s)$ is defined by:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{166}$$

This sum converges for all complex s with $\Re(s) > 1$. For complex s with $\Re(s) \leq 1$ a seperate formula is used, which is beyond the scope of this course.
We will use Parseval's theorem to compute $\zeta(2)$ and $\zeta(4)$.

For $\zeta(2)$ we use Parseval's theorem in conjunction with the Fourier expansion of $f(x) = x$ with period $L = 2\pi$. This was found to have real series coefficients of:

$$a_0 = 0, \tag{167}$$

$$a_n = 0, \tag{168}$$

$$b_n = \frac{2(-1)^n}{n} \tag{169}$$

Placing these in the real series version of Parseval's theorem we find:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \sum_{n=1}^{\infty} \frac{2}{n^2} \tag{170}$$

$$\frac{1}{2\pi}\left[\frac{x^3}{3}\right]_{-\pi}^{\pi} = \sum_{n=1}^{\infty} \frac{2}{n^2} \tag{171}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{172}$$

and so we find $\zeta(2) = \frac{\pi^2}{6}$.

For $\zeta(4)$ we use the Fourier series of x^2 with period $L = 2\pi$ in conjunction with the complex version of Parseval's theorem. We have found that the coefficients are:

$$c_n = \frac{2(-1)^n}{n^2} \tag{173}$$

However, note that this produces a divergent value for $n = 0$. This is due to the fact that to obtain the expression for c_n we used the integration:

$$\int e^{-inx} dx = \frac{e^{-inx}}{-in} \tag{174}$$

however in the case of $n = 0$, the exponential is replaced by a constant, so this expression is no longer valid. We must compute c_0 seperately.

$$c_0 = \int \pi{-}\pi x^2 dx = \frac{\pi^2}{3} \tag{175}$$

Placing these value into the complex Parsevals' theorem we find:

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} x^4 dx = \frac{\pi^4}{9} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{4}{n^4} \tag{176}$$

$$\frac{\pi^4}{5} = \frac{\pi^4}{9} + \sum_{n=-\infty, n\neq 0}^{\infty} \frac{4}{n^4} \tag{177}$$

$$\sum_{n=-\infty, n\neq 0}^{\infty} \frac{4}{n^4} = \frac{4\pi^4}{45} \tag{178}$$

The sum is even in n, so we simply have two copies of the sum over positive n:

$$\sum_{n=1}^{\infty} \frac{8}{n^4} = \frac{4\pi^4}{45} \tag{179}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^4} = \frac{\pi^4}{90} \tag{180}$$

So, we have $\zeta(4) = \frac{\pi^4}{90}$.

All values of the Riemann-Zeta function at even integers, $\zeta(2n)$, can be computed using the Fourier series of x^n.
There is no known closed form expression for the odd integer values, such as $\zeta(3)$.

2 Fourier Transform

The Fourier series is very useful for solving partial differential equations as derivatives are converted into multiplicative factors. However, the Fourier series is restricted to periodic functions. We would like to obtain an analogue of the series for non-periodic functions.

This can be done simply by altering the complex Fourier series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n x} \tag{181}$$

$$\lambda_n = \frac{2\pi n}{L} \tag{182}$$

If $f(x)$ is non-periodic we must lift the restriction of periodicity from the exponential functions. Currently the terms λ_n are restricted to multiples of the integers to ensure periodicity. We can remove this by replacing λ_n by a general real number k:

$$e^{i\lambda_n x} \to e^{ikx} \tag{183}$$

Once we have done this, a few other alterations need to be made to the Fourier series.
Firstly, the coefficients c_n rather than being a function of the integers n, becomes a function of the variable k, denoted $\tilde{f}(k)$. (Rather than $c(k)$).
Secondly, the sum over the discrete variable n, is changed to an integral over the continuous variable k. Overall, then the series changes as follows:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\lambda_n x} \to f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \tag{184}$$

This formula is known as the inverse Fourier Transform.

We must also alter the formula for the coefficients, we currently read:

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-i\lambda_n x} dx \tag{185}$$

Making the replacements outlined above, we get:

$$\tilde{f}(k) = C \int_{-\infty}^{\infty} f(x) e^{-ikx} dk \tag{186}$$

This formula being known as the Fourier Transform.
The value for the constant C is not fixed, as there is no natural non-periodic version of the $\frac{1}{L}$ from the complex Fourier series. The correct choice is $C = \frac{1}{2\pi}$.
Giving:

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \tag{187}$$

$$\tilde{f}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) e^{-ikx} dk \tag{188}$$

These are the formulae used in this course.
The reason for this choice of C is that it ensures the transform of $\tilde{f}(k)$ is $f(x)$, rather than a constant multiple of $f(x)$. For a given choice of C we obtain the following sequence of transformations:

$$f(x) \to \tilde{f}(k) \to \frac{C}{2\pi} f(x) \tag{189}$$

Hence only for $C = 2\pi$ does the second transform return $f(x)$ making it the inverse of the first.

In truth, the factor of $\frac{1}{2\pi}$, required to make the transforms inverses of each other, can be split between the two integrals in various ways. One common choice is the symmetric Fourier transform:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \tag{190}$$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx} dk \tag{191}$$

and the canonical Fourier transform, which places the factor 2π in the exponentials:

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{i2\pi kx} dk \tag{192}$$

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) e^{-i2\pi kx} dk \tag{193}$$

The canonical and symmetric Fourier transforms have the advantage of giving the both $f(x)$ and $\tilde{f}(k)$ the same $\mathcal{L}^2$-norm:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk \tag{194}$$

Often in physics, the $\mathcal{L}^2$-norm has the interpretation of the total energy of the system or represents the total probability (always = 1), which we would want to remain the same whether describing the function using k or x.
The formula we use has the advantage of giving simpler solutions to differential equations, without constantly having factors of 2π show up in solutions.

The advantages of the Fourier series carrier over to the Fourier transform, for example the conversion of differentiation into multiplication. If we take a function $f(x)$ then it's derivative has the transform:

$$\frac{d}{dx} f(x) = \frac{d}{dx} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \tag{195}$$

$$f^{'}(x) = \int_{-\infty}^{\infty} \tilde{f}(k) \frac{d}{dx} e^{ikx} dk \tag{196}$$

$$f^{'}(x) = \int_{-\infty}^{\infty} \left(ik\tilde{f}(k) \right) e^{ikx} dk \tag{197}$$

Hence we see that $f^{'}(x)$ has transform $ik\tilde{f}(k)$ and so derivatives $\frac{d}{dx}$ become multiplication by ik. Similarly $\frac{d}{dk}$ inverse transforms to $-ix$.

As an example of a Fourier transform we take the block function:

$$f(x) = 1, \quad |x| < 1 \tag{198}$$
$$f(x) = 0, \quad |x| \geq 1 \tag{199}$$
$$\tag{200}$$

The Fourier transform is computed using:

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{-ikx}dk \tag{201}$$

in our case this becomes:

$$\tilde{f}(k) = \frac{1}{2\pi}\int_{-1}^{1} e^{-ikx}dk \tag{202}$$
$$= \frac{1}{2\pi}\left[\frac{e^{-ikx}}{-ik}\right]_{-1}^{1} \tag{203}$$
$$= \frac{1}{2\pi}\frac{e^{ik} - e^{-ik}}{ik} \tag{204}$$
$$= \frac{1}{\pi}\frac{\sin(k)}{k} \tag{205}$$

So the Fourier transform of the block wave is: $\tilde{f}(k) = \frac{1}{\pi}\frac{\sin(k)}{k}$.
It can be seen from the example above that the evaluation of the Fourier transform is very similar to evaluating the coefficients of the complex Fourier series.

Before proceeding we will need to get some idea, although we will not provide rigorous proofs, of when the Fourier integral converges. First of all, let us take a look at the Fourier transform:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-ikx}dk \tag{206}$$
$$\tag{207}$$

The modulous of the integrand is:

$$|f(x)e^{-ikx}| = |f(x)| \tag{208}$$

since the modulous of e^{ikx} is 1. Taking the modulous of the function prevents any cancellations in the integral and represents the "worst case scenerio" for its convergence:

$$|f(x)| = |\int_{-\infty}^{\infty} \tilde{f}(k)e^{ikx}dk| \leq \int_{-\infty}^{\infty} |f(x)|dx \tag{209}$$

Hence the Fourier transform will be well-defined provided that:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty \tag{210}$$

That is if $f \in \mathcal{L}^1$.

However the existence of the Fourier transform $\tilde{f}(k)$, doesn't mean that $\tilde{f}(k)$ itself is in $\mathcal{L}^1$ and in general in won't be. Hence for several functions $f(x) \in \mathcal{L}^1$, although one can transform to $\tilde{f}(k)$ we can not transform back.

The functions for which the transformation in both directions is well defined will be a more restricted class objects than $\mathcal{L}^1$, they are known as the Schwartz functions.

To define the Schwartz functions we take a rough look at the properties of functions in $\mathcal{L}^1$. The defining property is:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$

This puts certain constraints on the decay of the function at large x. For instance $f(x)$ should decay faster than $\frac{1}{x}$. For example taking two functions whose behaviour at large x is (indicated by $\approx$)$g(x) \approx \frac{1}{x}$ and $h(x) \approx \frac{1}{x^2}$ we see:

$$\begin{aligned} \int_{-\infty}^{\infty} |g(x)| dx \approx \lim_{a\to\infty} \int^{a} \frac{1}{x} dx = \lim_{a\to\infty} \ln(a) \\ \int_{-\infty}^{\infty} |h(x)| dx \approx \lim_{a\to\infty} \int^{a} \frac{1}{x^2} dx = \lim_{a\to\infty} \frac{-1}{a} \end{aligned} \tag{211}$$

One can see that the first integral will diverge and the second does not. So a function needs to decay like $f(x)$ $\frac{1}{x^{1+\epsilon}}$, for some $\epsilon > 0$.

In order to make both Fourier transforms well defined, we will increase the requirements on the decay of the function. $\mathcal{L}^1$ only demands the function decay faster than $\frac{1}{x}$, so let us demand that it decay faster than $\frac{1}{x^n}$ for all n, that is faster than any inverse power.

However, this still isn't enough to ensure the existence of the inverse transform. We can phrase this definition as:

$$\lim\sup |x^n f(x)| < \infty, \quad \forall n \geq 0$$

However upon transforming to Fourier space x^n becomes $D^n = \frac{d^n}{dk^n}$ and we have:

$$\lim\sup |D^n \tilde{f}(k)| < \infty, \quad \forall n \geq 0$$

This tells use nothing about the decay rate of $\tilde{f}(k)$, simply that its derivatives are bounded.

So we will stregthen the condition to:

$$\lim\sup |x^n D^m f(x)| < \infty, \quad \forall n, m \geq 0$$

which states that $f(x)$ and all its derivatives decay faster than any inverse power of x.
This transforms to:

$$\lim\sup |k^m D^n \tilde{f}(k)| < \infty, \quad \forall n, m \geq 0$$

which the exact same condition, and hence $\tilde{f}(k)$ also decays quickly enough for the inverse transform to be well-defined.
Functions obeying:

$$\lim\sup |x^n D^m f(x)| < \infty, \quad \forall n, m \geq 0$$

are known as Schwartz functions.

However if the function is not a Schwartz function, although the integral of the Fourier and inverse Fourier transforms are not well-defined, we can nevertheless define a Fourier transform. We will take the example of the function, of k, $\delta(k) = \frac{1}{2\pi}$. We will attempt to look at its transform $\delta(x)$, known as the Dirac delta distribution (also known as the Dirac delta function).

Although we know the integral of the inverse transform will not converge to a well-defined function, we will attempt to evaluate:

$$\int_{-\infty}^{\infty} g(x)\delta(x)dx \tag{212}$$

With $g(x)$ any function that has a well-defined Fourier transform. First of all we remember that $\delta(x)$ is meant to be the inverse transform of $\frac{1}{2\pi}$:

$$\delta(x) = \int \frac{1}{2\pi} e^{ikx} dk$$

We replace δ by this integral in Eq.212:

$$\int g(x)\delta(x)dx = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \frac{1}{2\pi} g(x) e^{ikx} dx \right) dk$$

we evaluate the x integral, which is simply the definition of $\tilde{g}(-k)$:

$$\int g(x)\delta(x)dx = \int_{-\infty}^{\infty} \tilde{g}(-k)dk = \int_{-\infty}^{\infty} \tilde{g}(k)dk$$

Finally, if we look at the formula for the inverse Fourier transform, we see that this integral is $g(0)$ and so:

$$\int_{-\infty}^{\infty} g(x)\delta(x)dx = g(0) \tag{213}$$

This result however is quite unusual if we try to understand $\delta(x)$ as a function. Let us assume $\delta(x)$ is a function. First of all, if we look at functions for which $g(0) = 0$, then we find:

$$\int_{-\infty}^{\infty} g(x)\delta(x)dx = 0$$

In order for this integral to vanish for all functions with $g(0) = 0$, we would need:

$$\begin{aligned} \delta(x) &= 0, \quad x \neq 0 \\ &= C, \quad x = 0 \end{aligned}$$

That is, δ must vanish away from the origin. Now returning to any general function $g(x)$ we can compute:

$$\int_{-\infty}^{\infty} g(x)\delta(x)dx = \lim_{\epsilon \to 0} \left(\int_{-\infty}^{-\epsilon} g(x)\delta(x)dx + \int_{\epsilon}^{\infty} g(x)\delta(x)dx \right)$$

However both of these integrals must vanish, since δ vanishes away from the origin, and so:

$$\int_{-\infty}^{\infty} g(x)\delta(x)dx = 0 \tag{214}$$

in contradiction with what we have computed.

The resolution of the contradiction of our first result, Eq.213, and the result we just obtained in Eq.214 is that $\delta(x)$ is not a function. Rather it is an example of a more general class of objects known as distributions. The proper theory of distributions is beyond the scope of this course, but a brief explanation is that a distribution is a (linear) functional, that is a (linear) function of functions. That is:

$$T : g(x) \to C$$

where T is a distribution, $g(x)$ a function and C some constant. This is often written as $T(g(x))$.
One can form a distribution from a function $f(x)$ using the integral:

$$f(g(x)) = \int_{-\infty}^{\infty} f(x)g(x)dx \tag{215}$$

which maps a given function $g(x)$ to the constant given by the integral.
The Dirac delta distribution is not a function, as the map associated with it:

$$\delta(g(x)) = g(0)$$

cannot be obtained using the integral of $g(x)$ against some function $f(x)$ as in Eq.215.

The Dirac delta distribution can also be understood as the derivative of the Heaviside step function. This function is defined as:

$$\Theta(x) = 1, \quad x \geq 0 \tag{216}$$
$$= 0, \quad x < 0 \tag{217}$$

Evaluating the integral:

$$\int_{-\infty}^{\infty} \frac{d\Theta}{dx} f(x) dx \tag{218}$$

where $f(x)$ is a function of rapid decay. Using integration by parts with $u = f(x)$ and $dv = \frac{d\Theta}{dx} dx$ we find:

$$\int_{-\infty}^{\infty} \frac{d\Theta}{dx} f(x) dx = [f(x)\Theta(x)]_{-\infty}^{\infty} - \int_{\infty}^{\infty} \Theta(x) \frac{df}{dx} dx \quad = -[f(x)]_{0}^{\infty} = f(0) \tag{219}$$

So we see that:

$$\int_{-\infty}^{\infty} \frac{d\Theta}{dx} f(x) dx = 0 \tag{220}$$

hence, $\frac{d\Theta}{dx} = \delta(x)$.

Similarly we can compute the derivative of the Dirac delta function, suing the same trick:

$$\int_{-\infty}^{\infty} \frac{d\delta}{dx} f(x) dx = [f(x)\delta(x)]_{-\infty}^{\infty} - \int_{\infty}^{\infty} \delta(x) \frac{df}{dx} dx = -f^{'}(0) \tag{221}$$

So the Dirac delta distribution's derivative obeys:

$$\int_{-\infty}^{\infty} \frac{d\delta}{dx} f(x) dx = -f^{'}(0) \tag{222}$$

We can simply use integration by parts n-times in order to compute the n-th derivative. The result is quite simple:

$$\int_{-\infty}^{\infty} \frac{d^n\delta}{dx^n} f(x) dx = (-1)^n f^n(0) \tag{223}$$

We can also evaluate a translated version of the Dirac delta function using substitution:

$$\int_{-\infty}^{\infty} \delta(x-a) f(x) dx \tag{224}$$

using the substitution $u = x - a$ we get:

$$\int_{-\infty}^{\infty} \delta(u) f(u+a) du = f(a) \tag{225}$$

and hence:

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a) \tag{226}$$

One can also evaluate:

$$\int_{-\infty}^{\infty} \delta(ax)f(x)dx \tag{227}$$

using substitution. Specifically $u = ax$:

$$\int_{-\infty}^{\infty} \delta(u)f(u/a)\frac{du}{|a|} = \frac{f(0)}{|a|} \tag{228}$$

The presence of $|a|$ is due to the fact that negative values of a wil reverse the order of integration, flipping the order to its original form introduces a negative sign, cancelling the negative sign of a.

Finally we can also consider a Dirac delta distribution composed with a function $g(x)$:

$$\int_{-\infty}^{\infty} \delta(g(x))f(x)dx \tag{229}$$

We will assume for now that $g(x)$ is a monotonically increasing function. Substitution of $u = g(x)$ gives:

$$\int_{-\infty}^{\infty} \delta(u)f(g^{-1}(u))\frac{du}{|g'(g^{-1}(u))|} \tag{230}$$

Remembering that $x = g^{-1}(u)$ and that we have a modulous sign for the same reasons as the previous case, and so we have:

$$\int_{-\infty}^{\infty} \delta(u)f(g^{-1}(u))\frac{du}{|g'(g^{-1}(u))}| = \frac{f(g^{-1}(0))}{|g'(g^{-1}(0))|} \tag{231}$$

However $g^{-1}(0)$ is actually the specific value x_* for which $g(x_*) = 0$ so this is also written as:

$$\int_{-\infty}^{\infty} \delta(g(x))f(x)dx = \frac{f(x_*)}{|g'(x_*)|} \tag{232}$$

We finally prove an important property of the Fourier transform, the analogue of Parseval's theorem, known as Plancheral's theorem. This states that:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = 2\pi \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dx \tag{233}$$

First of all we replace $f(x)$ and its conjugate $\tilde{f}(k)$ in the integral:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} f(x) f^*(x) dx \tag{234}$$

with their Fourier transforms:

$$f(x) = \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \tag{235}$$

$$f^*(x) = \int_{-\infty}^{\infty} \tilde{f}^*(k) e^{-ikx} dk \tag{236}$$

which gives:

$$\int_{-\infty}^{\infty} f(x) f^*(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx} dk \int_{-\infty}^{\infty} \tilde{f}^*(q) e^{-iqx} dq dx \tag{237}$$

This can be rearranged as:

$$\int_{-\infty}^{\infty} f(x) f^*(x) dx = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(q) \int_{-\infty}^{\infty} e^{i(k-q)x} dx dq dk \tag{238}$$

The x-integral can be performed immediately as it simply defines the Dirac delta, that is:

$$\int_{-\infty}^{\infty} e^{i(k-q)x} dx = 2\pi \delta(k-q) \tag{239}$$

and so Eq.238 reduces to:

$$\int_{-\infty}^{\infty} f(x) f^*(x) dx = 2\pi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(q) \delta(k-q) dq dk \tag{240}$$

Performing the k-integral, the delta function just sets $k = q$ and so:

$$\int_{-\infty}^{\infty} f(x) f^*(x) dx = 2\pi \int_{-\infty}^{\infty} \tilde{f}(k) \tilde{f}^*(k) dk \tag{241}$$

$$= 2\pi \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk \tag{242}$$

Which completes the proof. Note that had we used the symmetric definition of the Fourier transform we would have obtained two factors of $\frac{1}{\sqrt{2\pi}}$ from both the k-integral and the q-integral. This would have cancelled the 2π in front and produced:

$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dx \tag{243}$$

which is often used in theoretical physics.

We end on the Fourier transform of the Gaussian:

$$f(x) = e^{-\frac{x^2}{\sigma}} \tag{244}$$

where σ gives the width of the Gaussian.
Although we could evaluate the Fourier transform directly, we will instead compute using the transformation properties of derivatives.
The Gaussian, $f(x)$ obeys the differential equation:

$$\frac{d}{dx}f(x) = -\frac{x}{\sigma}f(x) \tag{245}$$

transforming over to Fourier space, we already know that $f(x)$ becomes $\tilde{f}(k)$, $\frac{d}{dx}$ becomes ik, and x becomes $i\frac{d}{dk}$, hence this equation transforms to:

$$k\tilde{f}(k) = -\frac{1}{\sigma}\frac{d}{dk}\tilde{f}(k) \tag{246}$$

or:

$$\frac{d}{dk}\tilde{f}(k) = -k\sigma\tilde{f}(k) \tag{247}$$

which is simply the same equation as for the Gaussian in x-space, except with σ inverted, the solution is then:

$$\tilde{f}(k) = e^{-k^2\sigma} \tag{248}$$

a Gaussian with width $\frac{1}{\sigma}$. We then find an inverse relationship between the width of a Gaussian in k-space and x-space.
In general the width (average distance from average) of a function is calculated via:

$$\Delta x = \int_{-\infty}^{\infty} x^2|f(x)|^2dx \tag{249}$$

using coordinates where the average is at $x = 0$. We find that for the Fourier transform:

$$\Delta k\Delta x \geq \frac{1}{2} \tag{250}$$

hence the width of a function has an inverse relationship in k-space and x-space.
In quantum mechanics $p = \hbar k$ and so we recover the uncertainty principle:

$$\Delta k\Delta x \geq \frac{\hbar}{2} \tag{251}$$

3 Vector Calculus

3.1 Vector Fields and coordinates transformations.

We now move onto Vector Calculus, the study of vector-valued functions.

All the functions we will be dealing live either in two or three dimensions. Two/Three dimenionsal Euclidean space will be denoted: $\mathbb{E}^2$ or $\mathbb{E}^3$.
It should be noted, although this will be covered in more detail later in the course, that we often model Euclidean space using the coordintes (x, y, z). The space of this coordinates is denoted $\mathbb{R}^3$, a vector space. However this is just a coordinate system and should not be confused with the underlying space $\mathbb{E}^3$, (x, y, z) is just a method of labelling points in $\mathbb{E}^3$.

We begin by defining a few concepts that are the foundation of the rest of the material. The first is a scalar field.
Scalar Field: A function ϕ, from Euclidean space to the real numbers, that is: $\phi : \mathbb{E}^n \to \mathbb{R}$

When using a specific coordinate system, ϕ will appear as $\phi(x, y, z)$ a function of those three coordinates, for example:

$$\phi(x,y,z) = xy + z\phi(x,y,z) \qquad = x^2 + y + \sin(z) \tag{252}$$

$$\phi(x,y,z) = x^2yz + z\cos(y) \tag{253}$$

A mathematician would simply know this as a real valued function on $\mathbb{E}^3$.

The second concept is that of a path.
A path A function from the real numbers to Euclidean space, that is:
$\Gamma : \mathbb{R} \to \mathbb{E}^n$
This traces out a curve in Euclidean space, with each point, $p(t)$ on the curve being associated with a particular value of the path parameter t.
In coordinates a path will appear as coordinate function depending on time. For example:

$$\underline{x}(t) = (x(t), y(t), z(t)) = (t^2, t, t^3), \quad t \in [0,1] \tag{254}$$

This path begins at $(0, 0, 0)$ and ends at $(1, 1, 1)$.
We can also evaluate a scalar field on a path. For example:

$$\phi(x,y,z) = xy + z \tag{255}$$

$$\underline{x}(t) = (t^2, t, t^3), \quad t \in [0,1] \tag{256}$$

Here we just have our first scalar field above and the path given above. The values of the scalar field long the path are:

$$\phi(x(t), y(t), z(t)) = x(t)y(t) + z(t) = 2t^3 \tag{257}$$

Using this, we can also find the derivative of the function along the path:

$$\frac{d}{dt}\phi(x(t), y(t), z(t)) = 6t^2 \tag{258}$$

this being known as the path derivative. This should be distinguished from the partial derivative. For example if we take t to be time. The temporal partial derivative of the field above vanishes:

$$\frac{\partial}{\partial t}\phi(x, y, z) = \frac{\partial}{\partial t}(xy + z) = 0 \tag{259}$$

This is because the field only varies spatially. However the path derivative does not vanish, because the field still varies over time along the path.

The final notion is that of a vector field.
A Vector Field Is a function from Euclidean space to the real vector space of the same dimension.
$V : \mathbb{E}^n \to \mathbb{R}^n$.
Such a function essentially attaches a vector to each point of space.

With these concepts in place we can begin to generalise differentiation. There are four operations we can perform in this setting.

1. **The Gradient:**
The Gradient is typically denoted by $\nabla\phi$ and is calculated via the formula:

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right) \tag{260}$$

The meaning of this expression can seen by taking the path derivative of a scalar field, using the chain rule:

$$\frac{d}{dt}\phi = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt} \tag{261}$$

This is essentially a dot-product between the field and the path derivative of the path:

$$\frac{d}{dt}\phi = (\nabla\phi)\cdot\left(\frac{d\underline{x}}{dt}\right) \tag{262}$$

$\frac{d\underline{x}}{dt}$ gives the direction in which the path is changing, also known as the directional derivative.
This can be reexpressed as:

$$\frac{d}{dt}\phi = (\nabla\phi)\cdot\left(\frac{d\underline{x}}{dt}\right) = |\nabla\phi|\left|\frac{d\underline{x}}{dt}\right|\cos(\theta) \tag{263}$$

We can see here that the path derivative is at its maximum when the directional derivative is paralell to $\nabla\phi$. Hence $\nabla\phi$ points in the direction of greatest change of the field ϕ.

It should be noted that ∇ is a map from scalars to vectors.

2. **The Laplacian**
The operator is a map from scalar functions to scalar functions. It is defined as:

$$\triangle\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} \tag{264}$$

To find the meaning of this expression we turn to the discretised derivative. The discrete derivative can be written as:

$$\frac{\partial}{\partial x}\phi = \frac{\phi(x+h,y,z) - \phi(x,y,z)}{h} \tag{265}$$

or equivalently:

$$\frac{\partial}{\partial x}\phi = \frac{\phi(x+\frac{h}{2},y,z) - \phi(x-\frac{h}{2},y,z)}{h} \tag{266}$$

Applying this twice we get:

$$\frac{\partial^2}{\partial x^2}\phi = \frac{\phi(x+h,y,z) + \phi(x-h,y,z) - 2\phi(x,y,z)}{h^2} \tag{267}$$

or

$$\frac{\partial^2}{\partial x^2}\phi = \frac{2}{h^2}\left(\frac{\phi(x+h,y,z) + \phi(x-h,y,z)}{2} - \phi(x,y,z)\right) \tag{268}$$

The fraction inside the brackets is essnetially the average of the two values closest to the (x,y,z) in the x-direction.
Summing the second derivatives for the other two directions we find:

$$\triangle\phi = \frac{1}{6h^2}\left(\frac{\sum_{j=0}^{1}\sum_{i=1}^{3}\phi(\underline{x} + (-1)^j h\underline{e}_i)}{6} - \phi(x,y,z)\right) \tag{269}$$

We see that this is the difference between the average value of ϕ over the points closest to (x,y,z) and the value at (x,y,z), that is the deviation of $\phi(x,y,z)$ from the local average. If for example $\triangle\phi > 0$, the $\phi(x,y,z)$ is less than the local average.

This operator appear in several equations in mathematical physics for example Poisson's equation:

$$\triangle\phi = 4\pi G\rho \tag{270}$$

with ϕ the gravitational potential and ρ the mass density and G Newton's constant. The interpretation being the gravitational potential is less than the local average whenever some matter is present.

3. **The Divergence** The Divergence operator acts on a Vector field to produce a scalar. It is defined by:

$$\nabla \cdot \underline{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{271}$$

To give an interpretation to this, we will need a result known as Gauss' theorem, to be discussed later. For now, we simply say that it measures how much a given point acts as source or a sink for a vector field. For example, if $\nabla \cdot V > 0$ at some point (x, y, z) then the arrows of the vector field point away from that point, as if the vector field were emerging from it.

Given this interpretation we can then consider the electric field $\underline{E}$ from physics. Since charge creates electric fields we would expect an electric field to "emerge" from any point where there is some density of charge. Denoting charge density as ρ, we could guess

$$\nabla \cdot \underline{E} = \frac{\rho}{\epsilon_0} \tag{272}$$

with ϵ_0 a constant to the adjust units (if you measure electric charge and the electric field in the same units, it disappeares). This equation is in fact, the first Maxwell equation.
Now, magnetic fields do not have charges associated with them (although see later in the lectures). They always appear in a dipole like configuration, with the field running from the North to South poles of a magnet for example. There are always an equal number of lines flowing in (those coming from the south pole) and out (those flowing to the north pole). Hence, we would not expect a mangetic field to be either flowing into or out of any point. This can be encoded in the statement:

$$\nabla \cdot \underline{B} = 0 \tag{273}$$

this is the second Maxwell equation.

4. **The Curl** This final operation takes a vector field to another vector field. It is defined as:

$$\nabla \times \underline{V} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}, \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x}, \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \right) \tag{274}$$

again the full explanation of this operator will require a result known as Stoke's theorem. For now we say that the vector $\nabla \times \underline{V}$ at a point (x, y, z) conveys how much $\underline{V}$ is rotating about the point (x, y, z), specifically:

(a) It points perpendicular to the plane of rotation of the field $\underline{V}$ near the point (x, y, z). Which of the two perpendicular vectors it is parellel to is determined by the orientation of the flow.

(b) Its magnitude measures how strongly the field is rotating about the point (x, y, z).

Now using lead fillings it is often seen that a Magnetic field rotates about an electric current $\underline{J}$. From this we could conclude:

$$\nabla \times \underline{B} = \mu_0 \underline{J} \tag{275}$$

with μ_0 some constant to adjust the units. This is (almost) Maxwell's third equation.

It can also be observed that when one creates an alternating Magnetic current at a certain point, an electric field begins to be generated which flows/rotates about that point (electric induction). This can be encoded as:

$$\nabla \times \underline{E} = -\frac{\partial B}{\partial t} \tag{276}$$

These two quantities already have the same units. This is Maxwell's fourth equation.

The final component is that the opposite is true, a time varying electric field can create a rotating magentic field. Adding this fact to the third equation we get:

$$\nabla \times \underline{B} = \mu_0 \left(\underline{J} + \epsilon_0 \frac{\partial E}{\partial t} \right) \tag{277}$$

This is Maxwell's third equation. All four equations completely govern the dynamics of the electric and magnetic fields.

A final important point about Curl, is that it only really makes sense as an operation in three dimensions. In two dimensions, there is no direction perpendicular to the plane of the vector field and in four or higher dimensions there is no unique direction which is perpendicular. In three we have the unique relation among the coordinates, letting i, j, k denote the basis vectors and $\cdot$ denote the act of constructing a vector perpendicular (the cross product):

$$i \cdot j = k \tag{278}$$

$$k \cdot i = j \tag{279}$$

$$j \cdot k = i \tag{280}$$

This is essentially the quaternion algebra. There are other deeper reasons for the fact that Curl only exists in three dimensions, which we will explore when we get to coordinate transformations.

We will now look at coordinate transformations, starting with the two-dimensional case of cartesian and polar coordinates.
In two dimensions, we have the Cartesian coordinates (x, y), which indexes points on the plane using the vector space $\mathbb{R}^2$, via their horizontal and vertical displacement from some chosen "origin" point. However it also possible to describe the plane using an alternate coordinate system known as polar coordinates. In this case we label points via their distance, r from some chosen origin point, this being known as the radial distance. Of course there is a circle of points of equal distance r from the origin, which are distinguished by their location on the circle, indexed by the angle θ.
In polar coordinates a point is then labelled via (r, θ).
Assuming that they have a common choice of origin and that the unit circle is indexed so that $\theta = 0$ is the x-axis, then the relationship between the two coordinates is:

$$x = r\cos(\theta) \tag{281}$$
$$y = r\sin(\theta) \tag{282}$$
$$r = \sqrt{x^2 + y^2} \tag{283}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \tag{284}$$

With this relationship in place it isn't difficult to transfer a scalar function from one coordinate system to another via substitution. To take three examples:

$$\phi_1(x, y) = x^2 + y^2 \tag{285}$$
$$\phi_1(r, \theta) = r^2 \tag{286}$$
$$\phi_2(x, y) = \tan^{-1}\left(\frac{y}{x}\right)\sqrt{x^2 + y^2}^3 \tag{287}$$
$$\phi_2(r, \theta) = \theta r^3 \tag{288}$$
$$\phi_3(x, y) = \sin\left(\tan^{-1}\left(\frac{y}{x}\right)\right)^2 \tag{289}$$
$$\phi_3(r, \theta) = \sin(\theta)^2 \tag{290}$$

What requires much more attention is how one transforms Vector fields and the gradient, divergence and Curl operators.

To transform vector fields correctly we will look more closely at a vector field in Cartesian coordinates. This is given as:

$$\underline{V} = (V_x(x, y), V_y(x, y)) \tag{291}$$

this can be reexpressed as $\underline{V}$ being a combination of two basis vector fields:

$$\underline{V} = (V_x(x, y), V_y(x, y)) = (V_x)(1, 0) + (V_y)(0, 1) \tag{292}$$
$$= (V_x)\underline{e}_x + (V_y)\underline{e}_y \tag{293}$$

with $\underline{e}_x$ and $\underline{e}_y$ being the vector field with values $(1,0)$ and $(0,1)$ respectively at all points. Each of the basis vectors is assigns a vector pointing in the direction of a unit displacement of its associated coordinate. When we switch over to polar coordinates we will instead have to use $\underline{e}_r$ and $\underline{e}_\theta$, corresponding to radial and angular displacements.
We can see that it is not enough to simply change the coefficients V_x and V_y from functions of (x,y) to functions of (r,θ) we must also change the basis vectors in terms of which $\underline{V}$ is expanded.

To work out the transformation property, we will return to the idea of $\underline{e}_x$ being a small displacment in the x-coordinate. Hence we might imagine a relation:

$$\underline{e}_x = \frac{\partial}{\partial x} \tag{294}$$

as the derivative is also a variance in x.[4]
Taking the vector field $(1,0)$ in polar coordinates, we find:

$$\begin{aligned}(1,0) &= \underline{e}_r & (295)\\ &= \frac{\partial}{\partial r} & (296)\\ &= \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} & (297)\\ &= \cos(\theta)\frac{\partial}{\partial x} + \sin(\theta)\frac{\partial}{\partial y} & (298)\\ &= \cos(\theta)\underline{e}_x + \sin(\theta)\underline{e}_y & (299)\\ &= (\cos(\theta),\sin(\theta)) & (300)\end{aligned}$$

So we can see that the vector field we has the form $(1,0)$ in polar coordinates, has the form $(\cos(\theta),\sin(\theta))$ in cartesian.
Similarly for the angular vector:

$$\begin{aligned}\underline{e}_\theta &= \frac{\partial}{\partial \theta} & (301)\\ &= \frac{\partial x}{\partial \theta}\frac{\partial}{\partial x} + \frac{\partial y}{\partial \theta}\frac{\partial}{\partial y} & (302)\\ &= -r\sin(\theta)\frac{\partial}{\partial x} + r\cos(\theta)\frac{\partial}{\partial y} & (303)\\ &= -r\sin(\theta)\underline{e}_x + r\cos(\theta)\underline{e}_y & (304)\\ &= (-r\sin(\theta), r\cos(\theta)) & (305)\end{aligned}$$

Typically we want unit basis vectors, so instead we use:

$$\underline{e}_\theta = (-\sin(\theta),\cos(\theta)) \tag{306}$$

[4] In this course we see this as an analogy. However in differential geometry you will see that $\underline{e}_x = \frac{\partial}{\partial x}$. This is the correct definition of the basis vectors.

as our basis vector. We can then see that the relation between a vector in polar an cartesian coordinates is:

$$(V_r, V_\theta) = V_r(\cos(\theta), \sin(\theta)) + V_\theta(-\sin(\theta), \cos(\theta)) \tag{307}$$
$$= (V_r \cos(\theta) - \sin(\theta)V_\theta, V_r \sin(\theta) + V_\theta \cos(\theta)) \tag{308}$$

or in matrix notation:

$$\begin{bmatrix} V_x \\ V_y \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} V_r \\ V_\theta \end{bmatrix}$$

or in the opposite direction:

$$\begin{bmatrix} V_r \\ V_\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial r}{\partial x} & \frac{\partial r}{\partial y} \\ \frac{\partial \theta}{\partial x} & \frac{\partial \theta}{\partial y} \end{bmatrix} \begin{bmatrix} V_x \\ V_y \end{bmatrix}$$

This is how we transform a vector field from one coordinate system to another.

As a side note, one can think of objects which transform in the opposite way from vectors. That is:

$$\begin{bmatrix} W_r \\ W_\theta \end{bmatrix} = \begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial r}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix} \begin{bmatrix} W_x \\ W_y \end{bmatrix}$$

It might be that one of these objects $\mathbf{W}$ would have the same components in polar coordinates as a vector $\underline{V}$, but the difference between them would show up if you transformed to Cartesian coordinates. These objects are known as **one-forms**. This is why one must be careful with coordinates, they are simply a tool, two fundamentally different objects might appear identical in a given coordinate system.

Although probably the remit of differential geometry, we can imagine more general functions that vectors. Vector components are indexed by a single label V_i, we could imagine matrix functions M_{ij} requiring two indicies, giving an array of components. Or even more general functions M_{ijk} requiring several indicies. In each case one would have to specify whether the index transformed like a vector or a one form. The notation is:
M_{ij} is both tranform like one-forms.
M^i_j if one transforms like a vector and one like a one-form.
M^{ij} if both transform like vectors.

Such functions are known as $\binom{p}{q}$-tensors, indicating p of the indicies transform like vectors and q like one-forms.

We now move onto transforming the gradient and divergence operations.[5]

[5]We could transform curl as well, but it is simply very tedious to do so.

Gradient:
The gradient was defined as

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) \tag{309}$$

using the chain rule, we find:

$$\begin{aligned}
\nabla\phi &= \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}\right) && (310)\\
&= \left(\frac{\partial r}{\partial x}\frac{\partial\phi}{\partial r} + \frac{\partial\theta}{\partial x}\frac{\partial\phi}{\partial\theta}, \frac{\partial r}{\partial y}\frac{\partial\phi}{\partial r} + \frac{\partial\theta}{\partial y}\frac{\partial\phi}{\partial\theta}\right) && (311)\\
&= \left(\cos(\theta)\frac{\partial\phi}{\partial r} - \frac{1}{r}\sin(\theta)\frac{\partial\phi}{\partial\theta}, \sin(\theta)\frac{\partial\phi}{\partial r} + \frac{1}{r}\cos(\theta)\frac{\partial\phi}{\partial\theta}\right) && (312)\\
&= \frac{\partial\phi}{\partial r}(\cos(\theta), \sin(\theta)) + \frac{1}{r}\frac{\partial\phi}{\partial\theta}(-\sin(\theta), \cos(\theta)) && (313)
\end{aligned}$$

Each of the vector fields above is simply a basis vector in polar coordinates, so we have:

$$\begin{aligned}
\nabla\phi &= \frac{\partial\phi}{\partial r}\underline{e}_r + \frac{1}{r}\frac{\partial\phi}{\partial\theta}e_\theta && (314)\\
&= \left(\frac{\partial\phi}{\partial r}, \frac{1}{r}\frac{\partial\phi}{\partial\theta}\right) && (315)
\end{aligned}$$

Which is the expression for the graident in Polar coordinates.

Divergence:
The divergence is given by:

$$\nabla\cdot\underline{V} = \frac{1}{r}\frac{\partial(rV_r)}{\partial r} + \frac{1}{r}\frac{\partial V_\theta}{\partial\theta} \tag{316}$$

this can be obtain by using the chain rule in combination with the relations between V_x, V_y and V_r, V_θ.

In three dimensions there are three commonly used sets of coordinates. The Cartesian coordinate system, the cylindrical coordinate system and the spherical coordinate system.

Cylindrical coordinates:

Cylindrical coordinates essentially take the Cartesian coordinates (x, y, z) and transform the xy-plane to polar coordinates, leaving the z-axis as it is. This gives us three coordinates (ρ, ϕ, z).
The first coordinate ρ is the radial distance in the xy-plane. We must distinguish ρ, which measures the distance from the z-axis, from r, radial distance, which measures distance from the origin.
The second coordinate ϕ measures the anglular displacement of a point from the x-axis.[6] The z-coordinate is unchanged from Cartesian coordinates.

The range of each of these coordinates is:

$$\begin{aligned} &\rho \in [0, \infty) \\ &\phi \in [0, 2\pi) \\ &z \in \mathbb{R} \end{aligned}$$

we can transform to and from cylindrical coordinates using the relations:

$$\begin{aligned} x &= \rho \cos(\phi) \\ y &= \rho \sin(\phi) \\ z &= z \end{aligned}$$

and

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned} \tag{317}$$

Note that these are the same as the polar coordinate transformation rules. Since we have already worked out the transformation of the Gradient and Divergence operations in polar coordinates, we can simply transfer those results to work out their form in Cylindrical coordinates. These are:

$$\begin{aligned} \nabla\psi &= \left(\frac{\partial\psi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}, \frac{\partial\psi}{\partial z}\right) \\ \nabla \cdot \underline{V} &= \frac{1}{\rho}\frac{\partial(\rho V_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial V_\phi}{\partial\phi} + \frac{\partial V_z}{\partial z} \end{aligned}$$

[6]Since this is essentially the same angle as was used in two dimensional polar coordinates, many pure mathematics textbooks continue to use θ to denote it.

For completeness we will also give the Curl operation:

$$\nabla \times \underline{V} = \left(\frac{1}{\rho} \frac{\partial V_z}{\partial \phi} - \frac{\partial V_\phi}{\partial z} \right) \mathbf{e}_\rho + \left(\frac{\partial V_\rho}{\partial z} - \frac{\partial V_z}{\partial \rho} \right) \mathbf{e}_\phi + \frac{1}{\rho} \left(\frac{\partial (\rho V_\phi)}{\partial \rho} - \frac{\partial V_\rho}{\partial \phi} \right) \mathbf{e}_z \tag{318}$$

Spherical coordinates:

This coordinate system is based around radial distance, labelled r from some fixed origin point. All points with a fixed value of R form a sphere, we then use two angular coordinates denoted θ and ϕ to distinguish points on these spheres. The first angle, ϕ, is the same angle as used in cylindrical coordinates, the angle of displacement from the Cartesian x-axis. The second angle θ, measures the displacement from the z-axis, with $\theta = 0$ being the North pole of a sphere at fixed radius R. In geographical terms θ is the latitude of the sphere.[7] The ranges for the coordinates are:

$$r \in [0, \infty)$$
$$\theta \in [0, \pi)$$
$$\phi \in [0, 2\pi)$$

The second angle θ only needs to run to π, as $\theta = \pi$ is the south pole of the sphere and continuing θ beyond this point would cause points to be indexed twice.

The relation between spherical coordinates and cartesian coordinates is given by:

$$x = r \sin(\theta) \cos(\phi)$$
$$y = r \sin(\theta) \sin(\phi)$$
$$z = r \cos(\theta)$$

and

$$r = \sqrt{x^2 + y^2 + z^2}$$
$$\theta = \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}} \right)$$
$$\phi = \tan^{-1}\left(\frac{y}{x} \right)$$

[7] In pure mathematics, where θ is used for the angle of displacment against the x-axis, ϕ is used as the angle of displacement from the z-axis. Here we are using Physics notation.

The relation between its coordinates and those of Cylindrical coordinates is:

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\
\phi &= \tan^{-1}\left(\frac{y}{x}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
r &= \sqrt{\rho^2 + z^2} \\
\theta &= \tan^{-1}\left(\frac{\rho}{z}\right) \\
\phi &= \phi
\end{aligned}
$$

The angle ϕ remains unchanged as it is shared between the two coordinate systems.

Either of these relations can be used to derive the form of the various Vector Calculus operations. The divergence and the gradient are given by:

$$
\begin{aligned}
\nabla\psi &= \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin(\theta)}\frac{\partial\psi}{\partial\phi}\right) \\
\nabla\cdot\underline{V} &= \frac{1}{r^2}\frac{\partial(r^2 V_r)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{(\partial V_\theta \sin(\theta))}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial V_\phi}{\partial\phi}
\end{aligned}
\tag{319}
$$

For completeness the Curl operation is:

$$
\begin{aligned}
\nabla\times\underline{V} = &\frac{1}{r\sin(\theta)}\left(\frac{\partial\left(V_\phi\sin(\theta)\right)}{\partial\theta} - \frac{\partial V_\theta}{\partial\phi}\right)\mathbf{e}_r \\
&+ \frac{1}{r}\left(\frac{1}{\sin(\theta)}\frac{\partial V_r}{\partial\phi} - \frac{\partial\left(rV_\phi\right)}{\partial r}\right)\mathbf{e}_\theta \\
&+ \frac{1}{r}\left(\frac{\partial\left(rV_\theta\right)}{\partial r} - \frac{\partial V_r}{\partial\theta}\right)\mathbf{e}_\phi
\end{aligned}
\tag{320}
$$

3.2 Line, Surface and Volume Integrals

In three dimensions we can integrate objects various subsets of Euclidean space. We can take integrals over lines, surfaces and volumes. We begin with how one takes integrals over lines. One dimensional subsets of Euclidean space.

3.2.1 Line Integrals

As mentioned eariler a line/path is specified by parameterising the coordinates in terms of an extra variable t. As this parameter is varied it traces out a one-dimensional subset of three-dimensional Euclidean space, i.e. a line/path.
Examples of such path functions are:

1.

$$\underline{x}(t) = (a\cos(t), a\sin(t), 0) \tag{321}$$
$$t \in [0, 2\pi] \tag{322}$$

Here we see the coordinates have the following dependence on t:

$$x = a\cos(t)$$
$$y = a\sin(t)$$
$$z = 0$$

This implies that $x^2 + y^2 = a^2$, hence this is a paramterisation of a circle of radius a in the $z = 0$ plane. t then corresponds to the anglular coordinate on the circle.

2.

$$\underline{x}(t) = (a\cos(t), a\sin(t), t) \tag{323}$$
$$t \in [0, 6\pi] \tag{324}$$

Here we have the same relation $x^2 + y^2 = a^2$ in the xy-plane, however as t is varied we also move upward on the z-axis. This parameterisation then traces out a helix: circular in the xy-plane, but varying upward along the z-axis.

3. Here we look at two paths.

$$\underline{x}_1(t) = (t, t, t)\underline{x}_2(t) \qquad = (t, t^2, t^3) \tag{325}$$

in both cases $t \in [0, 1]$.
In this case both paths start ($t = 0$) at the origin $(0, 0, 0)$ and terminate at $(1, 1, 1)$. However in the case of the first path the relation between the first two coordinates is $y = x$, but for the second path $y = x^2$. Hence, in the xy-plane, the first path is a straight line, where as the second has some curvature.

We can integrate either Scalar fields or Vector fields along lines. The formulae for doing so are:

$$\int_a^b \phi(\underline{x}(t))\sqrt{\frac{d\underline{x}}{dt}}dt \tag{326}$$

for a scalar field and:

$$\int_a^b \underline{V}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt}dt \tag{327}$$

in both cases a, b are the limits on t and $\phi(\underline{x}(t)), \underline{V}(\underline{x}(t))$ denote the scalar and vector fields rewritten as a function of t.

(Examples)

3.2.2 Volume Integrals

Integrating over volumes in Cartesian coordinates is already familiar to you from MA1132. One handles a volume integral by performing three interated one-dimensional integrals:

$$\int_D \phi(x,y,z)dV = \int_e^f \int_{c(z)}^{d(z)} \int_{a(y,z)}^{b(y,z)} \phi(x,y,z)dxdydz \qquad (328)$$

Where D is the volume being integrated over. Often the algebraic relations specifying D will cause the limits of the coordinates to be functions of eachother. For instance when integrating over the volume enclosed by a cylinder of height 1 and radius 1, z would have limits $0,1$ and y would extend from -1 to 1, on either side of a cross section of a cylinder. However x and y being related via $x^2+y^2=1$, the x-cordinate would have maximum and minimum values: $-\sqrt{1-y^2}$ and $\sqrt{1-y^2}$ and integral of a scalar function would be computed via:

$$\int_{CylindricalVolume} \phi(x,y,z)dV = \int_0^1 \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \phi(x,y,z)dxdydz \qquad (329)$$

However looking at Eq.328, it isn't immediately obvious that this relation between volume integrals and iterated one-dimensional integrals should hold. The volume integral, roughly speaking, is defined via:

$$\int_D \phi(x,y,z)dV = \lim_{N\to\infty} \sum i = 1^N \phi(\underline{x}_i^*)\Delta V_i \qquad (330)$$

This basically the three-dimensional analogue of a Riemann sum. A region of volume A is partitioned into N subvolumes V_i with volume ΔV_i. The scalar field is evaluated at a fixed point inside each subvolume $\underline{x}_i^*$. The sum essentially computes the average value of the scalar field over the region. The integral is then defined as the limit of this average as the subvolumes go to zero size.
The interated one-dimensional integrals however correspond to sending each dimension of these subvolumes to zero sequentially, rather than shrinking the whole volume at once. A priori, there is no reason to expect that these two different limiting procedures should produce the same result.
Fortunately Fubini's theorem tells use that for any measurable function Eq.(328) holds. A vague definition of a measurable function is that it is a function, ϕ, whose inverse ϕ^{-1} maps open subsets of the real line to sets with well-defined volumes. That is, for all opensets I, $\phi^{-1}(I)$ has a volume. This is a concept from measure theory, which we won't explore in depth, only to say that the existence of non-measureable functions wasn't proven until 1905 and even then their existence depends on what axioms one uses when defining set theory. Suffice it to say, any function you encounter in mathematical physics will certainly be measureable.

Rather we look at how the iterated integral changes when we move to different coordinate systems. For simplicity we will use the same technique of looking at the two-dimensional case and simply quoting the results for three dimensions. In two dimensions, in Cartesian coordinates we have:

$$\int_D \phi(x,y,z)dA = \int_c^d \int_{a(y)}^{b(y)} \phi(x,y)dxdy \tag{331}$$

where we use dA as we are now integrating over areas.
We can use two different techniques to derive how the measure $dxdy$ changes when we move to polar coordinates.

Linear Algebra derivation:
We can imagine $dxdy$ as a small (infinitesmial) area, essentially the span of the (infinitesmial) vectors dx and dy. Then $drd\theta$ is the area spanned by the (infinitesmial) vectors dr and $d\theta$. To find the relation between these areas we look at the matrix that transforms one from Polar to Cartesian coordinates:

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

In Linear Algebra, a matrix dilates and rotates vectors. For a set of orthonormal vectors, the determinant of a matrix $\det A$ determines by what factor the matrix distorts the volume of the area they span.
The above transformation matrix has determinant r and so the area of the span of the vectors dr and $d\theta$ is altered by a factor of r upon transforming to Cartesian coordinates. Hence:

$$dxdy = rdrd\theta \tag{332}$$

Differential forms derivation:
Thinking of dx and dy instead as infinitesimial line segments, we can see $dxdy$ as infinitesmial area segment. The dx and dy lines segments can be related to the polar coordinate line segments via the chain rule:

$$dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta \tag{333}$$

$$dy = \frac{\partial y}{\partial r}dr + \frac{\partial y}{\partial \theta}d\theta \tag{334}$$

We have already computed the derivatives:

$$dx = \cos(\theta)dr - r\sin(\theta)d\theta \tag{335}$$

$$dy = \sin(\theta)dr + r\cos(\theta)d\theta \tag{336}$$

Hence:

$$dxdy = (\cos(\theta)dr - r\sin(\theta)d\theta)\,(\sin(\theta)dr + r\cos(\theta)d\theta) \tag{337}$$

$$= \cos(\theta)\sin(\theta)dr^2 - r^2\sin(\theta)\cos(\theta)d\theta^2 + r\cos(\theta)^2 drd\theta - r\sin(\theta)^2 d\theta dr \tag{338}$$

However dr^2 and $d\theta^2$ do not produce an area. A variation in r followed by another variation in r is still a one dimensional subset and hence as areas $dr^2 = 0$ and $d\theta^2 = 0$. So we have:

$$dxdy = r\cos(\theta)^2 drd\theta - r\sin(\theta)^2 d\theta dr \tag{339}$$

In integration we must remember that subsets are oriented, the integral from a to b is the negative of that from b to a. The area traced out via a variation in θ followed by one in r, that is $drd\theta$ is oriented in the opposite direction to the area produce by $d\theta dr$ that is $d\theta dr = -drd\theta$. So we have:

$$\begin{aligned} dxdy &= r\cos(\theta)^2 drd\theta - r\sin(\theta)^2 d\theta dr && (340)\\ &= r\cos(\theta)^2 drd\theta + r\sin(\theta)^2 drd\theta && (341)\\ &= rdrd\theta && (342) \end{aligned}$$

Note that this multiplication rule, $d\theta dr = -drd\theta$, where multiplication is anti-commutative, implies $dr^2 = 0$ and $d\theta^2 = 0$, since $dr^2 = drdr$ we have:

$$drdr = -drdr \tag{343}$$

which is only possible if $drdr = dr^2 = 0$. This multiplication rule is known as a Grassmann algebra and applying it to infinitesmial variations such as dx and dy is known as the theory of Differential Forms. One can see that this multiplication rule simply encodes the simple fact that areas/volumes are oriented in the theory of integration.

Now that we have the relation:

$$dxdy = rdrd\theta \tag{344}$$

or more accurately:

$$dxdy = \det A drd\theta \tag{345}$$

With A being:

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{bmatrix}$$

We can derive the results for Cylindrical and Spherical coordinates. We would simply have to compute the determinants of the relevant transformation matricies. For Cylindrical coordinates we have already done this as the transformation matrix:

$$\begin{bmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial z} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial z} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial z} \end{bmatrix}$$

becomes:

$$\begin{bmatrix} \cos(\phi) & -r\sin(\theta) & 0 \\ \sin(\phi) & r\cos(\phi) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Since the z-coordinate is unchanged, reducing this to the polar coordinate case essentially. Hence we have the relation:

$$dxdydz = \rho d\rho d\phi dz \tag{346}$$

For spherical coordinates we would simply evaluate a similar three-by-three matrix of derivatives (the transformation matrix) and find:

$$dxdydz = r^2 \sin(\theta) dr d\theta d\phi \tag{347}$$

The determinant of the transformation matrix, telling one how the measure changes between coordinates is known as the **Jacobian**.

3.2.3 Surface Integrals

It is also possible to integrate a Vector field over surface. This surface integral essentially measures the total flux of a vector field across the surface.

Just like Line integrals, in order to perform a surface integral one needs a paramterisation of the surface. For a line integral this depended on a single external parameter t, for a surface it will depend on two parameters s, t.

3.2.4 Surface parameterisations

A simple example of such a surface is the unit disk. We already had the example of a circle of radius a as an example of a path:

$$\underline{x}(t) = (a\cos(t), a\sin(t), t) \tag{348}$$

$$t \in [0, 2\pi] \tag{349}$$

To form the unit disk, we simply let the radius a be a variable rather than a constant:

$$\underline{x}(t, s) = (s\cos(t), s\sin(t), 0) \tag{350}$$

$$t \in [0, 2\pi] \tag{351}$$

$$s \in [0, 1] \tag{352}$$

The parametrisation of surfaces can also be obtained from the algebraic equations defining them. For example a sphere of Radius R is the set of points satisfying $x^2 + y^2 + z^2 = R^2$. We can form the parametric representation of the sphere by letting x and y be linear functions of t and s:

$$x = s \tag{353}$$

$$y = t \tag{354}$$

$$\tag{355}$$

Then since $z^2 = R^2 - x^2 - y^2$ we have two possible paramterisations of z:

$$z = \sqrt{R^2 - s^2 - t^2} \tag{356}$$

$$z = -\sqrt{R^2 - s^2 - t^2} \tag{357}$$

The first is used for the upper hemisphere and the second for the lower hemisphere. Hence the parameterisations are:

$$\underline{x}(t, s) = \left(s, t, \sqrt{R^2 - s^2 - t^2}\right) \tag{358}$$

$$s \in [-R, R] \tag{359}$$

$$t \in [-\sqrt{R^2 - s^2}, \sqrt{R^2 - s^2}] \tag{360}$$

for the upper hemisphere and:

$$\underline{x}(t,s) = \left(s, t, -\sqrt{R^2 - s^2 - t^2}\right) \tag{361}$$
$$s \in [-R, R] \tag{362}$$
$$t \in [-\sqrt{R^2 - s^2}, \sqrt{R^2 - s^2}] \tag{363}$$

for the lower hemisphere. Note the limits on t. It can only vary between these points as otherwise the square root defining z would become imaginary and we would no longer be describing points on the sphere[8].

Similarly we can take the surface given by $x^2 + z^2 = 9$, which is a cylinder of radius 3 lying along the y-axis. First of all the y coordinate is not constrained or related to the other variables, hence we can set it equal to one of the parameters $y = s$. x and z and then functions of the remaining parameter t. We can ignore y and the equation $x^2 + z^2 = 9$ can be seen as the equation for a circle of radius 3 in the xz-plane. Hence, looking at the expression for a circular path above, we have:

$$x = 3\cos(t) \tag{364}$$
$$z = 3\sin(t) \tag{365}$$
$$t \in [0, 2\pi] \tag{366}$$

Hence the paramterisation is:

$$\underline{x}(t,s) = (3\cos(t), s, 3\sin(t)) \tag{367}$$
$$t \in [0, 2\pi] \tag{368}$$

A final example is the paraboloid $z = 2x^2 + 2y^2$ between $z = 0$ and $z = 4$. It is quite easy to construct the paramterisation in this case:

$$x = s \tag{369}$$
$$y = t \tag{370}$$
$$z = 2s^2 + 2t^2 \tag{371}$$

With $s, t \in [0, 1]$ to ensure the limits on z and so the parameterisation is:

$$\underline{x}(s,t) = \left(s, t, 2s^2 + 2t^2\right) \tag{372}$$
$$s, t \in [0, 1] \tag{373}$$

3.2.5 Tangent vectors

Once we have the parameterisation of a surface we can form two new objects, the tangent vectors to that surface. These are given by the s and t partial derivatives of the parameterisation. For example:

[8] Instead we would be looking at a surface in $\mathbb{C}^n$

1. For the disk:

$$\frac{\partial \underline{x}}{\partial s} = (\cos(t), \sin(t), 0) \tag{374}$$
$$\frac{\partial \underline{x}}{\partial t} = (-s\sin(t), s\cos(t), 0) \tag{375}$$

2. For the cylinder:

$$\frac{\partial \underline{x}}{\partial s} = (0, 1, 0) \tag{376}$$
$$\frac{\partial \underline{x}}{\partial t} = (-3\sin(t), 0, 3\cos(t)) \tag{377}$$

3. For the paraboloid:

$$\frac{\partial \underline{x}}{\partial s} = (1, 0, 4s) \tag{378}$$
$$\frac{\partial \underline{x}}{\partial t} = (0, 1, 4t) \tag{379}$$

These tangent vectors are vector fields on the surface which point tangent to the surface. The both point in the direction in which their coordinate increases.

Associated with these two tangent vectors there are two tangent variations ds and dt however they are given be the formula:

$$\underline{ds} = \frac{\partial \underline{x}}{\partial s} ds \tag{380}$$
$$\underline{dt} = \frac{\partial \underline{x}}{\partial t} dt \tag{381}$$

these variations are associated with integrals along the surface.

3.2.6 Normal vectors

The last objects needed to define a Surface integral are the normal vector and the normal variation. The normal vector is the vector field which is orthogonal to the surface. We will denote it $\underline{n}$. Since the two vectors $\frac{\partial \underline{x}}{\partial s}$ and $\frac{\partial \underline{x}}{\partial t}$ each point along the surface, we can form a vector which points away from the surface by taking their cross-product[9]. We can take the cross-product in either order. For each of the examples above:

1. The disk:

$$\underline{n}_1 = \frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial t} = (0, 0, s) \tag{382}$$
$$\underline{n}_2 = \frac{\partial \underline{x}}{\partial t} \times \frac{\partial \underline{x}}{\partial s} = (0, 0, -s) \tag{383}$$

[9] As the cross-product $\underline{A} \times \underline{B}$ produces a vector orthogonal to both $\underline{A}$ and $\underline{B}$

2. The cylinder:

$$\underline{n}_1 = \frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial t} = (3\cos(t), 0, 3\sin(t)) \tag{384}$$

$$\underline{n}_2 = \frac{\partial \underline{x}}{\partial t} \times \frac{\partial \underline{x}}{\partial s} = (-3\cos(t), 0, -3\sin(t)) \tag{385}$$

3. The paraboloid:

$$\underline{n}_1 = \frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial t} = (-4s, -4t, 1) \tag{386}$$

$$\underline{n}_2 = \frac{\partial \underline{x}}{\partial t} \times \frac{\partial \underline{x}}{\partial s} = (4s, 4t, -1) \tag{387}$$

In the case of the cylinder for example $\underline{n}_1$ points outward from the surface and $\underline{n}_2$ points into the interior of the surface. There will always be two possible orthogonal directions to a given surface and one must determine which is the appropriate one to use in a given calculation.

Finally we have the normal variation $d\underline{S}$, this is used to calculate the flux of vectors fields across a surface, as it integrates components orthogonal to the surface. It is given by the formula:

$$d\underline{S} = \underline{n} dsdt = \frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial t} dsdt = \underline{ds} \times \underline{dt} \tag{388}$$

There will be two possible ways of defining $d\underline{S}$ depending on which way you take the cross-product.

3.2.7 Surface integral of vector fields

We can now compute the surface integral of a vector field. This is given by:

$$\int_S \underline{V} \cdot d\underline{S} = \int \int \underline{V}(\underline{x}(s,t)) \cdot \underline{n} dsdt \tag{389}$$

We can see this integrates over the surface the part of the vector field orthogonal to the surface. This can be seen as the amount of the vector field crossing the surface, known as *the flux* of the vector field. Let's look at a few examples.

1. The outward flux of the vector field $\underline{V} = \left(x^2, xyz, y^2\right)$ over the paraboloid given above.
   We have already computed the normal vector to this surface:

$$\underline{n} = \frac{\partial \underline{x}}{\partial t} \times \frac{\partial \underline{x}}{\partial s} = (4s, 4t, -1) \tag{390}$$

   we choose this ordering for the cross-product as it points outward from the surface. We also need to express the vector field $\underline{V}$ as a function of s and t, i.e. find $\underline{V}(\underline{x}(s,t))$. The parameterisation of this surface was:

$$\underline{x}(s,t) = \left(s, t, 2s^2 + 2t^2\right) \tag{391}$$

That is:

$$x = sy \qquad = tz = 2s^2 + 2t^2 \tag{392}$$

Hence the Vector field becomes:

$$\underline{V}(\underline{x}(s,t)) = \left(s^2, 2ts^3 + 2st^3, t^2\right) \tag{393}$$

The dot product is then:

$$\underline{V}(\underline{x}(s,t)) \cdot \underline{n} = 4s^3 + 8t^2s^3 + 8st^5 - t^2 \tag{394}$$

and so the surface integral is:

$$\int\int \underline{V}(\underline{x}(s,t) \cdot \underline{n} dsdt = \int_0^1 \int_0^1 \left(4s^3 + 8t^2s^3 + 8st^5 - t^2\right) dsdt = 2 \tag{395}$$

The limits on s and t being part of the original parameterisation.

2. The outward flux of the vector field $\underline{V} = \left(x^2, e^{zy}, xy\right)$ over a cylinder with the same radius as the one given above, but extending from $y = -3$ to $y = 3$.
   We have already computed the normal vector:

$$\underline{n} = \frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial t} \qquad = (3\cos(t), 0, 3\sin(t)) \tag{396}$$

The paramterisation was:

$$\underline{x}(t,s) = (3\cos(t), s, 3\sin(t)) \tag{397}$$

$$t \in [0, 2\pi] \tag{398}$$

In this case however the cylinder has finite extent on the y-axis, so its parameterisation is:

$$\underline{x}(t,s) = (3\cos(t), s, 3\sin(t)) \tag{399}$$

$$t \in [0, 2\pi] \tag{400}$$

$$s \in [-3, 3] \tag{401}$$

In either case we have for the individual coordinates:

$$x = 3\cos(t) \tag{402}$$

$$y = s \tag{403}$$

$$z = 3\sin(t) \tag{404}$$

$$t \in [0, 2\pi] \tag{405}$$

And so the vector field is:

$$\underline{V}(\underline{x}(s,t)) = \left(9\cos^2(t), e^{3s\sin(t)}, 3s\sin(t)\right) \tag{406}$$

The dot-product is then:

$$\underline{V}(\underline{x}(s,t)) \cdot \underline{n} = 27\cos^3(t) + 9s\sin^2(t) \tag{407}$$

and so the surface integral is:

$$\int\int \underline{V}(\underline{x}(s,t) \cdot \underline{n} dsdt = \int_0^{2\pi}\int_{-3}^{3}\left(27\cos^3(t) + 9s\sin^2(t)\right)dsdt = 648\cos^3(\pi^4) \tag{408}$$