MA2331: Coordinate systems and Vector Operations

The two dimensional case is ignored. To get the formulas in the two dimensional case remove the z coordinate from Cartesian or Cylindrical coordinates to get the formulae for Cartesian or Polar coordinates respectively. Remember, Curl does not exist in two dimensions.

1 Coordinate ranges

1. **Cartesian**

$$x \in \mathbb{R}$$
$$y \in \mathbb{R}$$
$$z \in \mathbb{R}$$

2. **Cylindrical**

$$\rho \in [0, \infty)$$
$$\phi \in [0, 2\pi)$$
$$z \in \mathbb{R}$$

3. **Spherical**

$$r \in [0, \infty)$$
$$\theta \in [0, \pi)$$
$$\phi \in [0, 2\pi)$$

2 Relations between coordinates

1. **Cartesian - Cylindrical**

$$x = \rho \cos(\phi)$$
$$y = \rho \sin(\phi)$$
$$z = z$$

2. **Cartesian - Spherical**

$$x = r \sin(\theta) \cos(\phi)$$
$$y = r \sin(\theta) \sin(\phi)$$
$$z = r \cos(\theta)$$

3. **Cylindrical - Cartesian**

$$\begin{aligned} \rho &= \sqrt{x^2 + y^2} \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \\ z &= z \end{aligned} \tag{1}$$

4. **Cylindrical - Spherical**

$$\begin{aligned} \rho &= r\sin(\theta) \\ \phi &= \phi \\ z &= r\cos(\theta) \end{aligned}$$

5. **Spherical - Cartesian**

$$\begin{aligned} r &= \sqrt{x^2 + y^2 + z^2} \\ \theta &= \cos^{-1}\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right) \\ \phi &= \tan^{-1}\left(\frac{y}{x}\right) \end{aligned}$$

6. **Spherical - Cylindrical**

$$\begin{aligned} r &= \sqrt{\rho^2 + z^2} \\ \theta &= \tan^{-1}\left(\frac{\rho}{z}\right) \\ \phi &= \phi \end{aligned}$$

3 Measure

1. **Cartesian**

$$dV = dxdydz \tag{2}$$

2. **Cylindrical**

$$dV = \rho d\rho d\phi dz \tag{3}$$

3. **Spherical**

$$dV = r^2 \sin^2(\theta) dr d\theta d\phi \tag{4}$$

4 Dot product

1. **Cartesian**

$$\underline{F} \cdot \underline{G} = F_x G_x + F_y G_y + F_z G_z \tag{5}$$

2. **Cylindrical**

$$\underline{F} \cdot \underline{G} = F_r G_r + \rho^2 F_\phi G_\phi + F_z G_z \tag{6}$$

3. **Spherical**

$$\underline{F} \cdot \underline{G} = F_r G_r + r^2 F_\theta G_\theta + r^2 \sin^2(\theta) F_\phi G_\phi \tag{7}$$

5 Gradient

1. **Cartesian**

$$\nabla\psi = \left(\frac{\partial\psi}{\partial x}, \frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial z}\right) \tag{8}$$

2. **Cylindrical**

$$\nabla\psi = \left(\frac{\partial\psi}{\partial\rho}, \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}, \frac{\partial\psi}{\partial z}\right) \tag{9}$$

3. **Spherical**

$$\nabla\psi = \left(\frac{\partial\psi}{\partial r}, \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \frac{1}{r\sin(\theta)}\frac{\partial\psi}{\partial\phi}\right) \tag{10}$$

6 Divergence

1. **Cartesian**

$$\nabla \cdot \underline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{11}$$

2. **Cylindrical**

$$\nabla \cdot \underline{F} = \frac{1}{\rho}\frac{\partial(\rho F_\rho)}{\partial\rho} + \frac{1}{\rho}\frac{\partial F_\phi}{\partial\phi} + \frac{\partial F_z}{\partial z} \tag{12}$$

3. **Spherical**

$$\nabla \cdot \underline{F} = \frac{1}{r^2}\frac{\partial(r^2 F_r)}{\partial r} + \frac{1}{r\sin(\theta)}\frac{(\partial F_\theta \sin(\theta))}{\partial\theta} + \frac{1}{r\sin(\theta)}\frac{\partial F_\phi}{\partial\phi} \tag{13}$$

7 Laplacian

All formulae can be derived from those above, combined with the definition $\nabla \cdot \nabla \psi = \nabla^2 \psi$

1. **Cartesian**

$$\nabla^2 \psi = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} \tag{14}$$

2. **Cylindrical**

$$\nabla^2 \psi = \frac{1}{\rho} \frac{\partial}{\partial \rho} \left(\rho \frac{\partial \psi}{\partial \rho} \right) + \frac{1}{\rho^2} \frac{\partial^2 \psi}{\partial \phi^2} + \frac{\partial^2 \psi}{\partial z^2} \tag{15}$$

3. **Spherical**

$$\nabla^2 \psi = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r^2 \sin(\theta)} \frac{\partial}{\partial \theta} \left(\sin(\theta) \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2(\theta)} \frac{\partial^2 \psi}{\partial \phi^2} \tag{16}$$

8 Curl

1. **Cartesian**

$$\begin{aligned} \nabla \times \underline{F} = & \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z} \right) \mathbf{e}_x \\ & + \left(\frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x} \right) \mathbf{e}_y \\ & + \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \mathbf{e}_z \end{aligned} \tag{17}$$

2. **Cylindrical**

$$\begin{aligned} \nabla \times \underline{F} = & \left(\frac{1}{\rho} \frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z} \right) \mathbf{e}_\rho \\ & + \left(\frac{\partial F_\rho}{\partial z} - \frac{\partial F_z}{\partial \rho} \right) \mathbf{e}_\phi \\ & + \frac{1}{\rho} \left(\frac{\partial \left(\rho F_\phi \right)}{\partial \rho} - \frac{\partial F_\rho}{\partial \phi} \right) \mathbf{e}_z \end{aligned} \tag{18}$$

3. **Spherical**

$$\nabla \times \underline{F} = \frac{1}{r\sin(\theta)}\left(\frac{\partial\left(F_\phi \sin(\theta)\right)}{\partial\theta} - \frac{\partial F_\theta}{\partial\phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin(\theta)}\frac{\partial F_r}{\partial\phi} - \frac{\partial\left(rF_\phi\right)}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial\left(rF_\theta\right)}{\partial r} - \frac{\partial F_r}{\partial\theta}\right)\mathbf{e}_\phi \tag{19}$$