2331 Tutorial Sheet 3.

November 29th 2012

The sheet on Vector Operations in various coordinates may be useful.

Questions

1. Verify Stoke's theorem in the case where the surface is one whose coordinates obey the constraints:

$$x^2 + y^2 = z^2 \tag{1}$$

$$0 \leq z \leq 4 \tag{2}$$

and the vector field:

$$\underline{F} = \left(x^2, y, x^2\right) \tag{3}$$

2. The Laplacian in $2, 3, 4$ dimensions in polar, spherical and hyperspherical coordinates are given by

$$\nabla^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\phi^2} \tag{4}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin(\theta)}\frac{\partial^2\psi}{\partial\phi^2} \tag{5}$$

$$\begin{aligned}\nabla^2\psi &= \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\frac{\partial\psi}{\partial r}\right) \\ &+ \frac{1}{r^2\sin^2(\chi)}\left(\frac{1}{\sin^2(\theta)}\frac{\partial^2\psi}{\partial\phi^2} + \frac{1}{\sin(\theta)}\frac{\partial}{\partial\theta}\left(\sin(\theta)\frac{\partial\psi}{\partial\theta}\right)\right) \\ &+ \frac{1}{r^2\sin^2(\chi)}\frac{\partial}{\partial\chi}\left(\sin^2(\chi)\frac{\partial\psi}{\partial\chi}\right)\end{aligned}$$

The four dimensional Laplacian has an extra angle χ, since in this case a surface of constant r is a three-dimensional object, a three sphere S^3.

In each case solve Laplace's equation for the gravitational potential:

$$\nabla^2\psi = -4\pi G\rho \tag{6}$$

where the mass density can be assumed to be (n-)spherically symmetric:

$$\rho = \rho(r) \tag{7}$$

In this case, one can take the potential to be spherically symmetric:

$$\psi = \psi(r) \tag{8}$$

Use Gauss' theorem to solve for the constant in the resulting solution.

Finally an orbiting body always feels a potential due to its own angular momentum. This potential is given by:

$$V_l(r) = \frac{l^2}{2mr^2} \tag{9}$$

in all dimensions. The sign is positive, so this tends to repel the orbiting body from the one generating the gravitational field.

The total potential felt by a particle (or planet) orbiting another body (e.g. a star) gravitationally is then the sum of the gravitational potential and the centrifugal potential:

$$V(r) = \psi(r) + \frac{l^2}{2mr^2} \tag{10}$$

Orbiting bodies will settle at the minimum of this potential. Find the local minima of this potential in each dimension.

If there aren't any, what does this say about gravitational orbits in that dimension.

3. Consider an infinitely long straight wire of radius R. As the wire itself is cylindrically invariant we shall use cylindrical coordinates, (ρ, ϕ, z). The z-axis is taken to be the axis of symmetry of the wire.

   The magnitude of current density vector $\underline{J}$ measures the rate of change of charge per area with respect to time and it points in the direction of the current flow through the wire. We shall assume the current running through the wire is axially symmetric and invariant under z-translations. That is, it does not depend on z or ϕ, but may only vary along the radius of the wire. Secondly we assume the current flows purely in the positive z-direction. Overall this means the current vector has the form:

$$\underline{J} = (J_\rho, J_\phi, J_z) = (0, 0, f(\rho)) \tag{11}$$

   Any current will generate a magnetic field via Maxwell's fourth equation. Assuming there are no electric fields, this equation has the form:

$$\nabla \times \underline{B} = \mu_0 \underline{J} \tag{12}$$

   As the current density has no z or ϕ dependence, neither will the magnetic field it generates. Hence the magnetic field will have the general form:

$$\underline{B} = (B_\rho(\rho), B_\phi(\rho), B_z(\rho)) \tag{13}$$

   (a) Use Maxwell's second equation:

$$\nabla \cdot \underline{B} = 0 \tag{14}$$

   to find the form of the first term B_ρ.

(b) Use Maxwell's fourth equation:

$$\nabla \times \underline{B} = \mu_0 \underline{J} \tag{15}$$

to obtain differential equations for B_ϕ and B_z. Solve the equation for B_z.

(c) Two of the components, B_ρ and B_z, have been solved for. However both solutions have a constant. To solve for the constant we calculate the energy of the magnetic field. The energy is given by:

$$\int \frac{\underline{B} \cdot \underline{B}}{2\mu_0} dV = \frac{1}{2\mu_0} \left(\int B_\rho^2 dV + \int \rho^2 B_\phi^2 dV + \int B_z^2 dV \right) \tag{16}$$

What values must the constants have for the B_ρ and B_z integrals to be finite (i.e. finite energy)?

(d) With B_ρ and B_z obtained, it only remains to solve for B_ϕ. Take the differential equation obtained in part(b) and solve it using an integral expression (i.e. fundamental theorem of calculus). Take the lower limit of integration to be $\rho = 0$. This will result in an integral expression for B_ϕ.

(e) Consider that the current density vector $\underline{J}$ is current per area and total current is denoted I. Use this fact to find an explicit form for B_ϕ outside the wire. (Outside R).

The magnetic field outside the wire has now been obtained. To solve for the magnetic field inside the wire it would be necessary to have an explicit form for the function $f(\rho)$.