# MA2331: Line Integrals

## 1 Scalar Fields

1. Integrate the scalar field

$$\phi(x, y, z) = x^2 + y^2 + zy \tag{1}$$

over the path:

$$\underline{x}(t) = \left(t, t^2, t\right) \quad t \in [0, 1] \tag{2}$$

*Solution:* The field itself, as a function of the path parameter, is given as:

$$\phi(x(t), y(t), z(t)) = t^2 + t^4 + t^3 \tag{3}$$

The tangent vector is:

$$\dot{\underline{x}}(t) = (1, 2t, 1) \tag{4}$$

$$|\dot{\underline{x}}(t)|^2 = 2 + 4t^2 \tag{5}$$

And so the path integral is:

$$\int_0^1 t^2 + t^3 + t^4|\sqrt{2 + 4t^2}|dt = \int_0^1 t^2 + t^3 + t^4 \left(\sqrt{2 + 2t^2}\right) dt \tag{6}$$

2. The scalar field

$$\phi(x, y, z) = x^2 + yz \tag{7}$$

Over the following path, a helix:

$$\underline{x}(t) = (\sin(t), \cos(t), t) \quad t \in [0, 2] \tag{8}$$

*Solution:* The scalar field is expressed on the path as:

$$\phi(x(t), y(t), z(t)) = \sin^2(t) + t\cos(t) \tag{9}$$

The tangent vector is:

$$\dot{\underline{x}}(t) = (\cos(t), -\sin(t), 1) \tag{10}$$

$$|\dot{\underline{x}}(t)|^2 = 2 \tag{11}$$

The line integral is then:

$$\int_0^2 \sin^2(t) + t\cos(t)|2|dt \approx 3.2 \tag{12}$$

# 2 Path Quantities - Time Integrals

1. The path of a particle is given by:
$$\underline{x}(t) = \left(e^t, t, t^3\right) \quad t \in [0,1] \tag{13}$$
What is the integral of the kinetic energy along this path?
*Solution:* The velocity along the path is
$$\underline{\dot{x}}(t) = \left(e^t, 1, 3t^2\right) \tag{14}$$
The kinetic energy is then:
$$\frac{m\underline{\dot{x}} \cdot \underline{\dot{x}}}{2} = e^{2t} + 1 + 9t^4 \tag{15}$$
The integral of which is:
$$\int_0^1 m(e^{2t} + 1 + 9t^4) \approx 6m \tag{16}$$

2. A particle of mass $m = 2$ travels in a potential energy field:
$$\psi(x, y, z) = \frac{-1}{\sqrt{x^2 + y^2 + z^2}} \tag{17}$$
Along the path:
$$\underline{x}(t) = \left(\frac{t}{\sqrt{3}}, \frac{t}{\sqrt{3}}, \frac{t}{\sqrt{3}}\right) \quad t \in [1,2] \tag{18}$$
The difference between the kinetic energy and potential energy, the Lagrangian is:
$$L = T - V = \frac{m\underline{\dot{x}} \cdot \underline{\dot{x}}}{2} - \psi(x, y, z) \tag{19}$$
Find the integral of the Lagrangian, the Action, over this path:
$$S[\Gamma] = \int_\Gamma L dt \tag{20}$$
*Solution:* There are two ways of doing this. One can stay in the Cartesian coordinate system or move to the spherical coordinate system. This can be done using the relation between the coordinates:
$$r = \sqrt{x^2 + y^2 + z^2} \tag{21}$$
$$\theta = \cos^{-1}\left(\frac{z}{r}\right) \tag{22}$$
$$\phi = \tan^{-1}\left(\frac{y}{r}\right) \tag{23}$$

This gives the path the following form in spherical coordinates:

$$\underline{x}(t) = (r(t), \theta(t), \phi(t)) = \left(t, \cos^{-1}\left(\frac{1}{\sqrt{3}}\right), \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\right) \tag{24}$$

The velocity vector is then simply:

$$\underline{\dot{x}}(t) = (1, 0, 0) \tag{25}$$

The kinetic energy is then simply:

$$\frac{m\underline{\dot{x}} \cdot \underline{\dot{x}}}{2} = 1 \tag{26}$$

and the potential energy is:

$$\psi(x, y, z) = \frac{-1}{\sqrt{x^2 + y^2 + z^2}} \tag{27}$$

$$\psi(r, \theta, \phi) = \frac{-1}{r} \tag{28}$$

$$\psi(r(t), \theta(t), \phi(t)) = \frac{-1}{t} \tag{29}$$

The integral of the Lagrangian over the path, the Action, is then:

$$\int_1^2 \left(1 + \frac{1}{t}\right) dt = \frac{17}{4} \tag{30}$$

**Note:** For this potential, there are trajectories with the same initial velocity and position, but with a smaller value for the Action. Classical Mechanics states that, for a given initial position and velocity, the path actually taken by a particle is the one with the smallest Action (Principle of Least Action).