1. Integrate the vector field $\underline{F} = x\underline{i} + xz\underline{j} + y^2\underline{k}$ along the following two paths from the point $(0,0,0)$ to the point $(1,1,1)$:

   (a) $\underline{x} = t\underline{i} + t\underline{j} + t\underline{k}$; $t \in [0,1]$,

   (b) $\underline{x} = t\underline{i} + t^2\underline{j} + t^3\underline{k}$; $t \in [0,1]$

   What do these two integrals tell you about $\underline{F}$?

   Solution: A line integral is given by the formula

$$\int_a^b \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt} dt$$

   So in the first case, the coordinates are given by the following functions: $x = t$, $y = t$, $z = t$. Therefore $\underline{F} = t\underline{i} + (t)(t)\underline{j} + (t)^2\underline{k}$. The derivative of the coordinate vector is $\underline{x}'(t) = \underline{i} + \underline{j} + \underline{k}$. So the line integral becomes

$$\int_a^b \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt} dt = \int_0^1 t + t^2 + t^2\, dt = \frac{7}{6}.$$

   For the second path, we have $x = t$, $y = t^2$, $z = t^3$. Then $\underline{F} = t\underline{i} + t(t^3)\underline{j} + (t^2)^2\underline{k}$. The deriviative of the coordinate function is $\underline{i} + 2t\underline{j} + 3t^2\underline{k}$. So the line integral becomes

$$\int_a^b \underline{F}(\underline{x}(t)) \cdot \frac{d\underline{x}}{dt} dt = \int_0^1 t + 2t^5 + 3t^6\, dt = \frac{53}{42}.$$

   The integral along the two paths is not equal; hence $\underline{F}$ is not path independent, and so, is not conservative.

2. Integrate the vector field $\underline{F} = z\underline{i} + x\underline{j} + x^2zy\underline{k}$ over the surface of the cylinder of radius 5 and of height 2 from the $x - y$ plane.

   Solution: First we must specify the embedding for the cylinder, that is, a parameterisation of the cylinder. We start with the body of the cylinder and exclude the two disks at each end. So in the $x - y$ plane, the cylinder is just a circle of radius 5. This is described by $x^2 + y^2 = 25$. This will be traced out by $x = 5\cos(t)$, $y = 5\sin(t)$ as t varies from 0 to 2π. The height of the cylinder is easy to describe, using $z = s$, as s varies from 0 to 2. So the cylinder, without the caps which close it at either end, is described by $\underline{x}(s,t) = (A\cos(t), B\sin(t), s)$. A surface integral is given by

$$\int_S \underline{F} \cdot \underline{dS} = \int_a^b \int_c^d \underline{F}(s,t) \cdot \left(\frac{\partial \underline{x}}{\partial t} \times \frac{\partial \underline{x}}{\partial s} \right) dtds.$$

In our case, the coordinates are given by $x = 5\cos(t)$, $y = 5\sin(t)$, $z = s$. So

$$\underline{F}(s,t) = s\underline{i} + 5\cos(t)\underline{j} + (5\cos(t))^2 s 5\sin(t)\underline{k}. \quad (1)$$

$$\frac{\partial \underline{x}}{\partial t} = (-5\sin(t), 5\cos(t), 0) \quad (2)$$

$$\frac{\partial \underline{x}}{\partial s} = (0,0,1). \quad (3)$$

Hence, the cross product is

$$\frac{\partial \underline{x}}{\partial t} \times \frac{\partial \underline{x}}{\partial s} = (5\cos(t), 5\sin(t), 0).$$

So the integral becomes

$$\int_0^{2\pi} \int_0^2 5s\cos(t) + 25\cos(t)\sin(t)\, dsdt = 0.$$

Now for the disks at each end. The first disk lies in the $x - y$ plane at $z = 0$. The circle which forms the boundary of this disk can be given parameterised by:

$$\underline{x}(t) = (\cos(t), \sin(t), 0) \quad (4)$$

We can vary the radius of this circle, from the boundary cirlce with radius 5, down to the centre of the disk at radius 0. This will cover the disk, hence:

$$\underline{x}(s,t) = (s\cos(t), s\sin(t), 0) \quad (5)$$

The other disk is the eact same except it is placed at $z = 2$:

$$\underline{x}(s,t) = (s\cos(t), s\sin(t), 2) \quad (6)$$

In both cases:

$$\frac{\partial \underline{x}}{\partial t} \times \frac{\partial \underline{x}}{\partial s} = (0, 0, s).$$

However for the bottom disk, this will point into the cylinder, so we use the other cross product instead:

$$\frac{\partial \underline{x}}{\partial s} \times \frac{\partial \underline{x}}{\partial t} = (0, 0, -s).$$

In both cases, the surface integral will only involve F_z, which is 0 for the bottom disk as $F_z = 0$ there. Hence there is only the top disk to worry about. In this case $F_z = 5s^3 \cos^2(t) \sin(t)$, and so the surface integral is:

$$\int_0^{2\pi} \int_0^5 2s^4 \cos^2(t) \sin(t) \, dsdt = 0$$

3. Check if the vector field $\underline{F} = (\frac{1}{x}, e^z, yze^{z-1} + \frac{2}{z}$ is conservative, and if so, construct its potential function.

   Solution: To check if a vector field is conservative, we match derivatives. Hence we should have

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}$$
$$\frac{\partial F_y}{\partial z} = \frac{\partial F_z}{\partial y}$$
$$\frac{\partial F_z}{\partial x} = \frac{\partial F_x}{\partial z}$$

   All of these derivatives are zero, except for the second line, which are non-zero and equal. The field is therefore conservative. To construct its potential function, we take one of these components, for example F_x, and integrate.

$$\phi(x, y, z) = \int F_x dx = \int \frac{1}{x} dx = \ln(x) + h(y, z).$$

   To determine $h(y, z)$, we differentiate ϕ and set this equal to another component, for example

$$\frac{\partial \phi}{\partial y} = \frac{\partial h}{\partial y} = F_y = e^z,$$

then integrating this produces $h(z, y) = ye^z + g(z)$. Again, differentiating ϕ and setting the result equal to the final component produces

$$\frac{\partial \phi}{\partial z} = \frac{\partial h}{\partial z} = yze^{z-1} + \frac{dg}{dz} = F_z = yze^{z-1} + \frac{2}{z}$$

Comparing these, we see that $\frac{dg}{dz} = \frac{2}{z}$, so $g(z) = \ln(z^2)$. Finally, $\phi(x, y, z) = ye^z + \ln(x) + \ln(z^2) = ye^z + \ln(xz^2)$

4. Calculate the curl of the vector field $\underline{F} = -\frac{y}{x^2+y^2}\underline{i} + \frac{x}{x^2+y^2}\underline{j}$. Then compute the integral of $\underline{F}$ around the unit circle. Finally, comment on the relation between these two results.

   Solution: The curl of $\underline{F}$ can be calculated easily and is given by

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y} = \frac{y^2 - x^2}{(x^2 + y^2)^2}.$$

   All other partial derivatives are zero, hence the curl is zero. The integral around the unit circle is defined by

$$\int_0^{2\pi} \underline{F} \cdot \frac{d\underline{x}}{dt} dt$$

   To proceed, we need a parameterisation of the circle. This is given by $\underline{x}(t) = (\cos(t), \sin(t), 0)$. For this parameterisation, $\underline{F}$ is given by

$$F_x = -\sin(t), \qquad F_y = \cos(t)$$

   The derivative of the coordinate vector is $\underline{x}' = (-\sin(t), \cos(t), 0)$, and so the integral is

$$\int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = \int_0^{2\pi} dt = 2\pi.$$

   One would expect that if the curl of $\underline{F}$ is zero, then $\underline{F}$ would be conservative, and hence, would evaluate to zero when integrated around closed paths. However, $\underline{F}$ is not defined everywhere in $\mathbb{R}^3$ due to the singularity at $x = y = 0$. This means that $\underline{F}$ is actually a vector field on $\mathbb{R}^3/C$, where C is the line given by $x = y = 0$, or the z-axis. This set is not simply connected, as paths above the z-axis cannot be smoothly distorted into paths below the z-axis. Hence, the curl of $\underline{F}$ being zero does not imply that $\underline{F}$ is conservative, and hence there is no reason for integrals along closed loops to be zero.

5. Integrate $\underline{F} = 3y\underline{i} + 6x\underline{j} + 9\ln(z)\underline{k}$ along the figure of eight given by the union of two circles of radius 1 in the $x-y$ plane, one above the x-axis and the other below.

Solution: Performing this line integral directly would be quite difficult, so instead, we use Stokes' theorem, which states that

$$\int_S \nabla \times \underline{F} \cdot \underline{dS} = \int_{\partial S} \underline{F} \cdot \underline{dl}$$

So we will use the left hand side of Stokes' theorem to calculate this integral. First of all, $\nabla \times \underline{F} = 3\underline{k}$. This is to be integrated over any surface whose boundary is the figure of eight. The simplest such surface is the union of two discs, one above the x-axis and one below. So

$$\int_{D_1 \cup D_2} 3\underline{k} \cdot \underline{dS} = \int_{D_1} 3\underline{k} \cdot \underline{dS} + \int_{D_2} 3\underline{k} \cdot \underline{dS}$$

Each of these integrals is identical as $\underline{dS}$ will just be $\underline{k}$ since $\underline{dS}$ is normal to the surface. Hence, these are really just area integrals, for two copies of the same shape.

$$\int_{D_1 \cup D_2} 3\underline{k} \cdot \underline{dS} = 2\int_{D_1} 3dA = 6\int_{D_1} dA = 6(2\pi) = 12\pi.$$