# Contiguity relations for hypergeometric integrals of type $(k, n)$

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# Introduction

We study hypergeometric functions by using twisted homology and cohomology groups.
Recently, some results for hypergeometric functions are applied to algebraic statistics (and physics?).
When we apply the results, we often have to write down them in explicit form. Twisted (co)homology groups are one of good tools to obtain explicit formulas.

Contents

- Gauss' hypergeometric function, contiguity relations
- Twisted cohomology groups
- Generalization — Aomoto-Gelfand hypergeometric function

# Gauss' hypergeometric function ${}_2F_1$

$$ {}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n. $$

$a, b, c \in \mathbb{C}$ are parameters, $x \in \mathbb{C}$ is a variable ($c \notin \mathbb{Z}_{\leq 0}$).
Here, $(\alpha)_n$ is the Pochhammer symbol defined as

$$ (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1) = \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} $$

$$ (\text{ for example, } (1)_n = n! \text{ }). $$

Note that

- $a \in \mathbb{Z}_{\leq 0}$ or $b \in \mathbb{Z}_{\leq 0} \implies {}_2F_1$ is a polynomial (we also call hypergeometric polynomial),
- $a, b \notin \mathbb{Z}_{\leq 0} \implies {}_2F_1$ converges on $\{x \in \mathbb{C} \mid |x| < 1\}$.

# Euler-type integral representation

If $\mathrm{Re}(a), \mathrm{Re}(c-a) > 0$, then

$$ {}_2F_1(a,b,c;x) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b} dt. $$

Note that the integrand is a multi-valued function on $\mathbb{C} - \{0, 1, 1/x\}$ ($t$-space).

**Proof.**

$$ {}_2F_1(a,b,c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n (1)_n} x^n = \frac{\Gamma(c)}{\Gamma(a)} \sum_{n=0}^{\infty} \frac{\Gamma(a+n)(b)_n}{\Gamma(c+n)(1)_n} x^n $$

$$ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \sum_{n=0}^{\infty} B(a+n, c-a) \frac{(b)_n}{(1)_n} x^n $$

$$ = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1} \left( \sum_{n=0}^{\infty} \frac{(b)_n}{n!} (xt)^n \right) dt = (\mathrm{RHS}). $$

□

# Contiguity relations

There are several properties of hypergeometric functions.
Today, we focus on the contiguity relations.

**What are contiguity relations?**

Behavior of ${}_2F_1(a,b,c;x)$ when $a$, $b$, $c$ are shifted by $\pm 1$.

We denote $F = {}_2F_1(a,b,c;x)$, $\partial_x = \dfrac{d}{dx}$, $\theta_x = x \cdot \partial_x$.

$$F|_{a \to a+1} = \frac{1}{a}(\theta_x + a)F,$$

$$F|_{a \to a-1} = \frac{1}{c-a}((1-x)\theta_x - bx + c - a)F,$$

$$F|_{c \to c+1} = \frac{c}{(c-a)(c-b)}((1-x)\partial_x + c - a - b)F,$$

$$F|_{c \to c-1} = \frac{1}{c-1}(\theta_x + c - 1)F.$$

Since $b$ and $a$ are symmetric, $F|_{b \to b+1}$ is obtained by replacing $a \leftrightarrow b$ in $F|_{a \to a+1}$.

$$F|_{a\to a+1} = \frac{1}{a}(\theta_x + a)F, \qquad F|_{c\to c-1} = \frac{1}{c-1}(\theta_x + c - 1)F,$$

$$F|_{a\to a-1} = \frac{1}{c-a}((1-x)\theta_x - bx + c - a)F,$$

$$F|_{c\to c+1} = \frac{c}{(c-a)(c-b)}((1-x)\partial_x + c - a - b)F$$

By using $(a+1)_n = (a+1)(a+2)\cdots(a+n) = \frac{a+n}{a}\cdot(a)_n$, $\theta_x x^n = nx^n$, and so on, blue relations are easily obtained.
How about green ones?

(I) Straightforward calculation ($\leftarrow$ if you know the answer.)

(II) Division in the ring $\mathbb{C}(x)\langle\partial_x\rangle$ of differential operators.

(III) Pfaffian system and inverse matrix.

(IV) Euler-type integral representation($\leftarrow$ today's topic)
$\rightarrow$ essentially, blue and green can be obtained by a same manner.

# (IV) Naive calculation (not so simple)

Integral representation of $F = {}_2F_1(a,b,c;x)$:

$$F = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b} dt$$
$$= \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a}(1-t)^{c-a}(1-xt)^{-b} \frac{dt}{t(1-t)}.$$

Since $\partial_x \cdot (1-xt)^{-b} = bt(1-xt)^{-b-1}$,

$$\partial_x F = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)} \int_0^1 t^{a}(1-t)^{c-a}(1-xt)^{-b} \frac{b\ dt}{(1-t)(1-xt)}.$$

We consider the case $c \to c+1$. Since $\Gamma$-part is easy, we see only integrations.

We put $U = t^a(1-t)^{c-a}(1-xt)^{-b}$.
Integration by part: (Note: $U|_{c\to c+1} = U\cdot(1-t)$)

$$F|_{c\to c+1} \leftrightarrow \int_0^1 U\cdot\frac{dt}{t} = \int_0^1 t^{a-1}(1-t)^{c-a}(1-xt)^{-b}dt$$
$$= \left[\frac{1}{a}U\right]_0^1 - \frac{1}{a}\int_0^1 U\left(-\frac{c-a}{1-t}+\frac{bx}{1-xt}\right)dt$$
$$= \int_0^1 U\left(\frac{c-a}{a}\frac{1}{1-t}-\frac{bx}{a}\frac{1}{1-xt}\right)dt.$$

Partial fraction decomposition:

$$F \leftrightarrow \int_0^1 U\frac{dt}{t(1-t)} = \int_0^1 U\left(\frac{1}{t}+\frac{1}{1-t}\right)dt,$$
$$\partial_x F \leftrightarrow \int_0^1 U\frac{b\ dt}{(1-t)(1-xt)} = \int_0^1 U\cdot\frac{b}{1-x}\left(\frac{1}{1-t}-\frac{x}{1-xt}\right)dt.$$

We eliminate $\frac{1}{1-t}$, $\frac{x}{1-xt}$.

Therefore, we obtain

$$\int_0^1 U \cdot \frac{dt}{t}$$
$$= \int_0^1 U \left( \frac{c-a-b}{c-b} \frac{dt}{t(1-t)} + \frac{1-x}{c-b} \frac{b\ dt}{(1-t)(1-xt)} \right),$$

$\downarrow$ (Consider $\Gamma$-factors.)

$$F|_{c\to c+1} = \frac{c(c-a-b)}{(c-a)(c-b)} F + \frac{c(1-x)}{(c-a)(c-b)} \partial_x F.$$

We regard them as a relation between the differential forms in some sense ($\rightarrow$ (de Rham) cohomology).

**Remark**

Though this computation is hard, twisted cohomology groups enable us to obtain such relations systematically, in the framework of the linear algebra.

# About twisted (co)homology groups

To derive contiguity relations, we use the theory of twisted cohomology groups.

- Twisted (co)homology groups are geometric tools to study special functions expressed by integrations. Aomoto applied this theory to study of hypergeometric functions.
- Intersection pairings of such (co)homology groups are defined in [Steenrod, 1943].
- By [Kita-Yoshida, 1994], intersection numbers of twisted homology groups can be evaluated in terms of homological intersection numbers and branches of the multi-valued function.
  $\longrightarrow$ Monodromy representations.
- By [Cho-Matsumoto, 1995] and [Matsumoto, 1998], intersection numbers of twisted cohomology groups is express by residues of logarithmic forms.
  $\longrightarrow$ Pfaffian equations, contiguity relations.

We consider (homology and) cohomology group based on the integral of a multi-valued function:

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt \ \big( = (\text{const.}) \cdot {}_2F_1 \big)$$
$$= \int_0^1 t^{a}(1-t)^{c-a}(1-xt)^{-b}\frac{dt}{t(1-t)} = \int_0^1 U\frac{dt}{t(1-t)}$$
$$\big(U = t^{a}(1-t)^{c-a}(1-xt)^{-b}\big).$$

We fix $x \neq 0, 1$.
We put $T = \mathbb{C} - \{0, 1, 1/x\} = \mathbb{P}^1 - \{0, 1, 1/x, \infty\}$ (: $t$-space).
$U(t) = t^{a}(1-t)^{c-a}(1-xt)^{-b}$ is a multi-valued function on $T$.

# Idea

We consider an integration of multi-valued function:

$$\int_\gamma U\varphi \qquad \left( \begin{array}{l} \gamma\text{: } k\text{-simplex} \\ \varphi\text{: } k\text{-form} \end{array} \right).$$

When we consider such an integral, the branch of $U$ on $\gamma$ is given. Thus, we regard $\gamma$ as a simplex loading a branch of $U$, and we denote this integral $\langle \varphi, \gamma \otimes U \rangle$ ($\leftarrow$ pairing).

By using this notation, Stokes' formula is expressed as follows (separate $U$ and differential forms):

$$\langle \psi, \partial\sigma \otimes U|_{\partial\sigma} \rangle = \int_{\partial\sigma} U\psi = \int_\sigma d(U\psi) = \int_\sigma (dU \wedge \psi + U d\psi)$$
$$= \int_\sigma U \cdot \left( d\psi + \frac{dU}{U} \wedge \psi \right) = \int_\sigma U \cdot \nabla(\psi) = \langle \nabla(\psi), \sigma \otimes U \rangle$$

(Usual) de Rham theory:

$$\text{Stokes' formula: } \int_{\partial\sigma} \psi = \int_{\sigma} d\psi$$

$$\text{boundary and coboundary: } \partial \longleftrightarrow d$$

Twisted de Rham theory:

$$\text{Stokes' formula: } \langle \psi, \partial^U(\sigma \otimes U) \rangle = \langle \nabla(\psi), \sigma \otimes U \rangle$$

$$(\partial^U(\sigma \otimes U) = \partial\sigma \otimes U|_{\partial\sigma})$$

$$\text{boundary and coboundary: } \partial^U \longleftrightarrow \nabla$$

To define the twisted cohomology group, we use $\nabla = d + \frac{dU}{U}\wedge$ instead of the exterior derivative $d$.
(The twisted homology group is defined by using $\partial^U$.)

# Definition of twisted cohomology group

We put

$$\Omega^l(T): \text{ the space of rational } l\text{-forms on } \mathbb{P}^1$$
$$\text{that have poles along } 0, 1, 1/x, \infty,$$

$$\omega = \frac{dU}{U} = a\frac{dt}{t} - (c-a)\frac{dt}{1-t} + bx\frac{dt}{1-xt} \in \Omega^1(T).$$

Since $\nabla = d + \omega\wedge$ satisfies $\nabla \circ \nabla = 0$, we have a complex

$$0 \to \Omega^0(T) \xrightarrow{\nabla} \Omega^1(T) \xrightarrow{\nabla} \Omega^2(T) = 0.$$

We call its cohomology group

$$H^l(\Omega^\bullet(T), \nabla) = \ker(\nabla : \Omega^l(T) \to \Omega^{l+1}(T))/\nabla(\Omega^{l-1}(T))$$

the $l$-th twisted cohomology group.

**Fact 1**

$$\dim H^l(\Omega^\bullet(T), \nabla) = \begin{cases} 0 & (l = 0) \\ 2 & (l = 1) \end{cases}$$

For example,

$$\varphi_1 = d\log\big(t/(1-t)\big) = \frac{dt}{t(1-t)},$$
$$\varphi_2 = d\log\big((1-xt)/(1-t)\big) = \frac{(1-x)dt}{(1-t)(1-xt)}$$

form a basis of $H^1(\Omega^\bullet(T), \nabla)$.

## Remark

Since $(0,1)\otimes U$ is a "twisted cycle", integrations on it depends only on cohomology classes:

$$\varphi = \psi \text{ in } H^1(\Omega^\bullet(T),\nabla) \Longrightarrow \int_0^1 U\varphi = \int_0^1 U\psi$$

By using

- $0=\nabla(1)=\omega = a\frac{dt}{t} - (c-a)\frac{dt}{1-t} + bx\frac{dt}{1-xt}$ in $H^1(\Omega^\bullet(T),\nabla)$
- partial fraction decomposition,

we have the following relation:

$$\frac{dt}{t} = \frac{c-a-b}{c-b}\varphi_1 + \frac{b}{c-b}\varphi_2 \qquad \text{in } H^1(\Omega^\bullet(T),\nabla).$$

By the above Remark, we have

$$\int_0^1 U\cdot\frac{dt}{t} = \int_0^1 U\left(\frac{c-a-b}{c-b}\varphi_1 + \frac{b}{c-b}\varphi_2\right).$$

$$\begin{aligned}\int_0^1 U\cdot\frac{dt}{t} &= \int_0^1 U\left(\frac{c-a-b}{c-b}\varphi_1+\frac{b}{c-b}\varphi_2\right)\\ &= \int_0^1 U\left(\frac{c-a-b}{c-b}\frac{dt}{t(1-t)}+\frac{b}{c-b}\frac{(1-x)dt}{(1-t)(1-xt)}\right)\\ &= \int_0^1 U\left(\frac{c-a-b}{c-b}\frac{dt}{t(1-t)}+\frac{1-x}{c-b}\frac{b\ dt}{(1-t)(1-xt)}\right).\end{aligned}$$

As seen before, this implies the contiguity relation

$$F|_{c\to c+1}=\frac{c(c-a-b)}{(c-a)(c-b)}F+\frac{c(1-x)}{(c-a)(c-b)}\partial_x F.$$

# Usual de Rham cohomology?

In fact, it is known that twisted cohomology group is also defined as the cohomology group with coefficients in a (non-trivial) local system of rank 1. This local system corresponds to the multi-valuedness of $U$.

As well-known, (usual) de Rham cohomology of $T = \mathbb{P}^1 - \{4\text{pts.}\} \underset{\text{homotopy}}{\sim} S^1 \vee S^1 \vee S^1$ is

$$\begin{array}{c} H^0(T, \mathbb{C}) \simeq \mathbb{C}, \quad H^1(T, \mathbb{C}) \simeq \mathbb{C}^3. \\ \updownarrow \text{different} \\ H^0(\Omega^\bullet(T), \nabla) \simeq 0, \quad H^1(\Omega^\bullet(T), \nabla) \simeq \mathbb{C}^2. \end{array}$$

# Intersection pairing

We can define the intersection pairing $\mathcal{I}$ between $H^1(\Omega^\bullet(T), \nabla)$ and $H^1(\Omega^\bullet(T), \nabla^\vee)$, where $\nabla^\vee = d - \omega\wedge$ (which corresponds to the dual local system).

It is known that there is an isomorphism

$$H^1(\Omega^\bullet(T), \nabla) \underset{\jmath}{\xrightarrow{\sim}} H^1(\mathcal{E}_c^0 \xrightarrow{\nabla} \mathcal{E}_c^1 \xrightarrow{\nabla} \mathcal{E}_c^2),$$

where $\mathcal{E}_c^k$ is the space of $C^\infty$ $k$-forms on $T$ with compact support. The intersection pairing is defined as

$$\mathcal{I}(\varphi, \psi) = \int_T \jmath(\varphi) \wedge \psi.$$

Note that we regard $T = \mathbb{C} - \{0, 1, 1/x\}$ as a 2-dimensional real manifold.

## Fact 2 ([Cho-Matsumoto, 1995])

There are explicit formulas of $\mathcal{I}(\varphi, \psi)$ for logarithmic differential forms $\varphi$, $\psi$. For example,

$$\Big(\mathcal{I}(\varphi_i, \varphi_j)\Big)_{i,j=1,2} = 2\pi\sqrt{-1} \cdot \begin{pmatrix} \frac{1}{a} + \frac{1}{c-a} & \frac{1}{c-a} \\ \frac{1}{c-a} & \frac{1}{-b} + \frac{1}{c-a} \end{pmatrix}$$

$$\text{(memo)} \qquad \begin{aligned} U &= t^{a}(1-t)^{c-a}(1-xt)^{-b} \\ \varphi_1 &= d\log\big(t/(1-t)\big) \\ \varphi_2 &= d\log\big((1-xt)/(1-t)\big) \end{aligned}$$

For $\varphi = \frac{dt}{t}$, we showed $\varphi = \frac{c-a-b}{c-b}\varphi_1 + \frac{b}{c-b}\varphi_2$, by using

- $0 = \nabla(1) = \omega$ in $H^1(\Omega^\bullet(T), \nabla)$, and
- partial fraction decomposition.

In fact, the intersection pairing $\mathcal{I}$ gives a more systematic method to obtain the relation.

Since $\varphi_1$ and $\varphi_2$ form a basis, we have $\varphi = \lambda_1\varphi_1 + \lambda_2\varphi_2$ for some $\lambda_1, \lambda_2$. It is known that

$$\Big(\mathcal{I}(\varphi,\varphi_1), \mathcal{I}(\varphi,\varphi_2)\Big) = 2\pi\sqrt{-1}\Big(\frac{1}{a}, 0\Big).$$

By $\varphi = \lambda_1\varphi_1 + \lambda_2\varphi_2$ and bilinearlity of $\mathcal{I}$, we also have

$$\Big(\mathcal{I}(\varphi,\varphi_1), \mathcal{I}(\varphi,\varphi_2)\Big) = (\lambda_1, \lambda_2)\cdot\Big(\mathcal{I}(\varphi_i,\varphi_j)\Big)_{i,j=1,2}.$$

Therefore, the coefficients $\lambda_1, \lambda_2$ is obtained as

$$(\lambda_1, \lambda_2) = \Big(\frac{1}{a}, 0\Big)\cdot\begin{pmatrix} \frac{1}{a} + \frac{1}{c-a} & \frac{1}{c-a} \\ \frac{1}{c-a} & \frac{1}{-b} + \frac{1}{c-a} \end{pmatrix}^{-1} = \Big(\frac{c-a-b}{c-b}, \frac{b}{c-b}\Big).$$

# Generalization — Aomoto-Gelfand hypergeometric function

(Aomoto (1970's –), Gelfand (1980's –), and many others... )

We consider a generalization based on the integral representation

$$\int_0^1 t^{a-1}(1-t)^{c-a-1}(1-xt)^{-b}dt.$$

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$$t^{\gamma_1}(1-t)^{\gamma_2}(1-xt)^{\gamma_3}$$
$$\longrightarrow 4 \text{ points } \{0,1,1/x,\infty\} \subset \mathbb{P}^1$$
$$\longrightarrow 4 \text{ hyperplanes } \subset \mathbb{P}^1$$

$$\Downarrow$$

hyperplane arrangement in $\mathbb{P}^k$

We consider $(k+n+2)$ hyperplanes in $\mathbb{P}^k$ defined by

$$L_0 = 1, \quad L_j = t_j \ (1 \le j \le k),$$
$$L_{k+j} = 1 + t_1 x_{1j} + \cdots + t_k x_{kj} \ (1 \le j \le n),$$
$$L_{k+n+1} = 1 + t_1 + \cdots + t_k,$$

where we regard $(L_0 = 0)$ defines the hyperplane at infinity. This arrangement corresponds to the following $(k+1) \times (k+n+2)$ matrix (each column corresponds to the above linear form):

$$\begin{array}{c} \\ 1 \\ t_1 \\ \vdots \\ t_k \end{array} \begin{array}{c} \begin{array}{cccccccc} L_0 & L_1 & \cdots & L_k & L_{k+1} & \cdots & L_{k+n} & L_{k+n+1} \end{array} \\ \begin{pmatrix} 1 & 0 & \cdots & 0 & 1 & \cdots & 1 & 1 \\ 0 & 1 & & 0 & x_{11} & \cdots & x_{1n} & 1 \\ \vdots & & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & x_{k1} & \cdots & x_{kn} & 1 \end{pmatrix} \end{array}$$

Today, we assume that these hyperplanes are in general position (i.e., every $(k+1)$-minor of this matrix is not $0$).
$x = (x_{ij})$ is $k \times n$ variables of hypergeometric function.

$\alpha = (\alpha_0, \alpha_1, \ldots, \alpha_{k+n+1})$ :parameters (we put $\alpha_0 = -\sum_{i=1}^{k+n+1} \alpha_i$.)

We assume $\alpha_i \notin \mathbb{Z}$ (In fact, we often consider integer cases in an application).

Let $U(t)$ be a multi-valued function in $t$ (here we fix $x$) defined by

$$U(t) = U_x(t) = \prod_{i=1}^{k+n+1} L_i(t)^{\alpha_i}.$$

(We regard $\alpha_0$ as the exponent of $L_0(=1)$. The above condition means homogeneity.)

The hypergeometric integral of type $(k+1, k+n+2)$ is defined by

$$F(\alpha; x) = \int_{\Delta} U(t) \cdot \frac{dt_1 \wedge \cdots \wedge dt_k}{t_1 \cdots t_k},$$
$$\Delta = \{(t_1, \ldots, t_k) \in \mathbb{R}^k \mid t_1 < 0, \ldots, t_k < 0,\ t_1 + \cdots + t_k > -1\}.$$

As an example, we consider the case of $k = n = 1$.

$$\begin{array}{c} \\ 1 \\ t \end{array} \begin{array}{c} \begin{array}{cccc} L_0 & L_1 & L_2 & L_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & x & 1 \end{pmatrix} \end{array} \qquad \begin{aligned} L_1(t) &= t, \\ L_2(t) &= 1 + xt, \\ L_3(t) &= 1 + t. \end{aligned}$$

The hypergeometric integral of type $(2, 4)$ is

$$\begin{aligned} F(\alpha; x) &= \int_{(-1,0)} t^{\alpha_1}(1 + xt)^{\alpha_2}(1 + t)^{\alpha_3} \frac{dt}{t} \\ &= (-1)^{\alpha_1} \int_{(0,1)} s^{\alpha_1}(1 - xs)^{\alpha_2}(1 - s)^{\alpha_3} \frac{ds}{s}. \qquad (s = -t) \end{aligned}$$

This is nothing but the integral representation of ${}_2F_1$.

# Twisted cohomology (higher dimensional case)

(Here, we fix $x$.) We put

$$T = T_x = \{t = (t_1, \ldots, t_k) \in \mathbb{C}^k \mid L_i(t) \neq 0\} \subset \mathbb{C}^k \subset \mathbb{P}^k$$
$$= \mathbb{P}^k - \bigcup_{i=0}^{k+n+1} (L_i = 0) \qquad \text{(complement of hyperplane arr.)}$$

(we also regard $(L_0 = 0)$ as the hyperplane at infinity (after the homogenization)), and put

$\Omega^l(T)$ : the space of rational $l$-forms on $\mathbb{P}^k$
that have poles along $\bigcup_i (L_i = 0)$,

$$\omega = \frac{dU}{U} = d\log\left(\prod_{i=1}^{k+n+1} L_i^{\alpha_i}\right) = \sum_{i=1}^{k+n+1} \alpha_i \frac{dL_i}{L_i} \in \Omega^1(T).$$

**Example ($k = n = 1$)**

$$\omega = d\log(t^{\alpha_1}(1+xt)^{\alpha_2}(1+t)^{\alpha_3}) = \alpha_1 \frac{dt}{t} + \alpha_2 \frac{xdt}{1+xt} + \alpha_3 \frac{dt}{1+t}$$

By using $\nabla = d + \omega\wedge : \Omega^l(T) \to \Omega^{l+1}(T)$, we obtain a complex

$$0 \to \Omega^0(T) \xrightarrow{\nabla} \Omega^1(T) \xrightarrow{\nabla} \cdots \xrightarrow{\nabla} \Omega^{k-1}(T) \xrightarrow{\nabla} \Omega^k(T) \to 0,$$

and its cohomology group

$$H^l(\Omega^\bullet(T), \nabla) = \ker(\nabla : \Omega^l(T) \to \Omega^{l+1}(T))/\nabla(\Omega^{l-1}(T)).$$

This is called the $l$-th twisted cohomology group.

### Fact 3 ([Aomoto-Kita, 2011])

$$\dim H^l(\Omega^\bullet(T), \nabla) = \begin{cases} 0 & (l \neq k) \\ \binom{k+n}{k} & (l = k) \end{cases}$$

We study only $H^{k}(\Omega^\bullet(T), \nabla) = \Omega^k(T)/\nabla(\Omega^{k-1}(T))$.
For $J = \{j_0, \ldots, j_k\} \subset \{0, 1, \ldots, k+n+1\}$ ($\leftarrow$ indices of hyperplanes), we define the logarithmic differential form as

$$\varphi\langle J\rangle = \varphi\langle j_0 \cdots j_k\rangle = \bigwedge_{p=1}^{k} d\log(L_{j_p}/L_{j_0}) \in \Omega^k(T)$$

### Fact 4 ([Aomoto-Kita, 2011], [Kita-Matsumoto, 1997])

(1) $H^k(\Omega^\bullet(T), \nabla)$ is spanned by $\varphi\langle J\rangle$'s.

(2) Especially, for a fixed pair $p \neq q$, we can take $\{\varphi\langle J\rangle \mid p \in J,\ q \notin J\}$ as a basis of $H^k(\Omega^\bullet(T), \nabla)$.

# Intersection pairing

We can also define the intersection pairing $\mathcal{I}(\cdot,\cdot)$ between $H^k(\Omega^\bullet(T),\nabla)$ and $H^k(\Omega^\bullet(T),\nabla^\vee)$. Note that $\mathcal{I}(\cdot,\cdot)$ is a non-degenerate bilinear form.

**Fact 5 ([Matsumoto, 1998])**

For $J=\{j_0,\dots,j_k\}$, $J'=\{j'_0,\dots,j'_k\}$, we have

$$\mathcal{I}(\varphi\langle J\rangle,\varphi\langle J'\rangle)$$
$$=(2\pi\sqrt{-1})^k\times\begin{cases}\dfrac{\sum_{j\in J}\alpha_j}{\prod_{j\in J}\alpha_j} & (J=J'),\\[2ex] \dfrac{(-1)^{p+q}}{\prod_{j\in J\cap J'}\alpha_j} & \left(\begin{array}{l}\#(J\cap J')=k,\\ J-\{j_p\}=J'-\{j'_q\}\end{array}\right),\\[2ex] 0 & (\text{otherwise}).\end{cases}$$

It is not so hard to evaluate $\mathcal{I}(\varphi\langle J\rangle,\varphi\langle J'\rangle)$ for logarithmic forms.

# To study hypergeometric integrals

- On a "twisted cycle" $\Delta \otimes U$,

$$\varphi = \psi \text{ in } H^k(\Omega^\bullet(T), \nabla) \Longrightarrow \int_\Delta U\varphi = \int_\Delta U\psi.$$

- $H^k(\Omega^\bullet(T), \nabla)$ is a finite dimensional vector space.
  $\longrightarrow$ We can study in the framework of the linear algebra.
- $\mathcal{I}$ is a non-degenerate bilinear form.
  $\longrightarrow$ It is useful as an inner product.

# Column vector to describe contiguity relations

We take a basis of $H^k(\Omega^\bullet(T), \nabla)$ by

$$\{\varphi\langle J\rangle \mid 0 \in J,\ k+n+1 \notin J\}.$$

By straightforward calculation, we have correspondence:

$$\text{integral of } \varphi\langle 01\cdots k\rangle \left(= \frac{dt}{t_1\cdots t_k}\right) \longleftrightarrow F(\alpha;x),$$

$$\text{integral of another } \varphi\langle 0\ j_1\cdots j_k\rangle \longleftrightarrow \text{ a partial derivative of } F(\alpha;x).$$

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For example, since $L_{k+1} = 1 + t_1x_{11} + \cdots + t_kx_{k1}$, we have

$$\frac{\partial}{\partial x_{11}}F(\alpha;x) = \int_\Delta \frac{\partial}{\partial x_{11}} \prod_{i=1}^{k+n+1} L_i^{\alpha_i} \frac{dt}{t_1\cdots t_k}$$

$$= \int_\Delta \alpha_k t_1 \cdot L_{k+1}^{\alpha_k - 1} \prod_{i\neq k+1} L_i^{\alpha_i} \frac{dt}{t_1\cdots t_k} = \int_\Delta \prod_i L_i^{\alpha_i} \cdot \alpha_k \cdot \frac{dt}{L_{k+1}\cdot t_2\cdots t_k}$$

$$= \int_\Delta \prod_i L_i^{\alpha_i} \cdot \frac{\alpha_k}{x_{11}} \cdot \varphi\langle 0, k+1, 2, \cdots, k\rangle.$$

Thus, we define the column vector $\mathbf{F}(\alpha; x)$ whose entries are the integrals of this basis.

We derive contiguity relations for the vector-valued function $\mathbf{F}(\alpha; x)$.

## Example ($k = n = 2$, of type $(3,6)$, 4 variables $(x_{11}, x_{12}, x_{21}, x_{22})$)

$$\dim H^2 = \binom{2+2}{2} = 6$$

$$\mathbf{F}(\alpha; x) = \int_\Delta \prod_{j=1}^{5} L_j^{\alpha_j} \cdot \begin{pmatrix} \varphi\langle 012\rangle \\ \varphi\langle 013\rangle \\ \varphi\langle 014\rangle \\ \varphi\langle 023\rangle \\ \varphi\langle 024\rangle \\ \varphi\langle 034\rangle \end{pmatrix} = \begin{pmatrix} F(\alpha; x) \\ \frac{x_{21}}{\alpha_3} \cdot \frac{\partial F(\alpha; x)}{\partial x_{21}} \\ \frac{x_{22}}{\alpha_4} \cdot \frac{\partial F(\alpha; x)}{\partial x_{22}} \\ \frac{-x_{11}}{\alpha_3} \cdot \frac{\partial F(\alpha; x)}{\partial x_{11}} \\ \frac{-x_{12}}{\alpha_4} \cdot \frac{\partial F(\alpha; x)}{\partial x_{12}} \\ \frac{x_{11}x_{22} - x_{12}x_{21}}{\alpha_3\alpha_4} \cdot \frac{\partial^2 F(\alpha; x)}{\partial x_{11}\partial x_{22}} \end{pmatrix}$$

## Remark

In fact, all of the first derivatives appear.

# Contiguity relations

We consider the parameter shift $\alpha_i \to \alpha_i + 1$. In this case, the parameter vector $\alpha$ changes into

$$\alpha^{(i)} = (\alpha_0 - 1, \alpha_1, \ldots, \alpha_{i-1}, \alpha_i + 1, \alpha_{i+1}, \ldots, \alpha_{k+n+1}).$$

By the definition, we have

$$F(\alpha^{(i)}; x) = \int_\Delta \prod_{j=1}^{k+n+1} L_j^{\alpha_j} \cdot \underbrace{L_i \cdot \underbrace{\varphi\langle 01 \cdots k \rangle}_{\in H^k(T_x, \nabla^{(i)})}}_{\in H^k(T_x, \nabla)}.$$

$$\nabla^{(i)} = d + d\log(\underset{U|_{\alpha_i \to \alpha_i + 1}\nearrow}{U} \cdot L_i)\wedge$$

**Proposition 6**

The multiplication by $L_i$ $\Omega^l(T_x) \to \Omega^l(T_x)$; $\varphi \mapsto L_i \cdot \varphi$ induces a linear map $\mathcal{A}_i : H^k(\Omega^\bullet(T), \nabla^{(i)}) \to H^k(\Omega^\bullet(T), \nabla)$.

We take the cohomology classes of

$$\{\varphi\langle J\rangle \mid 0 \in J,\ k+n+1 \notin J\} \quad (\leftarrow \text{corresponding to } \mathbf{F})$$

as bases of $H^k(\Omega^\bullet(T), \nabla^{(i)})$ and $H^k(\Omega^\bullet(T), \nabla)$.
Let $A_i(\alpha; x)$ be the representation matrix of
$\mathcal{A}_i : H^k(\Omega^\bullet(T), \nabla^{(i)}) \to H^k(\Omega^\bullet(T), \nabla)$. Thus we have

$$\begin{pmatrix} \vdots \\ L_i \cdot \varphi\langle J\rangle \\ \vdots \end{pmatrix} = A_i(\alpha; x) \cdot \begin{pmatrix} \vdots \\ \varphi\langle J\rangle \\ \vdots \end{pmatrix} \quad \text{in } H^k(\Omega^\bullet(T), \nabla).$$

By integrating on $\Delta \otimes U = \Delta \otimes \left(\prod_{j=1}^{k+n+1} L_j^{\alpha_j}\right)$, we obtain the contiguity relation:

$$\mathbf{F}(\alpha^{(i)}; x) = A_i(\alpha; x) \cdot \mathbf{F}(\alpha; x).$$

We want an explicit expression of the matrix $A_i(\alpha; x)$.

### Theorem 7 ([G.-Matsumoto, 2018])

The representation matrix $A_i(\alpha; x)$ is expressed as

$$A_i(\alpha; x) = C(\alpha^{(i)})P_i(\alpha^{(i)})^{-1}D_i(x)Q_i(\alpha)C(\alpha)^{-1},$$

where $D_i(x)$ is a diagonal matrix whose entries are ratios of determinants of some matrix, and

$$C(\alpha) = \Big(\mathcal{I}(\varphi\langle I\rangle, \varphi\langle J\rangle)\Big)_{\substack{0\in I, k+n+1\notin I\\ 0\in J, k+n+1\notin J}},$$

$$P_i(\alpha) = \Big(\mathcal{I}(\varphi\langle I\rangle, \varphi\langle J\rangle)\Big)_{\substack{0\in I, i\notin I\\ k+n+1\in J, 0\notin J}},$$

$$Q_i(\alpha) = \Big(\mathcal{I}(\varphi\langle I\rangle, \varphi\langle J\rangle)\Big)_{\substack{i\in I, 0\notin I\\ k+n+1\in J, 0\notin J}}.$$

(These intersection numbers can be evaluated by Fact 5)

**Sketch of Proof.**

First, we use other bases of the cohomology groups.
$\Longrightarrow$ the representation matrix is diagonal.
Next, we evaluate the matrices that express the changes of the bases. By using the intersection pairing ($\fallingdotseq$ inner product), we obtain these matrices. □

Note that in [Aomoto, 1975], contiguity relations are also derived in the frame work of twisted cohomology groups. However, he did not use the linear map $\mathcal{A}_i$ and the intersection pairing $\mathcal{I}$, and his calculation is harder than ours.

**Remark**

Recently, some physicists also study hypergeometric integrals and intersection theory. For example, in [Frellesvig, et al., 2019], contiguity relations are considered in a similar idea to this talk.

# Summary

- Twisted cohomology groups and the intersection pairing are useful tools to study hypergeometric functions.
- Aomoto-Gelfand hypergeometric function is a generalization of the integral representation of ${}_2F_1$.
- Some properties (today: contiguity relations) are understood by using twisted cohomology groups and intersection pairings.

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Thank you for your kind attention!

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