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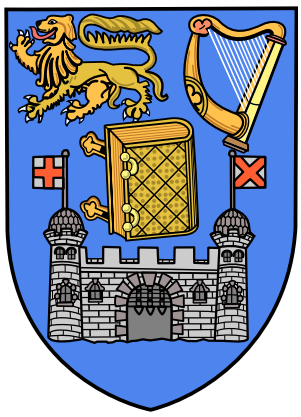

Trinity College

Perturbation Theory of Quantum Integrable Spin Chains

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Final Year Project, March 2017
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Declaration

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Abstract

Non-integrable perturbations were made to the Heisenberg XXX spin $\frac{1}{2}$ chain through the Lax operators and R matrices. The leading order perturbed Hamiltonian was obtained for general deformations, and the perturbative parts were split into local and generically non-local terms. The class of deformations that are integrable in the leading order were found from a deformed fundamental commutation relation, and the most general local deformation to the Hamiltonian obtained was found to depend on 14 parameters. The class of deformations contains the rescaled Lax operator, the inhomogeneous spin chain and the XXZ spin chain. For the simplified case where a reference state can be used and the R matrix takes a simple form, the eigenstates were constructed and a set of deformed Bethe ansatz equations were obtained.

Acknowledgements

A big thanks to Tristan for helping me with this project all year. Thanks to Jess as well I suppose.

Contents

1 Introduction

1.1 Motivation

Integrability is a very powerful concept in theoretical physics. In classical dynamics, while not the same as exactly solvable, it is a tool that helps make problems much more manageable. Quantum integrability is much less well defined. Nevertheless, it is still a very useful tool for lattice models in statistical physics and for quantum field theories.

Another very useful tool in physics is perturbation theory. We start with a problem to which we know the exact solution, and deviate slightly from it. It is split into a solvable part derived from the original problem and a perturbation part arising from the changes made. This method allows us to obtain approximate solutions to different and often more complex problems.

The main idea of this project is to combine these two concepts and make non-integrable perturbations to integrable systems. It might seem pointless to move away from the simple case, but if we can apply the standard framework for very small deviations from integrability, we can obtain approximate solutions to a whole new class of problems

This idea is already well established in classical dynamics through the KAM theorem. In a nutshell, the KAM theorem says that for sufficiently small deviations from integrability and under the right conditions, the deformed system will remain relatively well behaved. The simplest example of this is our solar system. It is a 10-body problem (including Pluto) which in theory is definitely not an integrable system. Yet we have been in stable orbit for 4.5 billion years; the solar system is somehow quasi-periodic. The main reasons behind this are that the mass of the Sun is much larger than the mass of the planets and the interactions between the planets is weak. This illustrates the point beautifully: consider a system that is well behaved, and move away from the ideal case slightly in order to determine the behaviour of a much more complicated, and often more physically relevant system.

Currently there is no quantum analogue of the KAM theorem. One possible approach that has recently been suggested is through quantum quenches [1]. Another approach is to perturb the system and identify a set of approximately conserved quantities [2], although this method is a computational one. In this project we aim to make a more systematic approach. It will allow us to not only find the perturbed Hamiltonian, but also construct the eigenstates. We also hope to be as general as possible with the perturbations. This project does not hope to formulate a quantum KAM theorem, but is certainly motivated by the lack of one.

1.2 Overview

The subject of integrability, especially quantum integrability, is a very advanced one by undergraduate standards. Learning about it was very challenging and took quite some time, so it is my intention that this report could be read and understood by an undergraduate student. As such, a discussion of $\mathfrak{su}(2)$ and tensor products is provided as they are an important tool for describing the systems considered in the project (and for notation purposes). A review of Lie algebras is included in Appendix A. The subject of classical and quantum integrability is then discussed. It is impossible to discuss everything relating to integrability, so we will only discuss the basic and relevant concepts, some of the most common systems and the approach used to solve them.

Then we introduce the Heisenberg XXX spin $\frac{1}{2}$ chain. We solve this in detail, as it is necessary for making comparisons later and is the best way to illustrate how the method works. We introduce deformations to the chain and solve the deformed system approximately using the standard methods, while keeping the deformations as general as possible. Finally, the possibilities for further research are discussed.

1.3 $\mathfrak{su}(2)$

$SU(2)$ is the Lie group of 2×2 unitary matrices with unit determinant. It is locally equivalent to the group of rotations in 3 dimensions, $SO(3)$, i.e. their Lie algebras are equivalent. It is important because its Lie algebra, $\mathfrak{su}(2)$, is used to describe spin angular momentum in quantum mechanics. $\mathfrak{su}(2)$ is described by

$$[S^a, S^b] = i\epsilon^{abc} S^c \quad , \tag{1}$$

where $\epsilon^{123} = 1$, and the rest of the terms are determined by skew-symmetry. S^a are the spin variables and in our (spin $\frac{1}{2}$) case, they are represented by the Pauli matrices,

$$\sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad , \tag{2}$$

where $S^a = \frac{1}{2}\sigma^a$ (we set $\hbar = 1$). We will work with the raising and lowering operators,

$$S^{\pm} = \frac{1}{2}\left(S_1 \pm iS_2\right) \quad , \tag{3}$$

along with S^3. Explicitly, $S^{\pm}$ are given by

$$S^+ = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad S^- = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \quad . \tag{4}$$

Since our system is a lattice of spin $\frac{1}{2}$ variables, each site is described by a copy of $\mathfrak{su}(2)$. The entire spin chain will be described by the product of these algebras.

1.4 Tensor Notation

Here we introduce some notions and notation relating to tensor products. The tensor product of two vector spaces $A \otimes B$ is a vector space with tensor products of elements $a \otimes b$, where $a \in A$ and $b \in B$. Here are a few basic properties of the tensor product,

$$\begin{aligned} (a \otimes b)(c \otimes d) &= ac \otimes bd \\ (a \otimes b)^{-1} &= a^{-1} \otimes b^{-1} \\ (a \otimes b)^T &= a^T \otimes b^T \end{aligned} \quad . \tag{5}$$

For 2×2 matrices $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$, their tensor product is a 4×4 matrix,

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{bmatrix} = \left[\begin{array}{cc|cc} a_{11}b_{11} & a_{11}b_{12} & a_{12}b_{11} & a_{12}b_{12} \\ a_{11}b_{21} & a_{11}b_{22} & a_{12}b_{21} & a_{12}b_{22} \\ \hline a_{21}b_{11} & a_{21}b_{12} & a_{22}b_{11} & a_{22}b_{12} \\ a_{21}b_{21} & a_{21}b_{22} & a_{22}b_{21} & a_{22}b_{22} \end{array}\right] \quad . \tag{6}$$

The horizontal and vertical lines are added to show that we can think of the tensor product as 4 2×2 blocks. We will generally omit the zeros in large matrices for the clarity. If A and B are $n \times n$ matrices, then $A \otimes B$ will be an $n^2 \times n^2$ matrix.

Consider the tensor product of spaces $V \otimes V$ and an operator $X \in \mathrm{End}(V)$ We will use a subscript i to denote operators that act non trivially only in the i^{th} space,

$$\begin{aligned} X_1 &\equiv X \otimes I \\ X_2 &\equiv I \otimes X \end{aligned} \quad . \tag{7}$$

In general, for a tensor product comprised of N vector spaces $V \otimes \ldots \otimes V$, we have

$$X_n \equiv I \otimes \ldots \otimes \underbrace{X}_{n} \otimes \ldots \otimes I, \qquad n \in \{1, \ldots, N\} \quad . \tag{8}$$

As a simple example, let us calculate the matrix representation of the permutation operator,

$$P = \frac{1}{2}\left(I \otimes I + \sum_{\alpha} \sigma^{\alpha} \otimes \sigma^{\alpha}\right) \quad , \tag{9}$$

which acts on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$. The Permutation operator acts on elements according to the following property,

$$P(a \otimes b) = b \otimes a \quad . \tag{10}$$

$I \otimes I$ is the identity matrix in 4 dimensions, and the products of Pauli matrices are given by

$$\sigma^1 \otimes \sigma^1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} & & & 1 \\ & & 1 & \\ \hline & 1 & & \\ 1 & & & \end{array}\right]$$

$$\sigma^2 \otimes \sigma^2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} = \left[\begin{array}{cc|cc} & & & -1 \\ & & 1 & \\ \hline & 1 & & \\ -1 & & & \end{array}\right] \quad , \tag{11}$$

$$\sigma^3 \otimes \sigma^3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \left[\begin{array}{cc|cc} 1 & & & \\ & -1 & & \\ \hline & & -1 & \\ & & & 1 \end{array}\right]$$

Thus, the representation of P is a 4×4 matrix,

$$P = \left[\begin{array}{cc|cc} 1 & & & \\ & & 1 & \\ \hline & 1 & & \\ & & & 1 \end{array}\right] \quad . \tag{12}$$

2 Integrability

2.1 Classical Integrability

In Hamiltonian dynamics, the concept of Liouville integrability is well defined; a system with as many conserved quantities as degrees of freedom is an integrable one [3]. Suppose we have a Hamiltonian system with a $2n$-dimensional phase space, Ω, parametrised by the canonical variables,

$$(q_1, \ldots, q_n, p_1, \ldots, p_n) \quad , \tag{13}$$

and the Hamiltonian $H = H(q_1, \ldots, q_n, p_1, \ldots, p_n)$ along with the Poisson brackets

$$\begin{aligned} \{q_i, q_j\} &= 0 \\ \{p_i, p_j\} &= 0 \qquad i, j \in \{1, \ldots, n\} \\ \{q_i, p_j\} &= \delta_{ij} \end{aligned} \quad . \tag{14}$$

Then the system is Liouville integrable if there exists a set of n independent conserved quantities (integrals of motion), F_i, such that

$$\begin{aligned} \dot{F}_i &= 0 \qquad i \in \{1, \ldots, n\} \\ \{F_i, F_j\} &= 0 \qquad i, j \in \{1, \ldots, n\} \end{aligned} \quad , \tag{15}$$

where the dot denotes differentiation with respect to time. Since all F_i are conserved we must have $\{H, F_i\} = 0$ and thus

$$H = H(\{F_i\}) \quad . \tag{16}$$

If a system is integrable and its phase space Ω is compact and connected then it will be diffeomorphic to an n-dimensional torus, $S^1 \times \cdots \times S^1$. Then one can introduce canonical coordinates $(\theta_1, \ldots, \theta_n, F_1, \ldots, F_n)$ where

$$0 \leq \theta_i \leq 2\pi, \qquad i \in \{1, \ldots, n\} \quad , \tag{17}$$

and F_i are integrals of motion. The equations of motion are now equivalent to

$$\begin{aligned} \dot{F}_i &= 0 \\ \dot{\theta}_i &= \omega_i(F_1, \ldots, F_n) \end{aligned} \tag{18}$$

for each $i \in \{1, \ldots, n\}$, and the system is solvable by quadratures (straightforward integration). This is where the name 'integrability' comes from.

The KAM theorem tells us that if an integrable system is subject to a small perturbation, then the motion will be confined on some deformed torus and the system will be quasi-periodic [4]. This is subject to certain other conditions, such as the frequencies, ω_i, being sufficiently incommensurate.

It becomes more difficult to satisfy the KAM theorem as the number of degrees of freedom increases, since as the dimension of the phase space increases, the volume occupied by the tori decreases.

Suppose that we can find a pair of $N \times N$ matrices (L, M) such that the equations of motion can be equivalently described by the Lax equation,

$$\frac{d}{dt}L = [L, M] \quad , \tag{19}$$

where $[.\ ,\ .]$ is the Lie bracket of some Lie algebra. Then for $n < N$ we can always get integrals of motion because

$$F_i = \mathrm{tr}(L^i) \tag{20}$$

is conserved;

$$\frac{d}{dt}F_i = i\,\mathrm{tr}\left(L^{i-1}\,[L, M]\right) = 0 \tag{21}$$

due to the linearity of the commutator and cyclicity of the trace. A stronger form of integrability is when the Lax pair depends on some complex parameter λ, known as the spectral parameter. Then there is a Lax pair at each value of λ, and the conversed quantities can be obtained by the expansion of $\mathrm{tr}(L^i)$.

2.2 Quantum Integrability

Naturally when we go to the quantum picture things become more complicated. There is no concrete definition of quantum integrability [5]. Some aspects of classical integrability transfer over nicely. Instead of requiring a set of conserved functions, we now require a set of commuting operators, as the Poisson bracket is replaced with the commutator. We also need to find the eigenstates of the Hamiltonian. One problem is that for quantum field theories, the number of degrees of freedom becomes infinite, and we require an infinite set of conserved charges. We will study the integrability of finite lattice models from statistical physics which under certain continuum limits produce integrable field theories, so we don't need to worry about the last point.

One of the most important equations for integrable systems is the Yang-Baxter equation [6]. Consider a complex vector space, V, and let $R(\lambda)$, known as the R matrix, be a function of $\lambda \in \mathbb{C}$. Then the Yang-Baxter equation, defined on $V^{\otimes 3}$, is

$$R_{12}(\lambda - \mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda - \mu) \quad , \tag{22}$$

where each R acts non-trivially on two copies of V, the indices indicating which two;

$$R_{12}(\lambda) = R(\lambda) \otimes I \quad , \tag{23}$$

etc. We will only consider $V = \mathbb{C}^2$. One such solution in this case is

$$R(\lambda) = \lambda I + iP = \left[\begin{array}{cc|cc} \lambda + i & & & \\ & \lambda & i & \\ \hline & i & \lambda & \\ & & & \lambda + i \end{array}\right] \quad , \tag{24}$$

where P is the permutation operator defined in the previous section. It is shown by explicit computation that this is a solution in the following section, and it is also shown using matrix representations later on. The Yang-Baxter equation is important because it leads to the objects which allow a system to be classified as integrable, i.e. it leads to operators, the traces of which are commuting, and hence a set of conserved charges.

Systems from statistical physics form one of the main areas of interest in the context of integrability, namely 1d spin chains and 2d lattice models. Of these, probably the most studied is the Heisenberg XXX spin $\frac{1}{2}$ chain, a 1d model of a magnet. Bethe solved the spin chain using what is now known as the Coordinate Bethe Ansatz (CBA), obtaining explicit eigenstates of the Hamiltonian which lead to evidence of particle-like excitations, magnons, in the system. In statistical physics, a model is regarded as exactly solvable if its partition function is known, which leads to the thermodynamic quantities. The first 2d model that was solved exactly was the Ising model on a square lattice (around the 1940s). Some 20 years later, the 6-vertex model was solved using a method similar to the CBA, and these methods were also important in obtaining the solution to the 8-vertex model. These 2d models will not be discussed in detail, but it is nice to see the link between 2d lattice models and 1d spin chains; the Ising model corresponds to the XY spin chain, the 6-vertex model corresponds to the XXZ spin chain and the 8-vertex model corresponds to the XYZ spin chain. These are all generalisations of the XXX spin chain, which is the focus of this project.

Today a more algebraic treatment, known as the quantum inverse scattering method or Algebraic Bethe Ansatz (ABA), is used to solve integrable systems. In a sense, this method generalises the creation-annihilation operator method for solving the quantum harmonic oscillator. The existence of a number of particle-like excitations in the system is assumed, which determines the eigenstates. For this to be true we obtain a set of algebraic equations, known as the Bethe ansatz equations (BAE), which must be satisfied. Solving them is a problem in itself. Obtaining a set of BAE for a system is roughly how we classify a system as quantum integrable.

In the next section, the ABA is shown in action for the XXX spin $\frac{1}{2}$ chain, and the BAE obtained are solved numerically.

3 Algebraic Bethe Ansatz

3.1 Heisenberg XXX Spin Chain

The ABA method is a more general version of the original CBA method for solving integrable systems. In this chapter we will demonstrate the ABA method for one of the most studied integrable models: the Heisenberg isotropic $\mathfrak{su}(2)$ spin chain [7]. First we must introduce the notion of quantum spin chains.

A quantum spin chain is a one dimensional array of L atoms, modelled by a lattice of length L with unit spacing between sites and periodic boundary conditions. The microscopic degrees of freedom are the quantum mechanical spins associated with each site on the chain. Each site has an associated Hilbert space, h_l, and a spin operator, $\vec{S}_l = (S_l^1, S_l^2, S_l^3)$, on h_l, each $\vec{S}_l$ generating a copy of $\mathfrak{su}(2)$.

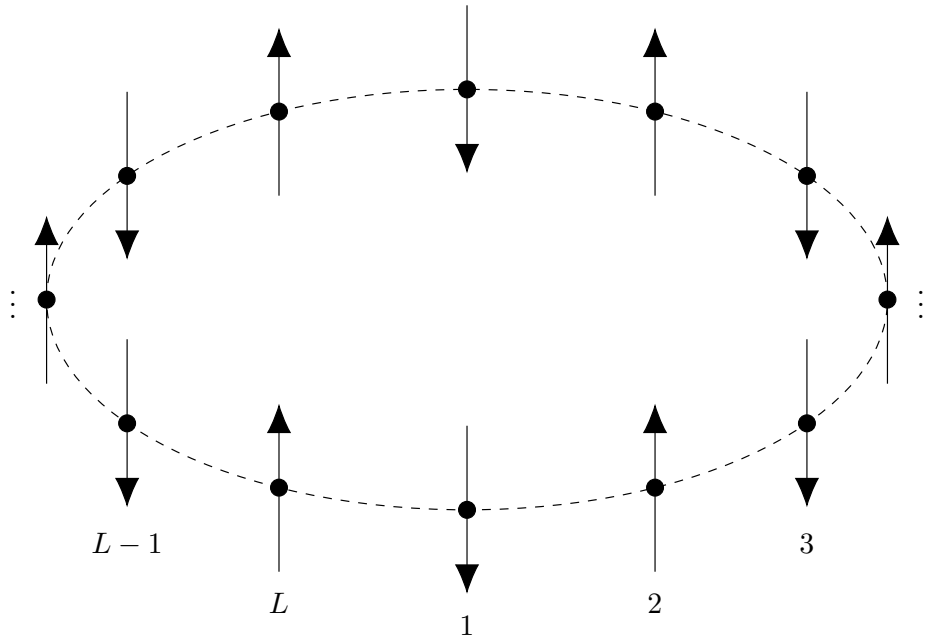

Figure 1: The $\mathfrak{su}(2)$ Heisenberg spin chain.

Since we focus on the spin $\frac{1}{2}$ case, each individual Hilbert space will be a copy of $\mathbb{C}^2$. The chain is ultralocal, so the Hilbert space for the entire chain is given by the product of the Hilbert spaces of the individual sites,

$$\mathcal{H} = h_1 \otimes \ldots \otimes h_L \quad . \tag{25}$$

We define the global spin operators on $\mathcal{H}$ in accordance with (8); the operator S_l^α acts on the entire

chain, but only non-trivially on the l^{th} site,

$$S_l^\alpha \equiv I_1 \otimes \ldots \otimes \underbrace{S_l^\alpha}_{l} \otimes \ldots \otimes I_L \quad . \tag{26}$$

The Lie algebra for the entire chain is given by the global relation between spin operators,

$$[S_l^\alpha, S_m^\beta] = i\delta_{lm}\epsilon^{\alpha\beta\gamma}S_l^\gamma \quad , \tag{27}$$

which is ultralocal since the spin operators at different sites commute. The observables we are interested in are the total spin,

$$S^\alpha = \sum_l S_l^\alpha \quad , \tag{28}$$

and the Hamiltonian,

$$H = -J\sum_l \vec{S}_l \cdot \vec{S}_{l+1} \quad . \tag{29}$$

The exchange coupling J sets the energy scale of the chain. For $J > 0$, the energy is minimised when the spins are aligned; this is the ferromagnetic case. For $J < 0$, the energy is minimised when the spins are anti-aligned. We will ignore the overall constant in this consideration, as well as add a constant term to the Hamiltonian for scaling purposes,

$$H_{\text{XXX}} = \sum_{l,\alpha} S_l^\alpha S_{l+1}^\alpha - \frac{L}{4} \quad . \tag{30}$$

Since each S_l^α contributes equally to the Hamiltonian, the Lie algebra is isotropic, and hence we call this model the XXX spin chain. The XXZ spin chain is a more general model where one of the operators contributes differently to the other two,

$$H_{\text{XXZ}} = \sum_{l,\alpha}\left(S_l^1 S_{l+1}^1 + S_l^2 S_{l+1}^2 + \Delta S_l^3 S_{l+1}^3\right) - \frac{L}{4} \quad , \tag{31}$$

where Δ is the anisotropy factor. XXZ reduces to XXX in the limit $\Delta \to 1$. An even more general XYZ spin chain is obtained when all three spin operators contribute differently.

3.1.1 The Commuting Operators

Our goal is to study the integrability of the XXX model (remarkably, the XXZ and XYZ models are also integrable [8][9]). Our first task is to show that the model is integrable. To do this we must find an operator that produces a set of commuting charges, the Hamiltonian being one of them. We will start with the R matrix. As mentioned in the previous section,

$$R_{a_1 a_2}(\lambda) = \lambda I_{a_1 a_2} + iP_{a_1 a_2} \quad , \tag{32}$$

is a solution to the Yang-Baxter equation, although not the only one. Subscripts a_1 and a_2 refer to spaces V_1 and V_2, which are both copies of $\mathbb{C}^2$, so $R_{a_1 a_2}$ acts on $V_1 \otimes V_2$. Now we can introduce the Lax operator,

$$L_{la}(\lambda) = R_{la}(\lambda - i/2) = (\lambda - i/2)I_{la} + iP_{la} \quad . \tag{33}$$

The subscript l refers to the Hilbert space associated with the l^{th} site. The subscript a refers to an auxiliary vector space V, which we will take to be another copy of $\mathbb{C}^2$. It is necessary to introduce an auxiliary space to connect the quantum spaces as they are all disconnected. This is essentially the purpose of the Lax operator; it is a connection on our chain, defining parallel transport between two adjacent sites.

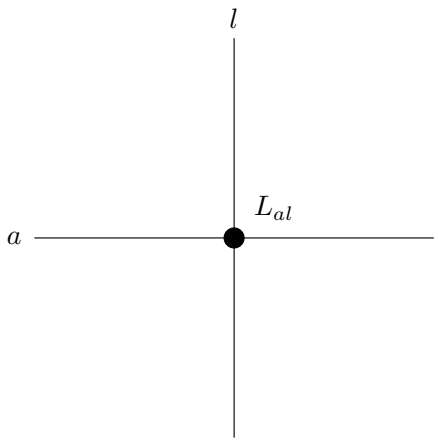

Figure 2: Graphical representation of the Lax operator.

Transport between non-adjacent sites can be described by the ordered product of the Lax operators of the sites and the ones between them. The ordered product of all of the Lax operators defines a monodromy around the chain,

$$T_{La}(\lambda) = L_{La}(\lambda) \dots L_{1a}(\lambda) \quad . \tag{34}$$

This is known as the monodromy matrix of the chain. The transfer matrix is obtained by taking the trace of the monodromy matrix over auxiliary space,

$$F_L(\lambda) = \mathrm{tr}_a(T_{La}(\lambda)) \quad . \tag{35}$$

The expansion of $F_L(\lambda)$ results in a set of $N-1$ commuting operators,

$$F_L(\lambda) = 2\lambda^L + \sum_{l=0}^{L-1} Q_l \lambda^l \quad , \tag{36}$$

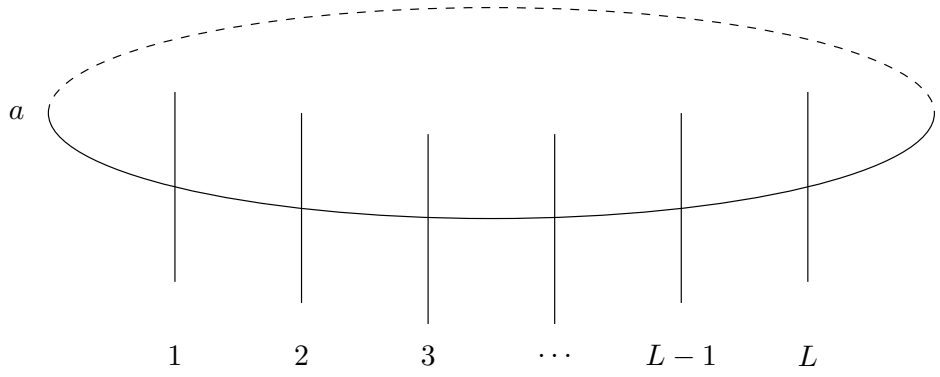

Figure 3: Graphical representation of the monodromy matrix.

and these Q_l form the set of conserved charges to which the Hamiltonian belongs. A component of the total spin will complete the set of conserved charges; it doesn't appear in the expansion because the spin operators are traceless. First we must show that the charges commute, which can be done by showing that F_L commutes with itself for any two values of the spectral parameter.

3.1.2 Fundamental Commutation Relations

In order to show the Hamiltonian belongs to the set of conserved charges we must first establish the commutation relation for the Lax operators. The products of Lax operators,

$$L_{la_1}(\lambda)L_{la_2}(\mu), \quad L_{la_2}(\mu)L_{la_1}(\lambda), \tag{37}$$

act on the product of spaces $h_l \otimes V_1 \otimes V_2$, and the matrix representations will be 16×16 matrices. We thus require 16 commutation relations, but these can be condensed into one relation using the Yang-Baxter equation,

$$R_{a_1a_2}(\lambda-\mu)L_{la_1}(\lambda)L_{la_2}(\mu) = L_{la_2}(\mu)L_{la_1}(\lambda)R_{a_1a_2}(\lambda-\mu) \quad . \tag{38}$$

This is the fundamental commutation relation (FCR) for the Lax operators, and can be verified by explicit computation. Explicitly we have

$$\begin{aligned} & [(\lambda-\mu)I_{a_1a_2} + iP_{a_1a_2}]\,[(\lambda-i/2)I_{la_1} + iP_{la_1}]\,[(\mu-i/2)I_{la_2} + iP_{la_2}] \\ = \quad & [(\mu-i/2)I_{la_2} + iP_{la_2}]\,[(\lambda-i/2)I_{la_1} + iP_{la_1}]\,[(\lambda-\mu)I_{a_1a_2} + iP_{a_1a_2}] \end{aligned} \quad , \tag{39}$$

and we can verify that this is true by comparing terms with the same number of permutation operators. For the terms without any permutation operators, this is immediate, as it is just the identity times the same constant in both cases. Note that this product is defined on $h_l \otimes V_1 \otimes V_2$, and for operators with two indices it is implied that they act non-trivially on the third space, i.e.

$$I_{a_1a_2} \equiv I_{a_1a_2} \otimes I_l = I_{a_1} \otimes I_{a_2} \otimes I_l \quad . \tag{40}$$

The proof is also immediate for the terms with one permutation operator, since they commute with the identity. Now for the terms with two permutation operators, we have

$$\begin{aligned} & (\lambda-\mu)P_{la_1}P_{la_2}+(\lambda-i/2)P_{a_1a_2}P_{la_2}+(\mu-i/2)P_{a_1a_2}P_{la_1} \\ = \quad & (\lambda-\mu)P_{la_2}P_{la_1}+(\lambda-i/2)P_{la_2}P_{a_1a_2}+(\mu-i/2)P_{la_1}P_{a_1a_2} \end{aligned} \quad , \tag{41}$$

and to prove this we must first establish some important properties of P. For a general operator we can rewrite (10), the main property, as

$$M_{a_1a_2}=P_{la_2}M_{la_1}P_{la_2} \quad . \tag{42}$$

This will also come in useful later. When this is applied to another permutation operator, we obtain a useful relation between permutation operators with one similar index,

$$P_{la_1}P_{la_2}=P_{a_1a_2}P_{la_1}=P_{la_2}P_{a_1a_2} \quad . \tag{43}$$

This will allow us to manipulate the indices in strings of permutation operators. Finally, permutation operators are symmetric in their indices,

$$P_{a_1a_2}=P_{a_2a_1} \quad , \tag{44}$$

and of course $P^2=I$. We use (43) on the terms which contain $P_{a_1a_2}$ to write the relation solely in terms of P_{la_1} and P_{la_2},

$$\begin{aligned} & (\lambda-\mu)P_{la_1}P_{la_2}+(\lambda-i/2)P_{la_2}P_{la_1}+(\mu-i/2)P_{la_1}P_{la_2} \\ = \quad & (\lambda-\mu)P_{la_2}P_{la_1}+(\lambda-i/2)P_{la_1}P_{la_2}+(\mu-i/2)P_{la_2}P_{la_1} \end{aligned} \quad , \tag{45}$$

and from this it is clear that the terms are equivalent. Finally, for the terms with three permutation operators we use

$$P_{a_1a_2}P_{la_1}P_{la_2}=P_{la_2}P_{la_1}P_{a_1a_2} \quad , \tag{46}$$

which is easily proven to be true using (43),

$$\begin{aligned} P_{a_1a_2}P_{la_1}P_{la_2} &= P_{la_2}P_{a_1a_2}P_{la_2} \\ &= P_{la_2}P_{la_2}P_{a_1a_2} \end{aligned} \quad . \tag{47}$$

Hence (38) indeed holds. Since this is a special case of the Yang-Baxter equation, we can also conclude that (24) is a solution to (22). We will call the FCR for Lax operators the RLL relation.

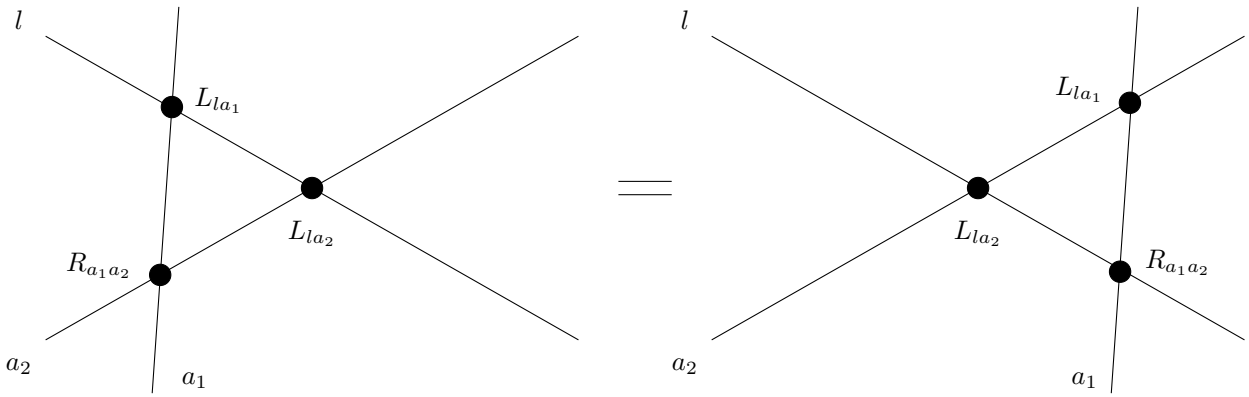

Figure 4: Graphical representation of the RLL relation [10].

Using (38) we can obtain a similar FCR for the monodromy matrices,

$$R_{a_1a_2}(\lambda-\mu)T_{a_1}(\lambda)T_{a_2}(\mu) = T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1a_2}(\lambda-\mu) \quad , \tag{48}$$

which we will call the RTT relation (the subscript L is dropped as it characteristic of the chain). This can be shown for $L = 2$ (two Lax operators), and the general case follows by induction. For $L = 2$ we have

$$\begin{aligned} R_{a_1a_2}L_{2a_1}\underbrace{L_{1a_1}L_{2a_2}}_{\text{commute}}L_{1a_2} &= \underbrace{R_{a_1a_2}L_{2a_1}L_{2a_2}}_{\text{FCR}}L_{1a_1}L_{1a_2} \\ &= L_{2a_2}L_{2a_1}\underbrace{R_{a_1a_2}L_{1a_1}L_{1a_2}}_{\text{FCR}} \\ &= L_{2a_2}\underbrace{L_{2a_1}L_{1a_2}}_{\text{commute}}L_{1a_1}R_{a_1a_2} \\ &= L_{2a_2}L_{1a_2}L_{2a_1}L_{1a_2}L_{1a_1}R_{a_1a_2} \end{aligned} \quad . \tag{49}$$

It is easy to see that this trick can be applied iteratively for any $L > 2$. Thus, we have a similar FCR for monodromy matrices acting on different auxiliary spaces.

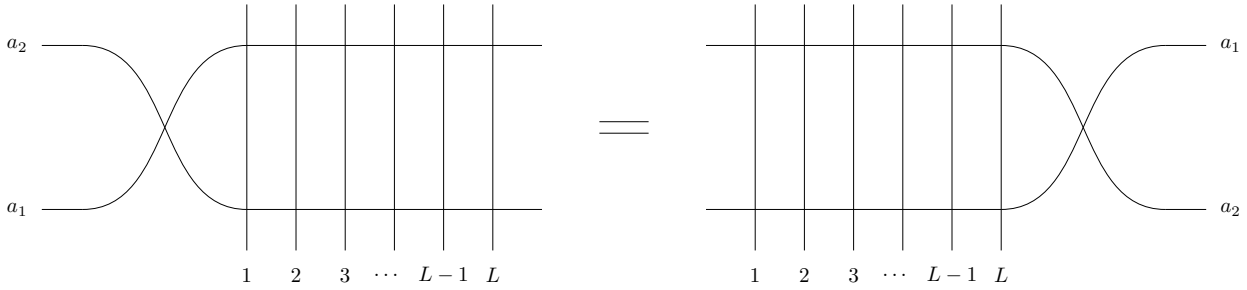

Figure 5: Graphical representation of the RTT relation; the train track argument.

From the RTT relation, it follows that the transfer matrices commute. Since R is invertible, we can left multiply by R^{-1} and then take the trace over both auxiliary spaces,

$$\mathrm{tr}_{a_1}\,\mathrm{tr}_{a_2}\left(T_{a_1}(\lambda)T_{a_2}(\mu)\right) = \mathrm{tr}_{a_1}\,\mathrm{tr}_{a_2}\left(R^{-1}_{a_1a_2}(\lambda-\mu)T_{a_2}(\mu)T_{a_1}(\lambda)R_{a_1a_2}(\lambda-\mu)\right) \quad , \tag{50}$$

and then from the cyclicity of the trace we obtain

$$[F(\lambda), F(\mu)] = 0 \quad , \tag{51}$$

where $F(\lambda)$ is traced over V_1, and $F(\mu)$ is traced over V_2. Since we have just shown that these operators commute, then if we show that the Hamiltonian belongs to the set of conversed charges obtained from $F(\lambda)$, we can conclude that our system is an integrable one.

3.1.3 Rewriting the Hamiltonian

We are now faced with the task of obtaining (30) from the expansion of the transfer matrices. The point $\lambda = i/2$ is important because

$$L_{la}(i/2) = iP_{la} \quad , \tag{52}$$

although this is no surprise since we defined the Lax operator from the R matrix; it corresponds to $R(0)$. From this we can conveniently expand $F(\lambda)$ about the point $\lambda = i/2$. At this point, $T(\lambda)$ is given by

$$T(i/2) = i^L P_{Na}\ldots P_{1a} \quad , \tag{53}$$

and so the trace at the point $\lambda = i/2$ is

$$\begin{aligned} F(i/2) &= i^L\,\mathrm{tr}_a\left(P_{La}\ldots P_{1a}\right) \\ &= i^L\,\mathrm{tr}_a\left(P_{12}\ldots P_{L-1L}P_{La}\right) \\ &= i^L P_{12}\ldots P_{L-1L} \end{aligned} \quad . \tag{54}$$

Here the identity (47) is used iteratively to rewrite the string of permutation operators. We only trace over the last permutation operator and

$$\begin{aligned} \mathrm{tr}_a(P_{la}) &= \mathrm{tr}_a\left(\frac{1}{2}\begin{bmatrix} I_l & 0 \\ 0 & I_l \end{bmatrix} + \frac{1}{2}\sum_\alpha \underbrace{\sigma_l^\alpha \otimes \sigma_a^\alpha}_{\text{traceless}}\right) \\ &= I_l \end{aligned} \quad . \tag{55}$$

Now that we have $F(i/2)$ in terms of permutation operators, its inverse is given by

$$F^{-1}(i/2) = i^{-L}P_{LL-1}\ldots P_{21} \quad . \tag{56}$$

We introduce the shift operator, U,

$$U = i^{-L}F(i/2) = P_{12}\ldots P_{L-1L} \quad , \tag{57}$$

which is unitary, and by (42) can be shown to shift the indices of operators,

$$U^{-1}M_lU = M_{l-1} \quad . \tag{58}$$

This allows us to introduce the (lattice) momentum observable,

$$U = e^{iP} \quad . \tag{59}$$

Momentum is the generator of translations and on our spin chain this is taken to mean a shift along one lattice site. This also belongs to the set of conserved charges generated by $F(\lambda)$.

The derivative of $T(\lambda)$ at the point $\lambda = i/2$ is given by

$$\frac{d}{d\lambda}T(\lambda)\Big|_{\lambda=i/2} = i^{L-1}\sum_l P_{La}\dots P_{l+1a}P_{l-1a}\dots P_{1a} \quad , \tag{60}$$

where the l^{th} permutation operator is missing. If we take the trace of this we will have the derivative of $F(\lambda)$ at the point $\lambda = i/2$. Using the same trick as before we get

$$\frac{d}{d\lambda}F(\lambda)\Big|_{\lambda=i/2} = i^{L-1}\sum_l P_{12}\dots P_{l-1l+1}\dots P_{L-1L} \quad , \tag{61}$$

the P_{l-1l+1} term arising because the l^{th} permutation operator was missing. If we multiply (61) by $F^{-1}(i/2)$ it will simplify things greatly, although it is not immediately obvious. We have

$$\begin{aligned}
\frac{d}{d\lambda}\ln(F(\lambda))\Big|_{\lambda=i/2} &= i^{-1}\sum_l P_{12}\dots P_{l-1l+1}\dots P_{L-1L}P_{LL-1}\dots P_{21} \\
&= i^{-1}\sum_l P_{12}\dots P_{l-1l+1}P_{l+1l}\dots P_{21} \\
&= i^{-1}\sum_l P_{12}\dots P_{ll+1}(\underbrace{P_{l-1l}P_{ll-1}}_{I})\dots P_{21} \\
&= i^{-1}\sum_l P_{ll+1}
\end{aligned} \quad . \tag{62}$$

In the second line, we use the commutation rule for the permutation operators to commute the P_{l-1l+1} and P_{ll+1}, creating an extra P_{l-1l}, which cancels with the one to the right of it. P_{ll+1} commutes with everything to the left of it, and the strings originally on either side of P_{ll+1} collapse to the identity. Recalling that P looks like (9), (62) conveniently looks very similar to the Hamiltonian, H_{XXX},

$$\begin{aligned}
H_{\text{XXX}} &= \sum_{l,\alpha} S_l^\alpha S_{l+1}^\alpha - \frac{L}{4} \\
&= \frac{1}{2}\sum_l P_{ll+1} - \frac{L}{2}
\end{aligned} \quad . \tag{63}$$

By using (62) we can finally write our Hamiltonian as

$$H_{\text{XXX}} = \frac{i}{2}\frac{d}{d\lambda}\ln(F(\lambda))\Big|_{\lambda=i/2} - \frac{L}{2} \quad , \tag{64}$$

which completes the proof that our system is integrable.

3.2 Bethe Ansatz Equations

Our next goal is to get the eigenvalues of $F(\lambda)$ using (48) and some properties of the Lax operators. This method is like a generalisation of the creation and annihilation operator method for solving the harmonic oscillator; we make use of a commutation relation such as $[a, a^\dagger] = 1$ and require the existence of a reference state $|\Omega\rangle$ such that $a\,|\Omega\rangle = 0$.

First we need to write down the required relations from the RTT relation. Recall that $R(\lambda)$ is given by (32) and its matrix representation on $V_1 \otimes V_2$ is

$$R_{a_1 a_2}(\lambda) = \left[\begin{array}{cc|cc} \lambda+i & & & \\ & \lambda & i & \\ \hline & i & \lambda & \\ & & & \lambda+i \end{array}\right] \quad . \tag{65}$$

We can also write $T(\lambda)$ in matrix form,

$$T(\lambda) = \begin{bmatrix} A(\lambda) & B(\lambda) \\ C(\lambda) & D(\lambda) \end{bmatrix} \quad . \tag{66}$$

Then on $\mathcal{H} \otimes V_1 \otimes V_2$, the T in (48) are explicitly given by

$$T_{a_1}(\lambda) = \left[\begin{array}{cc|cc} A(\lambda) & & B(\lambda) & \\ & A(\lambda) & & B(\lambda) \\ \hline C(\lambda) & & D(\lambda) & \\ & C(\lambda) & & D(\lambda) \end{array}\right], \quad T_{a_2}(\mu) = \left[\begin{array}{cc|cc} A(\mu) & B(\mu) & & \\ C(\mu) & D(\mu) & & \\ \hline & & A(\mu) & B(\mu) \\ & & C(\mu) & D(\mu) \end{array}\right] \quad . \tag{67}$$

We can now get some explicit relations for the entries of T from the RTT relation. Entry (1,3) in (48) gives

$$(\lambda - \mu + i)B(\lambda)A(\mu) = iB(\mu)A(\lambda) + (\lambda - \mu)A(\mu)B(\lambda) \quad , \tag{68}$$

which after rearranging and exchanging $(\lambda \leftrightarrow \mu)$ gives the first relation in (70). The second relation comes from entry (2,4),

$$(\lambda - \mu + i)B(\mu)D(\lambda) = (\lambda - \mu)B(\mu)D(\lambda) + iB(\lambda)D(\mu) \quad . \tag{69}$$

Also, we see that B commutes with itself. In summary, we have

$$\begin{aligned} [B(\lambda), B(\mu)] &= 0 \\ A(\lambda)B(\mu) &= f(\lambda - \mu)B(\mu)A(\lambda) + g(\lambda - \mu)B(\lambda)A(\mu) \quad , \\ D(\lambda)B(\mu) &= h(\lambda - \mu)B(\mu)D(\lambda) + k(\lambda - \mu)B(\lambda)D(\mu) \end{aligned} \tag{70}$$

where we introduce the notation

$$\begin{aligned} f(\lambda) &= \frac{\lambda - i}{\lambda} \qquad g(\lambda) = \frac{i}{\lambda} \\ h(\lambda) &= \frac{\lambda + i}{\lambda} \qquad k(\lambda) = -\frac{i}{\lambda} \end{aligned} \quad , \tag{71}$$

to simplify the relations.

To find $|\Omega\rangle$, we first make the argument that for each quantum space h_l there exists some vector $|\omega_l\rangle$ such that the Lax operator becomes upper triangular in auxiliary space when applied to it,

$$L_{la}(\lambda)\,|\omega_l\rangle = \begin{bmatrix} \lambda + i/2 & * \\ 0 & \lambda - i/2 \end{bmatrix} |\omega_l\rangle \quad , \tag{72}$$

the * referring to irrelevant terms (irrelevant in the context of finding the reference state). This is simply the basis vector,

$$|\omega_l\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad , \tag{73}$$

and our reference state is given by

$$|\Omega\rangle = |\omega_1\rangle \otimes \ldots \otimes |\omega_L\rangle \tag{74}$$

in $\mathcal{H}$. Thus,

$$T(\lambda)\,|\Omega\rangle = \begin{bmatrix} \alpha^L(\lambda) & * \\ 0 & \delta^L(\lambda) \end{bmatrix} |\Omega\rangle \quad , \tag{75}$$

where

$$\alpha(\lambda) = \lambda + i/2 \qquad \delta(\lambda) = \lambda - i/2 \quad . \tag{76}$$

Now we have

$$C(\lambda)\,|\Omega\rangle = 0 \qquad A(\lambda)\,|\Omega\rangle = \alpha^L(\lambda)\,|\Omega\rangle \qquad D(\lambda)\,|\Omega\rangle = \delta^L(\lambda)\,|\Omega\rangle \quad , \tag{77}$$

so $|\Omega\rangle$ is an eigenstate of both $A(\lambda)$ and $D(\lambda)$ and hence it is an eigenstate of $F(\lambda) = \mathrm{tr}_a(T(\lambda)) = A(\lambda) + D(\lambda)$.

It remains to find the rest of the eigenstates of $F(\lambda)$. We make the ansatz that they are of the form

$$|\Phi\{\lambda\}\rangle = B(\lambda_1)\ldots B(\lambda_m)\,|\Omega\rangle \quad , \tag{78}$$

for a set of parameters $\{\lambda = \lambda_1, \ldots, \lambda_m\}$. We can determine them by calculating $A\,|\Phi\rangle$ and $D\,|\Phi\rangle$, using (70) and the fact that $|\Omega\rangle$ is an eigenstate of A and D. Explicitly we have

$$\begin{aligned} A\,|\Phi\rangle &= A(\lambda)B(\lambda_1)\ldots B(\lambda_m)\,|\Omega\rangle \\ &= \alpha^L(\lambda)\left(\prod_{k=1}^{m} f(\lambda - \lambda_k)\right)|\Phi\rangle \\ &+ \sum_k M_k(\lambda, \{\lambda\})B(\lambda_1)\ldots\hat{B}(\lambda_k)\ldots B(\lambda_m)B(\lambda)\,|\Omega\rangle \end{aligned} \quad , \tag{79}$$

where $\hat{B}(\lambda_k)$ means that term is omitted. The first term comes from commuting $A(\lambda)$ through each $B(\lambda_k)$ using only the first term of the commutation relation. This is the term that we want to use since it contains $|\Phi\rangle$. The second term accounts for the $2^m - 1$ other terms generated from the commutation relation; they are not in terms of $|\Phi\rangle$ and the coefficient M_k is rather complicated. We can calculate the first one, M_1, and the rest are obtained by sending $\lambda_1 \to \lambda_k$. The second term obtained from commuting $A(\lambda)$ with $B(\lambda_1)$ is

$$g(\lambda - \lambda_1)B(\lambda)[A(\lambda_1)B(\lambda_2)\dots B(\lambda_m)]\,|\Omega\rangle \quad , \tag{80}$$

and now we need to commute $A(\lambda_1)$ with $B(\lambda_2)$, but the second term resulting from this, which contains $A(\lambda_2)$, will contribute to M_2, so we can ignore it. Repeating this process we get

$$M_1 = \alpha^L(\lambda_1)g(\lambda - \lambda_1)\left(\prod_{j=2}^{m} f(\lambda_1 - \lambda_j)\right) \quad , \tag{81}$$

and in general we have

$$M_k = \alpha^L(\lambda_k)g(\lambda - \lambda_k)\left(\prod_{j\neq k}^{m} f(\lambda_k - \lambda_j)\right) \quad . \tag{82}$$

Similarly, for D we have

$$\begin{aligned} D\,|\Phi\rangle &= D(\lambda)B(\lambda_1)\dots B(\lambda_m)\,|\Omega\rangle \\ &= \delta^L(\lambda)\left(\prod_{k=1}^{m} h(\lambda - \lambda_k)\right)|\Phi\rangle \\ &+ \sum_k N_k(\lambda, \{\lambda\})B(\lambda_1)\dots\hat{B}(\lambda_k)\dots B(\lambda_m)B(\lambda)\,|\Omega\rangle \end{aligned} \quad . \tag{83}$$

The coefficients N_k are determined as they were for M_k,

$$N_k = \delta^L(\lambda_k)k(\lambda - \lambda_k)\left(\prod_{j\neq k}^{m} h(\lambda_k - \lambda_j)\right) \quad . \tag{84}$$

$|\Phi\rangle$ will be an eigenstate of $F(\lambda)$ if the second terms in (79) and (83) vanish. So we will have

$$F(\lambda)\,|\Phi\rangle = (A + D)\,|\Phi\rangle = \Lambda(\lambda, \{\lambda\})\,|\Phi\rangle \quad , \tag{85}$$

with eigenvalue

$$\Lambda(\lambda, \{\lambda\}) = \alpha^L(\lambda)\left(\prod_{k=1}^{m} f(\lambda - \lambda_k)\right) + \delta^L(\lambda)\left(\prod_{k=1}^{m} h(\lambda - \lambda_k)\right) \quad , \tag{86}$$

if the following condition is met for each λ_k;

$$\alpha^L(\lambda_k)\left(\prod_{j\neq k}^{m} f(\lambda_k - \lambda_j)\right) = \delta^L(\lambda_k)\left(\prod_{j\neq k}^{m} h(\lambda_k - \lambda_j)\right) \quad , \tag{87}$$

where we used

$$g(\lambda) = -k(\lambda) \quad . \tag{88}$$

By rearranging and using the explicit expressions given in (71) and (76) we finally arrive at

$$\left(\frac{\lambda_k + i/2}{\lambda_k - i/2}\right)^L = \prod_{j\neq k}^{m} \frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i} \quad , \tag{89}$$

the Bethe Ansatz Equations (BAE). $|\Phi\rangle$ is known as the Bethe vector and λ_k are known as the Bethe roots. (87) is just the residue of Λ, so straight away as a result of (89) the poles in Λ will cancel and the eigenvalues will be analytic. BAE can also be used to take the large L limit and calculate thermodynamic quantities of the spin chain. To give more physical meaning to BAE, we can rewrite it as

$$e^{ip(\lambda_k)L} = \prod_{j\neq k}^{m} S(\lambda_k - \lambda_j) \quad , \tag{90}$$

where

$$S(\lambda) = \frac{\lambda + i}{\lambda - i} \tag{91}$$

is the two-particle scattering amplitude. The LHS is the phase factor picked up by the k^{th} magnon as it travels around the chain. The RHS accounts for the interactions between it and the rest of the magnons. The fact that the scattering amplitude for m magnons is factorised into 2-magnon scattering amplitudes is a result of the integrability of the model.

We can now find the momentum and energy of our chain. Recalling that the shift operator and momentum are related to $F(i/2)$, we have

$$U\,|\Phi\rangle = i^{-L}F(i/2)\,|\Phi\rangle = i^{-L}\Lambda(i/2)\,|\Phi\rangle = \prod_{k=1}^{m}\left(\frac{\lambda_k + i/2}{\lambda_k - i/2}\right)|\Phi\rangle \quad . \tag{92}$$

Taking the natural log of both sides, we get

$$P\,|\Phi\rangle = \sum_{j=1}^{m} p(\lambda_k)\,|\Phi\rangle \quad , \tag{93}$$

where

$$p(\lambda) = -i\ln\left(\frac{\lambda + i/2}{\lambda - i/2}\right) \quad , \tag{94}$$

is the momentum eigenvalue. We can relate this to the energy using (64), and by differentiating with respect to λ we obtain the energy eigenvalue,

$$H_{\text{XXX}}\,|\Phi\rangle = \sum_{k=1}^{m} \epsilon(\lambda_k)\,|\Phi\rangle \quad , \tag{95}$$

where

$$\epsilon(\lambda) = -\frac{1}{2}\left(\frac{1}{\lambda^2 + 1/4}\right) \quad . \tag{96}$$

This gives a quasiparticle interpretation of our states. Our quasiparticle is the magnon, a quantised spin wave with energy $\epsilon(\lambda)$ and lattice momentum $p(\lambda)$, λ effectively acting as the rapidity. It is clear to see that $B(\lambda)$ is the magnon creation operator and $C(\lambda)$ is the corresponding annihilation operator. The ansatz that we made is that our states are obtained from m excitations.

Taking the limit $\lambda \to \infty$, it can be shown that

$$\left[\frac{1}{2}\vec{\sigma_a} + \vec{S}, T_a(\lambda)\right] = 0 \quad . \tag{97}$$

From this we have

$$\begin{aligned} S^+ |\Omega\rangle &= 0 \\ S^3 |\Omega\rangle &= \frac{L}{2} |\Omega\rangle \end{aligned} \quad , \tag{98}$$

i.e. $|\Omega\rangle$ is a highest weight state, and hence

$$S^3 |\Phi\rangle = (L - \frac{m}{2}) |\Phi\rangle \quad . \tag{99}$$

It can also be shown that if BAE are satisfied then

$$S^+ |\Phi\rangle = 0 \quad , \tag{100}$$

and so all $|\Phi\rangle$ are highest weight states. This implies that there is a limit on the number of magnons we can have for a chain of fixed length, since the highest weight eigenvalue of S^3 is non-negative,

$$m \leq \frac{L}{2} \quad . \tag{101}$$

The lowest energy eigenstate is obtained when $m = L/2$ (rounding down when L is odd). For a more detailed treatment, see [7].

3.3 Solving BAE Numerically

We can solve (89) numerically for the Bethe roots for chains of finite length and hence find the energy eigenvalue. The eigenstate with lowest energy is the one where $m = L/2$ and all of the Bethe roots are real. The problem is that there are many complex roots (which describe bound states). When solving equations with Mathematica, if there are complex numbers in the equation it will search for complex solutions. Therefore we must rewrite BAE in a more 'real looking' form in order to solve for the real Bethe roots [11]. Taking the natural log of (89), we get

$$L \ln\left(\frac{\lambda_k + i/2}{\lambda_k - i/2}\right) = \sum_{j\neq k}^{m} \ln\left(\frac{\lambda_k - \lambda_j + i}{\lambda_k - \lambda_j - i}\right) + 2n_k\pi \quad , \tag{102}$$

where $0 \leq n_k \leq N-1$ defines the branch of the logarithms. To rewrite this, we can use the identity

$$\frac{1}{i} \ln\left(\frac{\lambda - ia}{\lambda + ia}\right) = 2\tan^{-1}\left(\frac{\lambda}{a}\right) \quad , \tag{103}$$

which will take care of the factors of i that cause Mathematica to search for complex roots. To prove this identity, consider the equation

$$\lambda = a\tan(w) = \frac{1}{i}\left(\frac{e^{2iw} - 1}{e^{2iw} + 1}\right) \quad . \tag{104}$$

Solving for e^{2iw} and taking the log of both sides gives

$$w = \frac{1}{2i} \ln\left(\frac{\lambda - ia}{\lambda + ia}\right) \quad , \tag{105}$$

but since we also have $w = \tan^{-1}(\lambda/a)$, we get

$$\frac{1}{2i} \ln\left(\frac{\lambda - ia}{\lambda + ia}\right) = \tan^{-1}\left(\frac{\lambda}{a}\right) \quad . \tag{106}$$

Applying this identity to (102) gives

$$L\tan^{-1}(2\lambda_k) - \sum_{j\neq k}^{m} \tan^{-1}(\lambda_k - \lambda_j) = \pi A_k \quad , \tag{107}$$

where A_k are quantum numbers that parametrise the λ_k, which come from the branch of the logarithms. It can be shown by examining the range of the function

$$\Phi(x) = \frac{1}{\pi}\left(L\tan^{-1}(2x) - \sum_{j=1}^{m}(x - x_j)\right) \tag{108}$$

that A_k takes values separated by integer steps in the range

$$\frac{L - 3m + 1}{2} \leq A_k \leq \frac{L - m - 1}{2} \quad . \tag{109}$$

This can be done by plotting $\Phi(x)$ over a sufficient interval for different values of L and m.

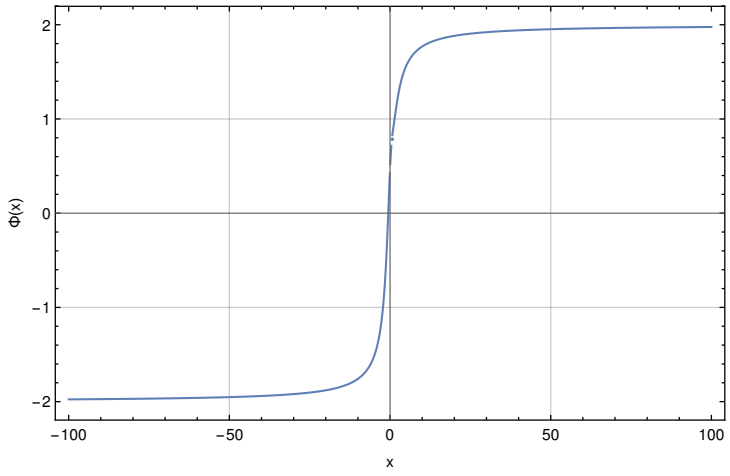

Figure 6: Plot of the function $\Phi(x)$ for $L = 7$ and $m = 3$. The asymptotic behaviour is clear and by repeating this for different values of L and m, the range of A_k can be determined.

We can solve (107) for real Bethe roots using **FindRoot** in Mathematica, and the energy is obtained from (96). As an example, the lowest energy for chains of lengths up to $L = 200$ is calculated. We see that the numerical solution agrees remarkably well with the energy obtained in the large L limit [12],

$$E \simeq -\ln(2)L \quad . \tag{110}$$

It is nice to see that after a very algebraic method of solving the system that we obtain good physical results.

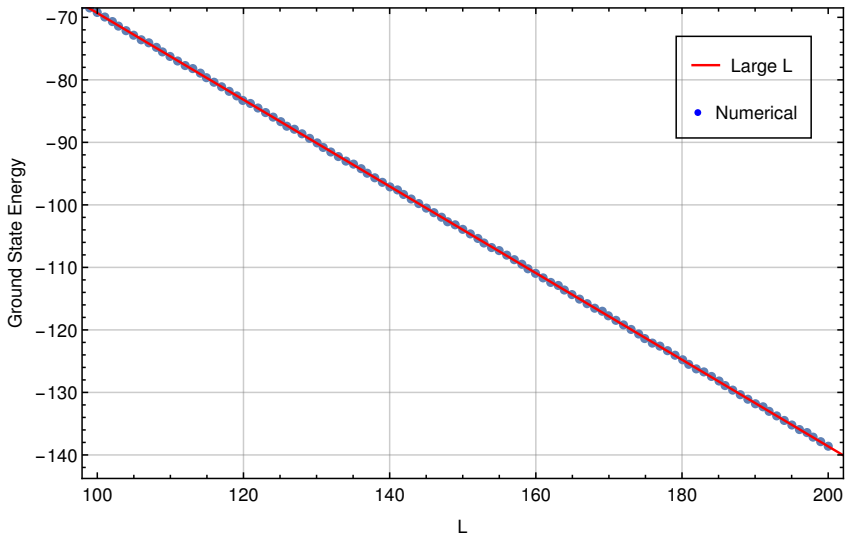

Figure 7: Ground state energy as a function of spin chain length.

4 Breaking Integrability

4.1 Finding the Perturbed Hamiltonian

In this section we introduce deformations to the spin chain. There are several ways to do this; by deforming the algebras, monodromy matrices, transfer matrices or by directly deforming the Hamiltonian. We will deform the Lax operator as it is perhaps the most fundamental object in our system.

$$L_{la} \to \widetilde{L}_{la} = L_{la} + \epsilon M_{la}(\lambda) \quad , \tag{111}$$

where $M_{la} \in h_l \otimes V$ and ϵ is a small parameter so that terms of order ϵ^2 and higher can be neglected. This deformation is introduced locally at each lattice site, although it may result in non-local deformations to the Hamiltonian. To see how this affects the system we need to find the perturbed Hamiltonian. Before we do this, we must find the resulting deformations to the monodromy and transfer matrices. We define the deformed monodromy matrix in the same way as before but in terms of the deformed lax operators,

$$\widetilde{T}(\lambda) = \widetilde{L}_{La} \dots \widetilde{L}_{1a} \quad . \tag{112}$$

By expanding and neglecting terms of order ϵ^2 and higher, an expression for $\widetilde{T}(\lambda)$ in terms of the original monodromy matrix plus leading order perturbative terms is obtained,

$$\widetilde{T}(\lambda) = T(\lambda) + \epsilon \sum_i L_{La} \dots M_{ia} \dots L_{1a} + \mathcal{O}(\epsilon^2) \quad , \tag{113}$$

where in the sum the i^{th} Lax operator is replaced by M_{ia}. Taking the trace as before gives

$$\widetilde{F}(\lambda) = F(\lambda) + \epsilon \sum_i \text{tr}_a(L_{La} \dots M_{ia} \dots L_{1a}) + \mathcal{O}(\epsilon^2) \quad . \tag{114}$$

The deformed Hamiltonian is written analogously to the original Hamiltonian,

$$\widetilde{H} = \frac{i}{2} \frac{d}{d\lambda} \ln(\widetilde{F}(\lambda))\Big|_{i/2} - \frac{N}{2} \quad , \tag{115}$$

and the main goal of this section is to write this in terms of the original Hamiltonian plus the leading order perturbative term,

$$\widetilde{H} = H_{\text{XXX}} + \epsilon f(M(\lambda)) + \mathcal{O}(\epsilon^2) \quad , \tag{116}$$

where $f(M(\lambda))$ has as simple a form as is possible. We differentiate the monodromy matrix,

$$\frac{d}{d\lambda}\widetilde{T}(\lambda)\Big|_{\lambda=i/2} = \frac{d}{d\lambda}\widetilde{T}(\lambda)\Big|_{\lambda=i/2} + \epsilon \sum_i \frac{d}{d\lambda} \left[L_{La} \dots M_{ia} \dots L_{1a}\right]_{\lambda=i/2} + \mathcal{O}(\epsilon^2) \quad . \tag{117}$$

The last term is complicated but we treat it using the methods used in the undeformed case. The deformation in (117) is given by

$$\sum_i \frac{d}{d\lambda}\left[L_{La}\dots M_{ia}\dots L_{1a}\right]_{\lambda=i/2}+\mathcal{O}(\epsilon^2)=\sum_i\left[\left(L'_{La}\right)L_{L-1a}\dots M_{ia}\dots L_{1a}+\dots+\right.\\ \left.+L_{La}\dots M'_{ia}\dots L_{1a}\right]_{\lambda=i/2} \quad , \tag{118}$$

where $'$ denotes differentiation with respect to λ. All of the terms in the sum are similar except for the one that includes M'_{ia}, so we will deal with it separately to the rest. At the point $\lambda=i/2$,

$$\sum_i\left[L_{La}\dots M'_{ia}\dots L_{1a}\right]_{\lambda=i/2}=i^{L-1}\sum_i P_{La}\dots M'_{ia}(i/2)\dots P_{1a} \quad . \tag{119}$$

Now taking the trace over auxiliary space and multiplying by $F^{-1}(i/2)$, (119) becomes

$$i^{-1}\sum_i \mathrm{tr}_a\left[P_{La}\dots P_{i+1a}M'_{ia}P_{i-1a}\dots P_{1a}\right]P_{LL-1}\dots P_{21} \quad , \tag{120}$$

and the permutation operators will collapse as they did before. We use (42) to rewrite the strings of permutations on either side of M'_{ia},

$$\begin{aligned} P_{la}\dots P_{i+1a}\dots P_{i+1a}&=P_{i+1a}P_{i+1i+2}\dots P_{L-1L}\\ P_{i-1a}\dots P_{1a}&=P_{12}\dots P_{i-2i-1}P_{i-1a}\end{aligned} \quad . \tag{121}$$

Now we factor the strings out of the trace to the right and left respectively, leaving only two permutation operators and M'_{ia} inside the trace. (120) becomes

$$\begin{aligned} &i^{-1}\sum_i P_{12}\dots P_{i-2i-1}\,\mathrm{tr}_a[P_{i+1a}M'_{ia}P_{i-1a}]P_{i+1i+2}\dots P_{L-1L}P_{LL-1}\dots P_{21}\\ &=i^{-1}\sum_i P_{12}\dots P_{i-2i-1}\,\mathrm{tr}_a[M'_{ii+1}P_{i+1i-1}P_{i+1a}]P_{i+1i}\dots P_{21}\end{aligned} \quad , \tag{122}$$

where we used the main property of the permutation operator to manipulate the indices inside the trace. We can move M'_{ii+1} to the left as it doesn't share any indices with the permutation operators to the left of it,

$$=i^{-1}\sum_i M'_{ii+1}P_{12}\dots P_{i-2i-1}P_{i+1i-1}P_{i+1i}\dots P_{21} \quad . \tag{123}$$

Now we just need to deal with the string of permutation operators,

$$\begin{aligned} P_{12}\dots P_{i-2i-1}P_{i+1i-1}P_{i+1i}\dots P_{21}&=P_{12}\dots[P_{i-2i-1}P_{ii+1}P_{ii-1}]P_{ii-1}\dots P_{21}\\ &=P_{ii+1}[P_{12}\dots P_{i-2i-1}P_{i-1i-2}\dots P_{21}]\\ &=P_{ii+1}\end{aligned} \quad , \tag{124}$$

so finally we can write (119) as

$$i^{-1}\sum_i \mathrm{tr}_a\left[P_{La}\dots P_{i+1a}M'_{ia}P_{i-1a}\dots P_{1a}\right]P_{LL-1}\dots P_{21}=i^{-1}\sum_i M'_{ii+1}(i/2)P_{ii+1} \quad . \tag{125}$$

One interesting observation is that this term contains only local interactions.

The remaining terms in (118) still need to be evaluated. Taking into account that $L'_{la}(\lambda) = I_l$ at any value of λ, we have

$$(L'_{La})\, L_{L-1a} \dots M_{ia} \dots L_{1a} + \dots = \sum_{j \neq i} L_{La} \dots L_{j+1a} L_{j-1a} \dots M_{ia} \dots L_{1a} \quad , \tag{126}$$

where the j^{th} Lax operator is omitted in each term in the sum. Now taking the trace over auxiliary space and multiplying by $F^{-1}(i/2)$ as before we get

$$i^{-2} \sum_{j \neq i} \mathrm{tr}_a [P_{La} \dots P_{j+1a} P_{j-1a} \dots M_{ia} \dots P_{1a}] P_{LL-1} \dots P_{21} \quad , \tag{127}$$

and similarly this simplifies down to

$$= i^{-2} M_{ii+1}(i/2) \sum_{j \neq i} P_{jj+1} P_{ii+1} \quad . \tag{128}$$

Since $P^2_{ii+1} = I$, we can add and subtract it to rewrite the sum,

$$i^{-2} M_{ii+1}(i/2) \left(\sum_{j \neq i} P_{jj+1} \right) P_{ii+1} = i^{-2} M_{ii+1}(i/2) \left(\sum_{j} P_{jj+1} P_{ii+1} - I \right) \quad , \tag{129}$$

and interestingly, this term contains generically non-local interactions. Finally, we have the deformed Hamiltonian,

$$\widetilde{H} = H_{\text{XXX}} + \frac{\epsilon}{2} \sum_i \left[M'_{ii+1}(i/2) P_{ii+1} - i M_{ii+1}(i/2) \left(\sum_j P_{jj+1} P_{ii+1} - I \right) \right] + \mathcal{O}(\epsilon^2) \quad . \tag{130}$$

What is interesting is that the deformation is split into local and generically non-local terms, the local terms depending on $M'(i/2)$ and the non-local terms depending on $M(i/2)$.

Since (130) is for a general deformation M, it is not necessarily integrable. Next we are faced with the task of finding the deformations that are integrable as well as the ones that are integrable in the leading order. We can then specify the entirely local and non-local deformations by requiring that $M(i/2) = 0$ and $M'(i/2) = 0$, respectively. We can also construct the eigenstates, leading to a set of deformed BAE.

4.2 FCR in the Deformed Case

We can search deformations that do not break integrability up to a certain order, that is,

$$[\widetilde{F}(\lambda), \widetilde{F}(\mu)] = 0 + \mathcal{O}(\epsilon^2) \quad . \tag{131}$$

This will also include deformations that are integrable. The simplest way to obtain operators that commute is to start from the ground up; if the deformed Lax operators satisfy some RLL relation, then we can show that the matrices also satisfy some RTT relation, and so their traces will commute (all up to $\mathcal{O}(\epsilon^2)$ of course). The new RLL relation looks like

$$\begin{aligned} R_{a_1a_2}(\lambda-\mu)\,[L_{na_1}(\lambda)+\epsilon M_{la_2}(\lambda)]\,[L_{la_2}(\mu)+\epsilon M_{la_2}(\mu)] = \\ [L_{la_2}(\mu)+\epsilon M_{la_2}(\mu)]\,[L_{la_1}(\lambda)+\epsilon M_{la_2}(\lambda)]\,R_{a_1a_2}(\lambda-\mu) \end{aligned} \quad . \tag{132}$$

Expanding this, we get

$$\begin{aligned} R_{a_1a_2}(\lambda-\mu)\,[L_{la_1}(\lambda)M_{la_2}(\mu)+M_{la_2}(\lambda)L_{la_2}(\mu)] = \\ [M_{la_2}(\mu)L_{la_1}(\lambda)+L_{la_2}(\mu)M_{la_2}(\lambda)]\,R_{a_1a_2}(\lambda-\mu)+\mathcal{O}(\epsilon^2) \end{aligned} \quad , \tag{133}$$

where the first terms from each side satisfy the original RLL relation (38). We must find the M that satisfy this relation. Here we sketch the method for obtaining all of the representations required to solve (133). For full details on solving (133) in Mathematica, see Appendix B.

4.2.1 Explicit Computation

Since the terms in (133) are defined on $V_1 \otimes V_2 \otimes h_l$, we can use matrix representations of the operators, and the problem reduces to a $2^3 \times 2^3 = 8 \times 8$ matrix equation. In theory we have 32 parameters (each deformation has 16 unique entries, and there are two of them depending on different spectral parameters) and 64 constraints. So our system is overdefined, however there is a large number of spurious constraints and we will get non-trivial solutions.

Each operator in the equation acts on different products of two of the three spaces in question. Thus, getting the matrix representations correct requires precise consideration; we need a systematic way of defining the operators on $V_1 \otimes V_2 \otimes h_l$. Recall that the Yang-Baxter equation takes the form

$$R_{12}(\lambda-\mu)R_{13}(\lambda)R_{23}(\mu) = R_{23}(\mu)R_{13}(\lambda)R_{12}(\lambda-\mu) \quad , \tag{134}$$

where we use a simpler notation for the subscripts: $R_{12} \equiv R_{a_1a_2}$, etc. In our case, subscripts 1 and 2 refer to the auxiliary spaces and subscript 3 refers to the quantum space. As discussed in the section on integrability our solution to this is given by

$$R(\lambda) = \lambda I + iP \quad . \tag{135}$$

R will be given explicitly by (65) as we have seen before, but it must also act trivially in the quantum space,

$$R_{12}(\lambda) \equiv R(\lambda) \otimes I = \left[\begin{array}{cc|cc} \lambda+i & & & \\ & \lambda & i & \\ \hline & i & \lambda & \\ & & & \lambda+i \end{array}\right] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc|cc} \lambda+i & & & & & \\ & \lambda+i & & & & \\ \hline & & \lambda & i & & \\ & & & \lambda & i & \\ \hline & & i & & \lambda & \\ & & & i & & \lambda \\ \hline & & & & \lambda+i & \\ & & & & & \lambda+i \end{array}\right] \quad . \tag{136}$$

Similarly,

$$R_{23}(\lambda) \equiv I \otimes R(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left[\begin{array}{cc|cc} \lambda+i & & & \\ & \lambda & i & \\ \hline & i & \lambda & \\ & & & \lambda+i \end{array}\right]$$

$$= \left[\begin{array}{c|cc|cc} \lambda+i & & & & \\ & \lambda & i & & \\ \hline & i & \lambda & & \\ & & & \lambda+i & \\ \hline & & & & \lambda+i \\ & & & & \\ \hline & & & & \lambda+i \\ & & & & \end{array}\right] \quad . \tag{137}$$

The representation of $R_{13}(\lambda)$ is not so straightforward to get since the space in which it acts trivially is sandwiched between the other two. We can make use of the fact that

$$R_{13}(\lambda) = P_{23} R_{12}(\lambda) P_{23} \quad . \tag{138}$$

Of course, we must first define the Permutation operator correctly,

$$P_{23} \equiv I \otimes P \quad . \tag{139}$$

Then we get

$$R_{23}(\lambda) = P_{23}R_{12}(\lambda)P_{23}$$

$$= \left[\begin{array}{cc|cc} \lambda+i & & & \\ & \lambda & i & \\ \hline & \lambda+i & & \\ & & \lambda & i \\ \hline & i & \lambda & \\ & & & \lambda+i \\ \hline & & \lambda & \\ & i & & \lambda+i \end{array}\right] , \tag{140}$$

and finally we have the correct representations of our operators acting on $V_1 \otimes V_2 \otimes h_n$. We obtain the RLL relation by setting

$$\begin{array}{l} \lambda \to \lambda - i/2 \\ \mu \to \mu - i/2 \end{array} . \tag{141}$$

We have previously shown that this relation holds for our spin chain, and we can now also show that it holds by explicit matrix multiplication. With the help of Mathematica, this is shown to be true. This is a good sanity check that our matrix representations are correct.

Now that we have a systematic method for obtaining matrix representations on $V_1 \otimes V_2 \otimes h_l$, we can write the representations of M and proceed to solve (133). Starting with the representation of M on the space $\mathbb{C}^2 \otimes \mathbb{C}^2$, let

$$M(\lambda) = \left[\begin{array}{cc|cc} a(\lambda) & b(\lambda) & e(\lambda) & f(\lambda) \\ c(\lambda) & d(\lambda) & g(\lambda) & h(\lambda) \\ \hline i(\lambda) & j(\lambda) & m(\lambda) & n(\lambda) \\ k(\lambda) & l(\lambda) & o(\lambda) & p(\lambda) \end{array}\right] , \tag{142}$$

where each 4 successive letters are the entries of a 2×2 block representing a quantum operator

which depends on λ. As above, we can easily define $M_{12}(\lambda)$ and $M_{23}(\lambda)$,

$$M_{12}(\lambda) = M(\lambda) \otimes I = \left[\begin{array}{cc|cc} a(\lambda) & b(\lambda) & e(\lambda) & f(\lambda) \\ c(\lambda) & d(\lambda) & g(\lambda) & h(\lambda) \\ \hline i(\lambda) & j(\lambda) & m(\lambda) & n(\lambda) \\ k(\lambda) & l(\lambda) & o(\lambda) & p(\lambda) \end{array}\right] \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \left[\begin{array}{cc|cc|cc|cc} a(\lambda) & & b(\lambda) & & e(\lambda) & & f(\lambda) & \\ & a(\lambda) & & b(\lambda) & & e(\lambda) & & f(\lambda) \\ \hline c(\lambda) & & d(\lambda) & & g(\lambda) & & h(\lambda) & \\ & c(\lambda) & & d(\lambda) & & g(\lambda) & & h(\lambda) \\ \hline i(\lambda) & & j(\lambda) & & m(\lambda) & & n(\lambda) & \\ & i(\lambda) & & j(\lambda) & & m(\lambda) & & n(\lambda) \\ \hline k(\lambda) & & l(\lambda) & & o(\lambda) & & p(\lambda) & \\ & k(\lambda) & & l(\lambda) & & o(\lambda) & & p(\lambda) \end{array}\right], \tag{143}$$

$$M_{23}(\lambda) = I \otimes M(\lambda) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \left[\begin{array}{cc|cc} a(\lambda) & b(\lambda) & e(\lambda) & f(\lambda) \\ c(\lambda) & d(\lambda) & g(\lambda) & h(\lambda) \\ \hline i(\lambda) & j(\lambda) & m(\lambda) & n(\lambda) \\ k(\lambda) & l(\lambda) & o(\lambda) & p(\lambda) \end{array}\right]$$

$$= \left[\begin{array}{cc|cc|cc|cc} a(\lambda) & b(\lambda) & e(\lambda) & f(\lambda) & & & & \\ c(\lambda) & d(\lambda) & g(\lambda) & h(\lambda) & & & & \\ \hline i(\lambda) & j(\lambda) & m(\lambda) & n(\lambda) & & & & \\ k(\lambda) & l(\lambda) & o(\lambda) & p(\lambda) & & & & \\ \hline & & & & a(\lambda) & b(\lambda) & e(\lambda) & f(\lambda) \\ & & & & c(\lambda) & d(\lambda) & g(\lambda) & h(\lambda) \\ \hline & & & & i(\lambda) & j(\lambda) & m(\lambda) & n(\lambda) \\ & & & & k(\lambda) & l(\lambda) & o(\lambda) & p(\lambda) \end{array}\right]. \tag{144}$$

Again, we use the same trick to obtain $M_{13}(\lambda)$,

$$M_{13}(\lambda) = P_{23} M_{12}(\lambda) P_{23}$$

$$= \left[\begin{array}{cc|cc|cc|cc} a(\lambda) & b(\lambda) & & & e(\lambda) & f(\lambda) & & \\ c(\lambda) & d(\lambda) & & & g(\lambda) & h(\lambda) & & \\ \hline & & a(\lambda) & b(\lambda) & & & e(\lambda) & f(\lambda) \\ & & c(\lambda) & d(\lambda) & & & g(\lambda) & h(\lambda) \\ \hline i(\lambda) & j(\lambda) & & & m(\lambda) & n(\lambda) & & \\ k(\lambda) & l(\lambda) & & & o(\lambda) & p(\lambda) & & \\ \hline & & i(\lambda) & j(\lambda) & & & m(\lambda) & n(\lambda) \\ & & k(\lambda) & l(\lambda) & & & o(\lambda) & p(\lambda) \end{array}\right], \tag{145}$$

and finally we have all of the ingredients to write (133) in matrix form. As mentioned previously, the system seems grossly overdefined. Remarkably though, (133) allows for deformations that depend on 8 parameters.

Note that $M(\lambda)$ and $M(\mu)$ are of the same form. This is because the deformations in (133) do not necessarily have to be the same. However, if we introduce the same deformation at each lattice site, then an $R\widetilde{T}\widetilde{T}$ relation would immediately follow if (133) is satisfied.

Using Mathematica, the set of M that satisfy (133) are of the form

$$M(\lambda) = \left[\begin{array}{cc|cc} g(\lambda+i/2)+y(\lambda+i/2)+z & ib & b(\lambda-i/2)+n(\lambda+i/2) & 0 \\ ic & g(\lambda-i/2)+y(\lambda-i/2)+z & i(g+x) & b(\lambda+i/2)+n(\lambda-i/2) \\ \hline c(\lambda-i/2)+o(\lambda+i/2) & i(g-x) & g(\lambda-i/2)-y(\lambda-i/2)+z & in \\ 0 & c(\lambda+i/2)+o(\lambda-i/2) & io & g(\lambda+i/2)-y(\lambda+i/2)+z \end{array}\right] \quad , \tag{146}$$

for 8 complex parameters. Here we made the assumption that M depends linearly on λ. Notice that this class of deformations contains both the identity matrix and the Lax operator by setting $z = 1$ or $g = 1$ respectively, and then setting the rest of the parameters to zero. This is a good sanity check since the case where $M \propto L$ should definitely be integrable. The case where $M \propto I$ should also be integrable, although it is less obvious. This will result in inhomogeneous spin chains; ones with Lax operators of the form

$$\widetilde{L}_{la}(\lambda) = (\lambda + \theta_l - i/2) + iP \quad . \tag{147}$$

The RLL relation for the inhomogeneous spin chain can be shown to hold either by straightforward calculation or by using the matrix representations, and hence it is also integrable.

In operator form, we can write $M(\lambda)$ as a 2×2 matrix,

$$M(\lambda) = \left[\begin{array}{c|c} i(bS^+ + cS^-) + (\lambda(g+y)+z)I & i(g+x)S^- + \lambda(b+n)I \\ +\frac{i}{2}(g+y)S^3 & +\frac{i}{2}(b-n)S^3 \\ \hline i(g-x)S^+ + \lambda(c+o)I & i(nS^+ + oS^-) + (\lambda(g-y)+z)I \\ +\frac{i}{2}(c-o)S^3 & -\frac{i}{2}(g-y)S^3 \end{array}\right] \quad , \tag{148}$$

which highlights the dependence on the generators $S^{\pm}$ and S^3.

Recall that the local terms in (130) depend on $M'(i/2)$ whereas the non-local terms depend on $M(i/2)$. If we examine (146), the only way to get $M(i/2) = 0$ is to set all of the parameters to zero. It would seem that our class of deformations doesn't include any purely local ones. This is not good because we showed previously that it includes the rescaled Lax operator, which does result in a local deformation. However there is one local term in the generically non-local part of

the perturbative Hamiltonian. When $M_{ii+1}(i/2) \propto P_{ii+1}$, which is the case when $M \propto L$, we have $P_{ii+1}^2 = 1$ in the 3$^{\text{rd}}$ part of (130), leaving a term that is local. So the local deformation where the Lax operator is rescaled is included in this class, but unfortunately it is the only one. Therefore this isn't the most general deformation we could have made; assuming that M depends linearly on λ is too restrictive, as is only deforming the Lax operators while keeping R fixed. This motivates the deformation of both L and R, and increasing the powers of λ in the deformations.

Looking at deformations with $M'(i/2) = 0$, we see that they are of the form

$$M(\lambda) = \left[\begin{array}{cc|cc} z & ib & -ib & 0 \\ ic & z & ix & ib \\ \hline -ic & -ix & z & -ib \\ 0 & ic & -ic & z \end{array}\right] \quad , \tag{149}$$

i.e. they are constant and depend on 4 parameters.

To recap, we have found a class deformations which is linear in λ that give commuting charges in the linear order. The only local deformation in this class is the rescaled Lax operator. We will improve on this by also deforming R and by increasing the powers of λ in the deformations. Before we do this it is worth applying the Bethe ansatz method to our model and constructing a set of deformed eigenstates.

4.3 BAE in the Deformed Case

If we write $\widetilde{T}(\lambda)$ in a similar form to (66),

$$\widetilde{T}(\lambda) = \begin{bmatrix} \widetilde{A}(\lambda) & \widetilde{B}(\lambda) \\ \widetilde{C}(\lambda) & \widetilde{D}(\lambda) \end{bmatrix} \quad , \tag{150}$$

where the entries are much more complicated in this case. In the previous section we showed that

$$R\widetilde{T}\widetilde{T} = \widetilde{T}\widetilde{T}R + \mathcal{O}(\epsilon^2) \quad , \tag{151}$$

where the R is the same as before, so we will have similar commutation relations for the entries of $\widetilde{T}(\lambda)$,

$$\begin{aligned} [\widetilde{B}(\lambda), \widetilde{B}(\mu)] &= 0 \\ \widetilde{A}(\lambda)\widetilde{B}(\mu) &= f(\lambda-\mu)\widetilde{B}(\mu)\widetilde{A}(\lambda) + g(\lambda-\mu)\widetilde{B}(\lambda)\widetilde{A}(\mu) \quad , \\ \widetilde{D}(\lambda)\widetilde{B}(\mu) &= h(\lambda-\mu)\widetilde{B}(\mu)\widetilde{D}(\lambda) + k(\lambda-\mu)\widetilde{B}(\lambda)\widetilde{D}(\mu) \end{aligned} \tag{152}$$

where f, g, h and k are again given by (71). If we also deform R, these commutation relations become much more complicated. First we must find a reference state, $|\Omega'\rangle$, such that

$$\widetilde{C}\,|\Omega'\rangle = 0 \quad . \tag{153}$$

We won't make the assumption that it is the same as the previous reference state, but we will assume that it can be written in the form

$$|\Omega'\rangle = |\omega_1'\rangle \otimes \ldots \otimes |\omega_L'\rangle \quad , \tag{154}$$

where we take each $|\omega_l'\rangle$ to be a general vector,

$$|\omega_l'\rangle = \begin{bmatrix} \rho \\ \sigma \end{bmatrix} \quad . \tag{155}$$

To satisfy (153), we can look at the action of the bottom left entry of each $M(\lambda)$ on each $|\omega_l'\rangle$. From (148) we have

$$\left[i(g-x)S^+ + \lambda(c+o)I + \frac{i}{2}(c-o)S^3\right]\begin{bmatrix} \rho \\ \sigma \end{bmatrix} = \begin{bmatrix} \lambda(c+o) + \frac{i}{2}(c-o)\rho \\ i(g-x)\sigma + \lambda(c+o) - \frac{i}{2}(c-o)\sigma \end{bmatrix} \quad , \tag{156}$$

and we want this to vanish for each l and for every value of λ. Since the deformation is the same at each site on the chain, if it works in this case, it will be true for each l. If we set $c = o = 0$, then (153) will actually be satisfied by the reference state $|\Omega\rangle$ for the undeformed chain. This makes life easier, but comes at the cost of fixing two parameters. Now we look at the action of the diagonal entries of $M(\lambda)$ on our reference state. We have

$$\left[ibS^+ + (\lambda(g+y)+z)I + \frac{i}{2}(g+y)S^3\right]|\omega_l\rangle = [(g+y)(\lambda+i/2)+z]\,|\omega_l\rangle \tag{157}$$

and

$$\left[inS^+ + (\lambda(g-y)+z)I - \frac{i}{2}(g-y)S^3\right]|\omega_l\rangle = [(g-y)(\lambda - i/2)+z]\,|\omega_l\rangle \quad , \tag{158}$$

since $S^+|\omega_l\rangle = 0$. Hence we have deformed versions of (77),

$$\widetilde{C}(\lambda)\,|\Omega\rangle = 0 \qquad \widetilde{A}(\lambda)\,|\Omega\rangle = \widetilde{\alpha}^L(\lambda)\,|\Omega\rangle \qquad \widetilde{D}(\lambda)\,|\Omega\rangle = \widetilde{\delta}^L(\lambda)\,|\Omega\rangle \quad , \tag{159}$$

where

$$\begin{aligned} \widetilde{\alpha}(\lambda) &= \alpha(\lambda) + \epsilon\,[(g+y)\alpha(\lambda)+z] \\ \widetilde{\delta}(\lambda) &= \delta(\lambda) + \epsilon\,[(g-y)\delta(\lambda)+z] \end{aligned} \quad , \tag{160}$$

for the same reference state $|\Omega\rangle$. Hence $|\Omega\rangle$ is an eigenstate of $\widetilde{F}(\lambda)$ with eigenvalue $\widetilde{\alpha}^L(\lambda) + \widetilde{\delta}^L(\lambda)$. Since the commutation relations are of the same form as before we proceed to construct the rest of the eigenstates in a similar fashion, the difference being the different explicit forms of the entries of $\widetilde{T}$. We make the ansatz that the eigenstates are of the form

$$|\Phi'\{\lambda\}\rangle = \widetilde{B}(\lambda_1)\ldots\widetilde{B}(\lambda_m)\,|\Omega\rangle \quad , \tag{161}$$

for parameters $\{\lambda = \lambda_1 \ldots \lambda_m\}$. Then $|\Phi'\rangle$ is an eigenstate of $\widetilde{F}(\lambda)$ with eigenvalue

$$\widetilde{\Lambda}(\lambda,\{\lambda\}) = \widetilde{\alpha}^L(\lambda)\left(\prod_{k=1}^{m} f(\lambda-\lambda_k)\right) + \widetilde{\delta}^L(\lambda)\left(\prod_{k=1}^{m} h(\lambda-\lambda_k)\right) \tag{162}$$

if the following BAE are satisfied,

$$\left(\frac{\lambda_k + i/2 + \epsilon\,[(g+y)(\lambda_k+i/2)+z]}{\lambda_k - i/2 + \epsilon\,[(g-y)(\lambda_k-i/2)+z]}\right)^L = \prod_{j\neq k}^{m}\frac{\lambda_k-\lambda_j+i}{\lambda_k-\lambda_j-i} \quad . \tag{163}$$

Only the LHS is deformed, which makes sense because we only deformed L. The RHS is determined by the entries of R, so deforming R will lead to deformations in the RHS.

4.4 Deforming the R Matrix

The results of the previous two sections suggest that both L and R should be deformed in order to obtain the most general deformations possible. We expect this will lead to more local deformations to the Hamiltonian as well as deformations to the interactions described in the BAE. We introduce the deformation to R in the same way as before,

$$R_{la} \to \widetilde{R}_{la} = R_{la} + \epsilon N_{la}(\lambda) \quad , \tag{164}$$

and the calculations from the previous sections must be repeated, accounting for the additional deformation $N(\lambda)$. First we need to find deformations that result in commuting transfer matrices up to linear order. These are determined by a FCR which is similar to (133), although more complicated,

$$\begin{aligned} &N_{a1a2}(\lambda-\mu)L_{la_1}(\lambda)L_{la_2}(\mu) + R_{a_1a_2}(\lambda-\mu)\left[L_{la_1}(\lambda)M_{la_2}(\mu) + M_{la_2}(\lambda)L_{la_2}(\mu)\right] = \\ &L_{la_2}(\mu)L_{la_1}(\lambda)N_{a1a2}(\lambda-\mu) + \left[M_{la_2}(\mu)L_{la_1}(\lambda) + L_{la_2}(\mu)M_{la_2}(\lambda)\right]R_{a_1a_2}(\lambda-\mu) + \mathcal{O}(\epsilon^2) \end{aligned} \quad . \tag{165}$$

This is solved using the methods described in section 4.2.1. This case is considerably more complicated, and so more care is needed when solving with Mathematica. The previous case was simple enough to be solved using the **Solve** function, but there are too many variables in this case, and Mathematica will solve for the constants in terms of λ and μ. To avoid this, we note that each entry in (165) is a polynomial in λ and μ, and solve each order separately. The Mathematica code used to solve (165) is included in Appendix B. The solutions obtained are considerably more complicated,

$$M(\lambda) = \left[\begin{array}{cc|cc} b+a\lambda & c+d\lambda & -c+\frac{i}{2}d+q(\lambda+i/2) & 0 \\ e+f\lambda & g+h\lambda & j+k\lambda & c+\frac{i}{2}d+l(\lambda-i/2) \\ \hline -e+\frac{i}{2}f+m(\lambda+i/2) & -ia+2b-2g-j+in-k\lambda & g+\frac{i}{2}h+n(\lambda-i/2) & -c+d\lambda+q(\lambda+i/2)-l(\lambda-i/2) \\ 0 & e+\frac{i}{2}f+o(\lambda-i/2) & -e+f\lambda+m(\lambda+i/2)-o(\lambda-i/2) & b-a(\lambda+i)+(h+n)(\lambda+i/2) \end{array}\right] \quad , \tag{166}$$

$$N(\lambda) = \left[\begin{array}{cc|cc} b-g-\frac{i}{2}(a+h)+p(\lambda+i) & d\lambda & -d\lambda & 0 \\ f\lambda & (-a+h+p)\lambda & b-g-\frac{i}{2}(a+h)+ip+k\lambda & -(d+q-l)\lambda \\ \hline -f\lambda & b-g\frac{i}{2}(a+h)+ip-k\lambda & (a-h+p)\lambda & (d+q-l)\lambda \\ 0 & -(f+m-o)\lambda & (f+m-o)\lambda & b-g-\frac{i}{2}(a+h)+p(\lambda+i) \end{array}\right] \quad . \tag{167}$$

As the deformations get more complicated, we must be careful about how we count the number of free parameters in them. We will take the number of free parameters to be the number of conditions required to set the deformations to zero (which is not necessarily the same as sending all of the constants to zero), plus 1 for the rescaling of the Lax operator and R matrix. In this case, we get 16 free parameters, which is twice as many as before. To count the number of free parameters in

the local deformations, we apply the conditions required to set $M(i/2) = 0$ and then we count the number of conditions required to set $M'(i/2) = 0$. Again we add 1 to account for rescaling. There are 8 free parameters in the local deformations, which is a good improvement on the previous case,

$$M(\lambda) = (\lambda - i/2) \left[\begin{array}{cc|cc} a & ic & -ic & 0 \\ ie & ig & ij & l \\ \hline -ie & -ij & ig - 2a & -l \\ 0 & o & -o & a \end{array}\right] \quad . \tag{168}$$

We can do better by increasing the order of the λ dependence. After solving the FCR where the λ dependence goes up to quadratic order, the deformations obtained were found to depend on 22 parameters, and the local deformations were found to depend on 14, which is a significant improvement. When the dependence was increased to cubic, there were 23 parameters and still 14 parameters for the local deformations. When the λ dependence was increased to quartic, the number of free parameters increased by 1 again, and the number of free parameters in the local deformations remained at 14. It appears that 14 free parameters in the local deformations is the best we can do.

The R matrix for the XXZ model is

$$R_{\mathrm{XXZ}}(\lambda) = \left[\begin{array}{cc|cc} \sinh(\eta(\lambda+i)) & & & \\ & \sinh(\eta\lambda) & i\sin(\eta) & \\ \hline & i\sin(\eta) & \sinh(\eta\lambda) & \\ & & & \sinh(\eta(\lambda+i)) \end{array}\right] \quad , \tag{169}$$

where $\Delta = \cosh(\eta)$ in (31). Expanding for small η gives

$$R_{\mathrm{XXZ}}(\lambda) - \eta R(\lambda) \approx \frac{\eta^3}{6} \left[\begin{array}{cc|cc} (\lambda+i)^3 & & & \\ & \lambda^3 & -i & \\ \hline & -i & \lambda^3 & \\ & & & (\lambda+i)^3 \end{array}\right] \quad , \tag{170}$$

which is contained in the deformations with cubic λ dependence. Hence, the cubic deformations include the XXZ model, which is a good sanity check. The general form of these deformations is not presented here, as they become significantly more complicated when the powers of λ increase. The main point of this illustration is that our methods give sensible results, which is a good sign.

4.4.1 BAE

We proceed to construct the eigenstates and get a new set of BAE. One problem that arises from deforming R is that it seriously complicates the commutation relations that we used to obtain the BAE. To make the problem solvable using the same approach as before we fix some of the parameters so that $\widetilde{R}$ has the same form as R. By setting $\widetilde{i} = l$, $h = a$, and d, f, k and m to zero, we get

$$\widetilde{R}(\lambda) = \left[\begin{array}{cc|cc} \widetilde{a}(\lambda) & & & \\ & \widetilde{b}(\lambda) & \widetilde{c}(\lambda) & \\ \hline & \widetilde{c}(\lambda) & \widetilde{b}(\lambda) & \\ & & & \widetilde{a}(\lambda) \end{array}\right] \quad , \tag{171}$$

where

$$\begin{aligned} \widetilde{a}(\lambda) &= \lambda + i + \epsilon(b - g - ia + p(\lambda + i)) \\ \widetilde{b}(\lambda) &= \lambda + \epsilon(\lambda p) \\ \widetilde{c}(\lambda) &= i + \epsilon(b - g - ia + ip) \end{aligned} \quad . \tag{172}$$

This gives us a set of commutation relations that have the same form as before,

$$\begin{aligned} [\widetilde{B}(\lambda), \widetilde{B}(\mu)] &= 0 \\ \widetilde{A}(\lambda)\widetilde{B}(\mu) &= \widetilde{f}(\lambda - \mu)\widetilde{B}(\mu)\widetilde{A}(\lambda) + \widetilde{g}(\lambda - \mu)\widetilde{B}(\lambda)\widetilde{A}(\mu) \\ \widetilde{D}(\lambda)\widetilde{B}(\mu) &= \widetilde{h}(\lambda - \mu)\widetilde{B}(\mu)\widetilde{D}(\lambda) + \widetilde{k}(\lambda - \mu)\widetilde{B}(\lambda)\widetilde{D}(\mu) \end{aligned} \quad , \tag{173}$$

where $\widetilde{A}$, $\widetilde{B}$ and $\widetilde{D}$ are entries of $\widetilde{T}$, which is not the same as the previous case, and

$$\begin{aligned} \widetilde{f}(\lambda) &= \frac{\widetilde{a}(-\lambda)}{\widetilde{b}(-\lambda)} \qquad \widetilde{g}(\lambda) = -\frac{\widetilde{c}(-\lambda)}{\widetilde{b}(-\lambda)} \\ \widetilde{h}(\lambda) &= \frac{\widetilde{a}(\lambda)}{\widetilde{b}(\lambda)} \qquad \widetilde{k}(\lambda) = -\frac{\widetilde{c}(\lambda)}{\widetilde{b}(\lambda)} \end{aligned} \quad . \tag{174}$$

The rest of the derivation is the same as before. First, we must set $e = o = 0$ so we can use the reference state $|\Omega\rangle$,

$$\widetilde{C}(\lambda)\,|\Omega\rangle = 0 \quad . \tag{175}$$

We also have

$$\widetilde{A}(\lambda)\,|\Omega\rangle = \widetilde{\alpha}^L(\lambda)\,|\Omega\rangle \qquad \widetilde{D}(\lambda)\,|\Omega\rangle = \widetilde{\delta}^L(\lambda)\,|\Omega\rangle \quad , \tag{176}$$

where

$$\begin{aligned} \widetilde{\alpha}(\lambda) &= \alpha(\lambda) + \epsilon\,[a\lambda + b] \\ \widetilde{\delta}(\lambda) &= \delta(\lambda) + \epsilon\left[g + \frac{i}{2}a + n(\lambda - i/2)\right] \end{aligned} \quad . \tag{177}$$

If we construct the eigenstates

$$|\Phi'\{\lambda\}\rangle = \widetilde{B}(\lambda_1)\ldots\widetilde{B}(\lambda_m)\,|\Omega\rangle \quad , \tag{178}$$

then again $|\Phi'\rangle$ is an eigenstate of $\widetilde{F}(\lambda)$ with eigenvalue

$$\widetilde{\Lambda}(\lambda,\{\lambda\}) = \widetilde{\alpha}^L(\lambda)\left(\prod_{k=1}^{m}\widetilde{f}(\lambda-\lambda_k)\right) + \widetilde{\delta}^L(\lambda)\left(\prod_{k=1}^{m}\widetilde{h}(\lambda-\lambda_k)\right) \tag{179}$$

if a new set of BAE are satisfied,

$$\left(\frac{\widetilde{\alpha}(\lambda_k)}{\widetilde{\delta}(\lambda_k)}\right)^L = \prod_{j\neq k}^{m}\frac{\widetilde{h}(\lambda_k-\lambda_j)}{\widetilde{f}(\lambda_k-\lambda_j)} \quad . \tag{180}$$

This is given explicitly as

$$\left(\frac{\lambda_k+i/2+\epsilon\left[a\lambda_k+b\right]}{\lambda_k-i/2+\epsilon[g+\frac{i}{2}a+n(\lambda_k-i/2)]}\right)^L = \prod_{j\neq k}^{m}\frac{\lambda_k-\lambda_j+i+\epsilon\left[b-g-ia+p(\lambda_k-\lambda_j+i)\right]}{\lambda_k-\lambda_j-i-\epsilon\left[b-g-ia-p(\lambda_k-\lambda_j-i)\right]} \quad . \tag{181}$$

These equations are not as symmetric as the previous ones, but this is expected as the deformations weren't written in as symmetric a form. While it appears that these equations include deformations to the magnon interactions, they can be reduced to a form similar to (163) by clever relabelling of the parameters. This will not happen with higher powers of λ in the deformations. Let us take XXZ as an example. The approximate BAE are given by

$$\frac{\widetilde{\alpha}^L(\lambda_k)}{\widetilde{\delta}^L(\lambda_k)} = \prod_{j\neq k}^{m}\frac{6\eta(\lambda_k-\lambda_j+i+\epsilon\eta^3(\lambda_k-\lambda_j+i)^3}{6\eta(\lambda_k-\lambda_j-i+\epsilon\eta^3(\lambda_k-\lambda_j-i)^3} \quad , \tag{182}$$

where $\widetilde{\alpha}(\lambda)$ and $\widetilde{\delta}(\lambda)$ are also cubic in λ. Their explicit form is irrelevant for this argument; these equations have deformations to the RHS which can't be removed by relabelling the parameters.

5 Discussions and Conclusions

A discussion on integrability in physics was provided, and the algebraic Bethe ansatz method for solving integrable systems was presented. The general leading order Hamiltonian for deformations to the Lax operators and R matrices was obtained; the perturbative parts were split into local and generically non-local parts. The deformed FCR was solved to leading order, and the deformations that preserve integrability to leading order were thus obtained. The deformations produced sensible results, as they were found to include the case of rescaling the Lax operator, the inhomogeneous spin chain, and the XXZ spin chain.

For systems that are integrable to leading order, it was found that just making a linear (in the spectral parameter) deformation to the Lax operators was too restrictive. When both the Lax operator and R matrix were deformed linearly, the local deformations were found to depend on 8 free parameters. This was increased to 14 when the dependence on the spectral parameter in the deformations was increased to quadratic order. This was not improved any further by adding powers of the spectral parameter to the deformations, up to order 5. We expected that generically, the most general local deformation could depend on 16 parameters. One possible reason we could only obtain 14 is that we started from the XXX spin chain, which is highly degenerate. Perhaps making deformations to a more general model, such as the XYZ spin chain would improve upon this. A summary of the results obtained for the various types of deformations studied is provided in the table below.

Deformation	λ Dependence	Parameters	Local
L	λ	8	1
L, R	λ	16	8
L, R	λ^2	22	14
L, R	λ^3	23	14
L, R	λ^4	24	14
L, R	λ^5	25	14

Finally, it was shown that for some simple cases, i.e. when the standard reference state can be used and when the R matrix has the same form as the undeformed one, that the eigenstates of the deformed system can be constructed, and a set of leading order deformed Bethe ansatz equations can be obtained. For linear deformations, they can be reduced to a case where only the LHS is deformed. For higher order deformations, they contained deformations on both sides, i.e. the side which describes transport around the chain and the side which describes magnon interaction are

both deformed.

In conclusion, given a small deformation to the Lax operators and R matrices in the XXX spin chain, the leading order perturbed Hamiltonian can be obtained, and in some simple cases a set of eigenstates can be constructed, subject to a set of deformed Bethe ansatz equations. It was hoped in this project that some similar aspects of the KAM theorem could be observed in quantum systems. While it was not the aim to define a quantum KAM theorem, we made small deformations to a quantum system and obtained sensible results, which is a step in the right direction.

5.1 Further Research

There are many different directions this project could take in terms of further research. Continuing along the same track, one could check that other deformations are reproduced, such as the XYZ spin chain. The deformed BAE could be solved, either numerically or explicitly for a small number of magnons. It would be interesting to take the thermodynamic limit and investigate if there are differences in thermodynamic properties between the XXX spin chain and one with non-integrable perturbations. It may also be interesting to study the relation between spin chains and 2d statistical models when deformations are introduced.

Another interesting aspect worth looking into is introducing different types of deformations to the chain, perhaps by deforming the monodromy matrices or transfer matrices. The method of deformation treated in this project generally lead to both local and non-local interactions. Deforming the monodromy matrices may provide another way of introducing long-range interactions to the chain; they are typically introduced through the Hamiltonian. Spin chains with long-range interactions aren't typically discussed in this context (see [13], for example), so it could be interesting to deform the monodromy matrix to introduce long-range interactions, and then obtain a set of BAE.

Of course, it would also be interesting to apply these methods to other models. Perhaps by starting from a less degenerate model, such as the XYZ spin chain, a more general class of deformations could be obtained. Other 1d models such as the Hubbard model or the Bose gas (Lieb-Linger model) could also be studied. If these methods can be successfully applied to other systems, it may be possible to make a more general and stronger statement about non-integrable perturbations to quantum integrable models.

Appendix A: Lie Algebras Review

Lie groups and Lie algebras are essential for describing our systems [14]. Lie groups were first studied by the Norwegian mathematician Sophus Lie around the end of the 19^{th} century. A Lie group G is a set with a group structure that is also a smooth manifold (rigorous definitions are not presented here, nor are the basic definitions i.e. of groups, manifolds, etc.). Informally, it is a group of continuous symmetries. This is very important since generally the only problems which are reasonably solvable are the ones with a high degree of symmetry.

Every system which has a set of continuous symmetries has an associated Lie Group. One of the simplest examples is S^2, the unit sphere in 3 dimensions. It is invariant under continuous rotations, and so the associated Lie group is $SO(3)$, the group of rotations in 3-dimensional space.

The Lie algebra of $\mathfrak{g}$ of G is equivalent to the tangent space at I, the identity:

$$\mathfrak{g} \simeq T_I G \quad . \tag{183}$$

Suppose we can parametrise group elements sufficiently close to the identity by N real parameters, α^a such that

$$g(\alpha)|_{\alpha=0} = I \tag{184}$$

Then given some g in an infinitesimal neighbourhood of the identity we can express it in terms of its Taylor expansion. We have

$$g(\alpha) = I + \alpha^a \frac{\partial}{\partial \alpha^a} g(\alpha)\bigg|_{\alpha=0} + \mathcal{O}(\alpha^2) \quad , \tag{185}$$

where the summation over repeated is implied unless otherwise specified. We usually write this as

$$g(d\alpha) = I + i\alpha^a T^a + \mathcal{O}(\alpha^2) \quad , \tag{186}$$

where

$$T^a \equiv -i \frac{\partial}{\partial \alpha^a} g(\alpha)\bigg|_{\alpha=0} \tag{187}$$

are the generators of the algebra, and the factor of i is included for convenience since if g is a unitary matrix, then T^a will be hermitian.

We can define a finite transformation, i.e. an element of G far from the identity, by making an infinite number of infinitesimal transformations:

$$g(\alpha) = \lim_{k\to\infty} \left(I + i\frac{\alpha^a}{k} T^a \right)^k \equiv e^{i\alpha^a T^a} \quad . \tag{188}$$

This is known as the exponential parametrisation. Since we are working with linear operators, the group multiplication is not so straightforward. Using (188), we have in general

$$e^{i\alpha^a T^a} e^{i\beta^b T^b} \neq e^{i(\alpha^a + \beta^a)Ta} \tag{189}$$

for two group elements generated by different linear combinations of generators. Since G is closed under group multiplication we can however, say that

$$e^{i\alpha^a T^a} e^{i\beta^b T^b} = e^{i\delta^a T^a} \tag{190}$$

for some parameters δ^a. Since everything is smooth, we can determine δ^a by expanding both sides of (190) and equating powers of α^a and β^a. Taking the natural logarithm of (190) gives

$$i\delta^a T^a = \ln\left(1 + \underbrace{e^{i\alpha^a T^a} e^{i\beta^b T^b} - 1}_{x}\right) \quad . \tag{191}$$

Expanding x gives

$$x = i\alpha^a T^a + i\beta^b T^b - \alpha^a T^a \beta^b T^b - \frac{1}{2}\left((\alpha^a T^a)^2 + (\beta^b T^b)^2\right) + \dots \quad , \tag{192}$$

and using this in the expansion of (191) we get

$$\begin{aligned} i\delta^a T^a &= x - \frac{1}{2}x^2 + \dots \\ &= i\alpha^a T^a + i\beta^b T^b - \alpha^a T^a \beta^b T^b \\ &\quad - \frac{1}{2}(\alpha^a T^a)^2 - \frac{1}{2}(\beta^b T^b)^2 + \dots \\ &\quad + \frac{1}{2}(\alpha^a T^a + \beta^b T^b)^2 + \dots \end{aligned} \quad . \tag{193}$$

The terms of quadratic order almost cancel, but since T^a and T^b don't commute, the cross terms will remain. We can write (190) in terms of the commutator of T^a and T^b though;

$$i\delta^a T^a = i\alpha^a T^a + i\beta^b T^b - \frac{1}{2}\left[\alpha^a T^a, \beta^b T^b\right] + \dots \quad . \tag{194}$$

Thus we have

$$\left[\alpha^a T^a, \beta^b T^b\right] = -2i(\delta^c - \alpha^c - \beta^c)T^c + \dots \equiv i\gamma^c T^c \quad , \tag{195}$$

and since this must hold for any α^a and β^b, we have

$$\gamma^c = \alpha^a \beta^b f^{abc} \quad , \tag{196}$$

and hence

$$[T^a, T^b] = i f^{abc} T^c \quad . \tag{197}$$

(197) is known as the Lie bracket of the Lie algebra. It is determined uniquely by the structure constant f^{abc}, which is antisymmetric under exchange of indices. The commutator in (197) is skew-symmetric and obeys the Jacobi identity.

Appendix B: Solving the Yang-Baxter Equation with Mathematica

Here we describe the method for solving the Yang-Baxter equation using Mathematica, which was used to confirm that (24) is a solution and to find the deformed versions of L and R. The case where both L and R are deformed is presented here, although the other cases use the same method.

First we define the permutation operators and R matrices as described in section 4.2.

```
In[1]:= P := 1/2 (TensorProduct[IdentityMatrix[2], IdentityMatrix[2]] +
            TensorProduct[( {{0, 1},{1, 0}} ), ( {{0, 1},{1, 0}} )] +
            TensorProduct[( {{0, -I},{I, 0}} ), ( {{0, -I},{I, 0}} )] +
            TensorProduct[( {{1, 0},{0, -1}} ), ( {{1, 0},{0, -1}} )])//ArrayFlatten
        P12 := TensorProduct[P, IdentityMatrix[2]]//ArrayFlatten;
        P23 := TensorProduct[IdentityMatrix[2], P]//ArrayFlatten;
        P13 := P23 . P12 . P23;

        R[λ_] := λ (TensorProduct[IdentityMatrix[2],
            IdentityMatrix[2]] // ArrayFlatten) + I P;
        R12[λ_] := TensorProduct[R[λ], IdentityMatrix[2]]//ArrayFlatten;
        R23[λ_] := TensorProduct[IdentityMatrix[2], R[λ]]//ArrayFlatten;
        R13[λ_] := P23.R12[λ].P23;
```

Now we can define the Lax operators.

```
In[2]:= L13[λ_] := R13[λ - I/2];
        L23[λ_] := R23[λ - I/2];
```

Then we define the deformations to L and R. We assume their dependence on λ to be linear.

```
In[3]:= A = Table[Symbol["a" <> ToString@i <> ToString@j], {i, 4}, {j, 4}];
        B = Table[Symbol["b" <> ToString@i <> ToString@j], {i, 4}, {j, 4}];
        m[λ_] := λ A + B;
        m23[λ_] := TensorProduct[IdentityMatrix[2], m[λ]]//ArrayFlatten;
        m12[λ_] := TensorProduct[m[λ], IdentityMatrix[2]]//ArrayFlatten;
        m13[λ_] := P23.m12[λ].P23;

        X = Table[Symbol["c" <> ToString@i <> ToString@j], {i, 4}, {j, 4}];
        Y = Table[Symbol["d" <> ToString@i <> ToString@j], {i, 4}, {j, 4}];
        n[λ_] := λ X + Y;
        n23[λ_] := TensorProduct[IdentityMatrix[2], n[λ]]//ArrayFlatten;
```

```
n12[λ_] := TensorProduct[n[λ], IdentityMatrix[2]]//ArrayFlatten;
n13[λ_] := P23.n12[λ].P23;
```

We define the deformed FCR as a function to be solved.

```
In[4]:= YBE[λ_, μ_] := n12[λ - μ].L13[λ].L23[μ] + R12[λ - μ].(L13[λ].m23[μ] +
                       m13[λ].L23[μ]) - L23[μ].L13[λ].n12[λ - μ] -
                       (m23[μ].L13[λ] + L23[μ].m13[λ]).R12[λ - μ]
```

We can proceed to solve for the deformations using the **Solve** function. This will work in the simpler cases: for the undeformed case where only L is deformed. The problem arises when both L and R are deformed. It is a very large system, and Mathematica will assume that the constants we solve for can depend on λ and μ, leading to some very nonlinear deformations. One way around this is to note that entry as a polynomial in λ and μ,

$$\mathrm{YBE}(\lambda,\mu) = A + B\lambda + C\mu + D\lambda\mu + E\lambda^2 + F\mu^2 \quad , \tag{198}$$

where A, B, C, D, E and F are matrices, and then solve each order sequentially.

```
In[5]:= Sol1 = Solve[YBE[0, 0] == 0, Join[Flatten[A], Flatten[B], Flatten[X],
          Flatten[Y]]];
        a1[λ_, μ_] := YBE[λ, μ]/.Flatten[Sol1]//Simplify
        Sol2 = Solve[(D[a1[λ, μ], μ]/.{λ→ 0,μ→ 0}) == 0, Join[Flatten[A],
          Flatten[B], Flatten[X], Flatten[Y]]];
        b1[λ_, μ_] := a1[λ, μ]/.Flatten[Sol2]//Simplify
        Sol3 = Solve[(D[b1[λ, μ], λ]/.{λ→ 0,μ→ 0}) == 0, Join[Flatten[A],
          Flatten[B], Flatten[X], Flatten[Y]]];
        c1[λ_, μ_] := b1[λ, μ] /. Flatten[Sol3]//Simplify
        Sol4 = Solve[(D[c1[λ, μ], λ, μ]/.{λ→ 0,μ→ 0}) == 0, Join[Flatten[A],
          Flatten[B], Flatten[X], Flatten[Y]]];
        d1[λ_, μ_] := c1[λ, μ]/.Flatten[Sol4]//Simplify
        Sol5 = Solve[(D[d1[λ, μ], {λ,2}]/.{λ→ 0,μ→ 0}) == 0, Join[Flatten[A],
          Flatten[B], Flatten[X], Flatten[Y]]];
        e1[λ_, μ_] := d1[λ, μ]/.Flatten[Sol5]//Simplify
        Sol6 = Solve[(D[e1[λ, μ], {μ,2}]/.{λ→ 0,μ→ 0}) == 0, Join[Flatten[A],
          Flatten[B], Flatten[X], Flatten[Y]]];
        f1[λ_, μ_] := e1[λ, μ]/.Flatten[Sol6]//Simplify
```

Now we apply these constraints to the deformations.

```
In[6]:= mm[λ_] := m[λ]/.Flatten[Sol1]/.Flatten[Sol2]/.Flatten[Sol3]/.Flatten[Sol4]
                  /.Flatten[Sol5]/.Flatten[Sol6]// Simplify
        nn[λ_] := n[λ]/.Flatten[Sol1]/.Flatten[Sol2]/.Flatten[Sol3]/.Flatten[Sol4]
                  /.Flatten[Sol5]/.Flatten[Sol6]// Simplify

        mm23[λ_] := TensorProduct[IdentityMatrix[2], mm[λ]]//ArrayFlatten;
        mm12[λ_] := TensorProduct[m[λ], IdentityMatrix[2]]//ArrayFlatten;
        mm13[λ_] := P23.m12[λ].P23;

        nn23[λ_] := TensorProduct[IdentityMatrix[2], nn[λ]]//ArrayFlatten;
        nn12[λ_] := TensorProduct[q[λ], IdentityMatrix[2]]//ArrayFlatten;
        nn13[λ_] := P23.nn12[λ].P23;
```

We can check that these deformations satisfy the FCR.

```
In[7]:= X = (q12[λ - μ].L13[λ].L23[μ] + R12[λ - μ].(L13[λ].m23[μ] + m13[λ].L23[μ]) -
            (L23[μ].L13[λ].q12[λ - μ] (m23[μ].L13[λ] + L23[μ].m13[λ]).R12[λ - μ]))
            //Simplify;
        X === ConstantArray[0, {8, 8}]

Out[7]= True
```

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