MA3431: Classical Field Theory

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Contents

Chapter 1

Lagrangian & Hamiltonian mechanics

1.1 Fundamentals

1.1.1 The Lagrangian

The **Lagrangian** L of a system is given by $T - V$, the difference between its kinetic energy and its potential energy.

1.1.2 Hamilton's principle

Hamilton's principle states that

$$\delta \int_{t_1}^{t_2} L\, dt = 0$$

for a trajectory.

1.1.3 Euler–Lagrange equations

Hamilton's principle leads to the **Euler–Lagrange equations**:

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \text{ for each } i = 1, 2, ..., N$$

(Derivation in Section 1.4)

1.1.4 Symmetry and conservation laws

If for some particular q_i, $\frac{\partial L}{\partial q_i} = 0$, then from EL equations $\dot{p}_i = 0$, so p_i is independent of t. In other words, 'spatial translation symmetry $\rightarrow$ momentum conservation'.

Other examples include 'rotational symmetry $\rightarrow$ angular momentum conservation' and 'time transformation $\rightarrow$ energy conservation'.

1.1.5 The Hamiltonian

Defining p_i as $\frac{\partial L}{\partial \dot{q}_i}$, the **Hamiltonian** H of a system is given by $p_i q_i - L$ (summing over i). Its time derivative is:

$$\begin{aligned}
\frac{dH}{dt} &= \frac{d}{dt}(p_i \dot{q}_i - L) \\
&= \frac{dp_i}{dt}\dot{q}_i + p_i \frac{d\dot{q}_i}{dt} - \frac{\partial L}{\partial q_i}\frac{\partial q_i}{\partial t} - \frac{\partial L}{\partial \dot{q}_i}\frac{\partial q_i}{\partial t} - \frac{\partial L}{\partial t} \\
&= \dot{p}_i \dot{q}_i + p_i \ddot{q}_i - \dot{p}_i \dot{q}_i - p_i \ddot{q}_i - \frac{\partial L}{\partial t} \text{ (by EL equations)} \\
&= \frac{\partial L}{\partial t}
\end{aligned}$$

So H is conserved if L has no explicit time dependence.

1.1.6 Hamilton's dynamical equations

From the definition of the Hamiltonian, we have:

$$\delta H = \dot{q}_i \delta p_i - \dot{p}_i \delta q_i$$

and considering $H = H(p_i, q_i)$, we have

$$\delta H = \frac{\partial H}{\partial p_i}\delta p_i + \frac{\partial H}{\partial q_i}\delta q_i.$$

Comparing terms gives Hamilton's dynamical equations:

$$\frac{\partial H}{\partial p_i} = \dot{q}_i \quad \& \quad \frac{\partial H}{\partial q_i} = -\dot{p}_i$$

1.2 The wave equation

1.2.1 Derivation

Consider a 1D system of N masses m connected by springs of spring constant k. The displacement of each mass is denoted by $\phi_i(t)$ and at equilibrium they are separated by a distance a. The force on mass i is

$$\underbrace{-k(\phi_i - \phi_{i-1})}_{\text{from left spring}} + \underbrace{k(\phi_{i+1} - \phi_i)}_{\text{from right spring}}$$

The Lagrangian for the system is:

$$\begin{aligned}
L &= T - V \\
&= \sum_{i=1}^{N} \tfrac{1}{2} m \dot{\phi}_i^2 - \tfrac{1}{2} k (\phi_{i+1} - \phi_i)^2
\end{aligned}$$

EL equations give equations of motion for the system:

$$m\ddot{\phi}_i = -k\left[(\phi_i - \phi_{i-1}) - (\phi_{i+1} - \phi_i)\right]$$

Taking the limit as $N \to \infty$, keeping mN and $a(N-1)$ constant gives:

$$\mu \frac{d}{dt}(\dot{\phi}) - ka\frac{\partial^2 \phi}{\partial x^2} = 0$$

where $\mu \equiv \frac{m}{a}$. We let $\nu \equiv ka$, so:

$$\mu \frac{\partial^2 \phi}{\partial t^2} - \nu \frac{\partial^2 \phi}{\partial x^2} = 0$$

This is the 1D **wave equation**; the physical interpretation is that any perturbation will propagate through the system as a wave.

1.2.2 Notation

The 1D wave equation can be more conveniently expressed as:

$$\mu \partial_t^2 \phi - \nu \partial_x^2 \phi = 0.$$

In general, the wave equation in 3D is:

$$\partial_t^2 \phi - (\partial_x^2 + \partial_y^2 + \partial_z^2)\phi = 0$$

or, more conveniently:

$$\partial_t^2 \phi - (\nabla^2)\phi = 0$$

or, even *more* conveniently:

$$\Box \phi = 0.$$

∇^2 is called the **Laplacian operator** and $\Box$ is called the **d'Alembertian operator.**

1.3 The Lagrangian density

The Lagrangian of the discrete 1D system discussed in Section 1.2.1 can be written as a sum of parts:

$$L = \sum_{i=1}^{N} a\mathcal{L}$$

where $\mathcal{L} = \frac{1}{2}\mu\dot{\phi}_i^2 - \frac{1}{2}\frac{k}{a}(\phi_{i+1} - \phi_i)^2$. $\mathcal{L}$ is called the **Lagrangian density** of the system. As $a \to 0$,

$$L = \int \mathcal{L}\, dx$$

and $\phi_i(t) \to \phi(t,x)$, $\frac{\phi_{i+1} - \phi_i}{a} \to \partial_x \phi$.

In the simple linear case,

$$\mathcal{L} = \frac{1}{2}\mu(\partial_t \phi)^2 - \frac{1}{2}\nu(\partial_x \phi)^2$$

Recall the EL equations:

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right) + \partial_x \left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

substituting $\mathcal{L}$ into the EL equations reveals:

$$\mu \partial_t(\partial_t \phi) - \nu \partial_x(\partial_x \phi) = 0$$

$$\Rightarrow \mu \partial_t^2 \phi - \nu \partial_x^2 \phi = 0.$$

Generally $\mathcal{L}$ refers to a *3D* Lagrangian density. In the 3D version of the 'masses connected by springs' example, $\mathcal{L}$ would be given by:

$$\mathcal{L} = \tfrac{1}{2}\mu(\partial_t \phi)^2 - \tfrac{1}{2}\nu(\nabla \phi)^2.$$

1.4 Deriving the Euler–Lagrange Equations

We will derive the EL equations for a scalar field $\phi(t,x)$. Given $\mathcal{L}(\partial_t \phi, \partial_x \phi, \phi, t, x)$, we use the variational principle $\delta S = \delta \int_\Gamma \mathcal{L}\, dt\, dx = 0$.

Since $\mathcal{L}$ is invariant under time translation $t \mapsto t + \tau$ and space translation $x \mapsto x + \alpha$, there is no explicit dependence on t or x. Hence $\mathcal{L} = \mathcal{L}(\partial_t \phi, \partial_x \phi, \phi)$, and:

$$\delta \mathcal{L} = \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta(\partial_t \phi) + \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \delta(\partial_x \phi) + \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi$$

So returning to the variational principle:

$$\begin{aligned}
\delta S &= \int\limits_\Gamma \delta \mathcal{L}\, dt\, dx \\
&= \int \left[\int \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \partial_t(\delta \phi)\, dt \right] dx + \int \left[\int \frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \partial_x(\delta \phi)\, dx \right] dt + \iint \frac{\partial \mathcal{L}}{\partial \phi} \delta \phi\, dt\, dx.
\end{aligned}$$

Integrating by parts gives the first term:

$$\int \frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \partial_t(\delta \phi)\, dt = \left[\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \delta \phi \right]_{t_1}^{t_2} - \int\limits_{t_1}^{t_2} \partial_t \left[\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right] \delta \phi\, dt$$

and a similar expression for the second term.

The boundary variations, i.e. $\delta\phi(t_1, x)$, $\delta\phi(t_2, x)$, $\delta\phi(t, x_1)$, $\delta\phi(t, x_2)$, are zero, which means that the $[\,]_{t_1}^{t_2}$ and $[\,]_{x_1}^{x_2}$ terms vanish. This gives:

$$\delta S = \int \left(-\partial_t \left[\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right] \delta \phi - \partial_x \left[\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right] \delta \phi \right) dt\, dx$$

Since $\delta S = 0$ for any $\delta \phi$, the integrand multiplying $\delta \phi$ must be zero, i.e.

$$\partial_t \left(\frac{\partial \mathcal{L}}{\partial(\partial_t \phi)} \right) + \partial_x \left(\frac{\partial \mathcal{L}}{\partial(\partial_x \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0,$$

the Euler–Lagrange equations for a scalar field $\phi(t,x)$.

In 1+3 dimensions, y and z terms appear. We can write:

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \right) - \frac{\partial \mathcal{L}}{\partial \phi} = 0$$

where ∂_μ is the operator $\left(\frac{1}{c}\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$.

The field can have many components: ϕ^α, $\alpha \in \{1, 2, ..., n\}$ and EL equations may be written for each α.

Chapter 2

Relativistic field theory

2.1 Lorentz transformations

For an event (t, x) in frame S, its coordinates (t', x') in frame S' which moves relative to S with velocity v in the x direction are obtained by **Lorentz transformation**:

$$\begin{aligned} t' &= \gamma(t - vx/c^2) \\ x' &= \gamma(x - vt) \end{aligned}$$

where $\gamma^{-2} = 1 - v^2/c^2$.

2.1.1 Rapidity

The Lorentz transformations can be rewritten:

$$\begin{aligned} ct' - x' &= \sqrt{\frac{1 + v/c}{1 - v/c}}(ct - x) \\ ct' + x' &= \sqrt{\frac{1 - v/c}{1 + v/c}}(ct + x) \end{aligned}$$

Or, defining the **rapidity** ζ as $\log \sqrt{\frac{1+v/c}{1-v/c}}$,

$$\begin{aligned} ct' - x' &= e^{\zeta}(ct - x) \\ ct' + x' &= e^{-\zeta}(ct + x) \end{aligned}$$

Note that for successive Lorentz transformations,

$$\begin{aligned} ct'' - x'' &= e^{\zeta(v')}(ct' - x') \\ &= e^{\zeta(v')}e^{\zeta(v)}(ct - x) \\ &= e^{\zeta(v')+\zeta(v)}(ct - x) \end{aligned}$$

i.e. one can ADD RAPIDITIES, by contrast with velocities.

2.1.2 Hyperbolic angles

We let $\beta = v/c$. Expanding ζ gives:

$$\begin{aligned}
\zeta(\beta) = \frac{1}{2}\log\left(\frac{1+\beta}{1-\beta}\right) &= \tfrac{1}{2}\big[\log(1+\beta) - \log(1-\beta)\big] \\
&= \tfrac{1}{2}\big[(\beta - \tfrac{1}{2}\beta^2 + \tfrac{1}{3}\beta^3 - ...) - (-\beta - \tfrac{1}{2}\beta^2 - \tfrac{1}{3}\beta^3 - ...)\big] \text{ for } |\beta| < 1 \\
\Rightarrow \zeta(\beta) &= \beta + \tfrac{1}{3}\beta^3 + \tfrac{1}{5}\beta^5 + ... \text{ for } |\beta| < 1.
\end{aligned}$$

So when $v \ll c$, $\zeta \approx v/c$.

How does β depend on ζ? Working from the definition of ζ,

$$\begin{aligned}
e^{2\zeta} &= \frac{1+\beta}{1-\beta} \\
1+\beta &= (1-\beta)e^{2\zeta} \\
\beta(1+e^{2\zeta}) &= e^{2\zeta} - 1 \\
\beta &= \frac{e^{\zeta} - e^{-\zeta}}{e^{\zeta} + e^{-\zeta}} = \frac{2\sinh\zeta}{2\cosh\zeta} = \tanh\zeta.
\end{aligned}$$

This of course means that $\zeta = \operatorname{arctanh}\beta$.

2.1.3 Matrix form

From the definition of γ,

$$\begin{aligned}
v &= c\tanh\zeta \\
\gamma^{-2} &= 1 - \tanh^2\zeta \\
&= \operatorname{sech}^2\zeta \\
\gamma^2 &= \cosh^2\zeta \\
\gamma &= \cosh\zeta
\end{aligned}$$

We rewrite the Lorentz transformations as:

$$\begin{aligned}
ct' &= \gamma(ct - \beta x) \\
x' &= \gamma(x - \beta ct)
\end{aligned}$$

and, noting that $\beta\gamma = \tanh\zeta\cosh\zeta = \sinh\zeta$, this gives:

$$\begin{aligned}
ct' &= ct\mathrm{ch}\zeta - x\sin\zeta \\
x' &= x\mathrm{ch}\zeta - ct\mathrm{sh}\zeta \\
y' &= y \\
z' &= z
\end{aligned}$$

where $\mathrm{ch} \equiv \cosh$ and $\mathrm{sh} \equiv \sinh$.

We now write (ct, x, y, z) as the vector (x^0, x^1, x^2, x^3), allowing the Lorentz boost along the x^1 axis with velocity v to be expressed as a matrix Λ operating on the vector:

$$\Lambda = \begin{bmatrix} \mathrm{ch}\zeta & -\mathrm{sh}\zeta & 0 & 0 \\ -\mathrm{sh}\zeta & \mathrm{ch}\zeta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

This is a *hyperbolic rotation* in the x^0-x^1 plane.

Any proper Lorentz transformation can be written as a product of a rotation $\Lambda^{(R)}$ which aligns z-axis with direction of motion, a boost along the new z-direction Λ^B, and another rotation $\Lambda^{(R')}$, i.e. $\Lambda = \Lambda^{(R')}\Lambda^B\Lambda^{(R)}$.

2.2 Tensors

2.2.1 Light rays

A light ray passing through the origin has coordinates which obey:

$$c^2t^2 - x^2 - y^2 - z^2 = 0$$

A 4-vector x^μ is called:

- time-like if $x^\mu x_\mu > 0$
- light-like if $x^\mu x_\mu = 0$
- space-like if $x^\mu x_\mu < 0$

where $x^\mu = (x^0, x^1, x^2, x^3)$ and $x_\mu = (x^0, -x^1, -x^2, -x^3)$.

2.2.2 Metric tensor

For the purposes of this course, $g^{00} = 1$, $g^{11} = -1$, $g^{22} = -1$, $g^{33} = -1$, and 0 otherwise. The indices ensure that signs are accounted for; to raise index x_μ for $\mu \in \{0, 1, 2, 3\}$:

$$x_\mu = g^{\mu\nu}x_\nu = g^{\mu 0}x_0 + g^{\mu 1}x_1 + g^{\mu 2}x_2 + g^{\mu 3}x_4$$

(summing over repeated index ν on the left). This gives $x^0 = x_0$ and $x^{1,2,3} = -x_{1,2,3}$.

2.2.3 Lorentz transformations and tensors

For x'^μ in S', x^μ in S:

$$x'^\mu = \Lambda^\mu_\alpha x^\alpha$$

A light-ray obeys both:

$$\begin{aligned} g_{\mu\nu}x'^\mu x'^\nu &= 0 \text{ and} \\ g_{\alpha\beta}x^\alpha x^\beta &= 0. \end{aligned}$$

So for all x^α and x^β,

$$\begin{aligned} g_{\mu\nu}\Lambda^\mu_\alpha\Lambda^\nu_\beta x^\alpha x^\beta &= g_{\mu\nu}x^\mu x^\nu \\ &= g_{\alpha\beta}x^\alpha x^\beta \end{aligned}$$

So the **Lorentz group** is defined by matrices which obey:

$$g_{\mu\nu}\Lambda^\mu_\alpha\Lambda^\nu_\beta = g_{\alpha\beta}.$$

Examples:

- Any 4-vector A^μ transforms the same way as x^μ: $A'^\mu = \Lambda^\mu_\nu A^\nu$.
- The EM tensor $F^{\mu\nu}$ transforms as $F'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta F^{\alpha\beta}$.
- The angular momentum tensor $M^{\lambda\mu\nu}$ transforms as: $M'^{\lambda\mu\nu} = \Lambda^\lambda_\alpha \Lambda^\mu_\beta \Lambda^\nu_\gamma M^{\alpha\beta\gamma}$

Note: Must observe 'grammar' of indices; i.e. one lower λ on left $\Rightarrow$ one lower λ on right; lower *and* upper μ on left $\Rightarrow$ no μ on right, etc. For example:

$$g_{\lambda\mu} g^{\mu\nu} = g^\nu_\lambda$$

2.2.4 Lorentz scalars

$$g_{\alpha\beta} a^\alpha b^\beta = a^\alpha b_\alpha$$

and

$$a'^\mu b'_\mu = a^\mu b_\mu,$$

which is a *scalar*, and invariant under LT. We call $a^\mu b_\mu$ a **Lorentz scalar**.

Analogous to $\vec{a} \cdot \vec{b}$, which is invariant under rotation in 3D.

2.2.5 Invariant tensors

Scalar time increment: $d\tau$ is defined:

$$c^2 d\tau^2 \equiv g_{\mu\nu} dx^\mu dx^\nu$$
$$(= dx^\mu dx_\mu)$$

$c^2 d\tau^2$ is invariant under LT, and can be used to construct a **4-velocity:**

$$V^\mu = \frac{dx^\mu}{d\tau}$$

Metric tensor: $g'^{\mu\nu} = \Lambda^\mu_\alpha \Lambda^\nu_\beta g^{\alpha\beta} = g^{\mu\nu}$

Levi–Civita tensor: An important tensor, defined as:

$$\epsilon^{\alpha\beta\gamma\delta} = \begin{cases} +1 & \text{if } \alpha\beta\gamma\delta \text{ is an even permutation} \\ -1 & \text{if } \alpha\beta\gamma\delta \text{ is an odd permutation} \\ 0 & \text{otherwise.} \end{cases}$$

Under LT, ϵ transforms:

$$\begin{aligned} \epsilon'^{\mu\nu\rho\sigma} &= \Lambda^\mu_\alpha \Lambda^\nu_\beta \Lambda^\rho_\gamma \Lambda^\sigma_\delta \epsilon^{\alpha\beta\gamma\delta} \\ &= \epsilon^{\alpha\beta\gamma\delta} \det \Lambda \end{aligned}$$

(by definition of the determinant.)

2.2.6 Proper and improper Lorentz transformations

If $\det \Lambda = 1$, Λ is called a *proper* LT, and if $\det \Lambda = -1$, Λ is called an *improper* LT. So ϵ is invariant under a proper LT, and for an improper LT, $\epsilon' = -\epsilon$.

The **parity transformation** Λ^P flips the sign of all spatial coordinates:

$$\Lambda^P = \mathrm{diag}(1, -1, -1, -1)$$

and the **time reversal transformation** Λ^T flips the sign of the time coordinate:

$$\Lambda^T = \mathrm{diag}(-1, 1, 1, 1)$$

Any LT can be written as the product of a proper LT and one of $\{1, \Lambda^P, \Lambda^T, \Lambda^{PT}\}$. The Lorentz group $O(1,3)$ has four disconnected subsets corresponding to LTs of the form Λ, $\Lambda\Lambda^P$, $\Lambda\Lambda^T$, $\Lambda\Lambda^{PT}$ for Λ proper.

The subgroup consisting of *orthochronous* LTs (i.e. those which preserve time direction) is denoted $O^+(1,3)$.

2.2.7 4-velocity and 4-momentum

Recall:

$$\begin{aligned}
c^2 d\tau^2 &= dx_\mu dx^\mu \\
d\tau^2 &= dt^2 - \frac{d\vec{x}^2}{c^2} \\
d\tau^2 &= dt^2 \left(1 - \frac{1}{c^2}(\frac{d\vec{x}}{dt})^2\right) \\
&= dt^2 \left(1 - \frac{v^2}{c^2}\right) \\
&= dt^2 \left(1 - \beta^2\right) \\
\Rightarrow dt &= \gamma dt \\
\frac{d}{d\tau} &= \gamma \frac{d}{dt}
\end{aligned}$$

4-velocity is defined as:

$$\begin{aligned}
\frac{dx^\mu}{d\tau} &= \gamma \frac{dx^\mu}{dt} \\
&= \gamma \frac{d}{dt}(ct, \vec{x}) \\
&= \gamma(c, \vec{v}).
\end{aligned}$$

4-momentum of a particle with mass m is defined as:

$$p^\mu = m \frac{dx^\mu}{d\tau}$$

and it transforms under LT as: $p'^\mu = \Lambda^\mu_\nu p^\nu$.

The time component of p^μ is p^0, with expansion: $p^0 = m\gamma c = mc(1 - \vec{v}^2/c^2)^{\frac{1}{2}}$.

$$\begin{aligned}
cp^0 &= mc^2 \left[1 + \left(-\frac{1}{2}\right)\left(-\frac{\vec{v}^2}{c^2}\right) + \frac{1}{2!}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{\vec{v}^2}{c^2}\right)^2 + \dots\right] \\
cp^0 &= mc^2 + \frac{1}{2}m\vec{v}^2\left(1 + \frac{3}{4}\frac{\vec{v}^2}{c^2} + \dots\right) \\
&= \text{rest energy} + \frac{1}{2}m\vec{v}^2 + \text{higher powers.}
\end{aligned}$$

The constant mc^2 has no dynamical effect, since EL equations only depend on partial derivatives.

If we denote cp^0 by $\mathcal{E}$, the relativistic energy of a particle with mass m:

$$\mathcal{E} = mc^2 + \frac{1}{2}m\vec{v}^2 + \frac{3}{8}m\frac{\vec{v}^2}{c^4} + \dots$$

Thus $p^0 = \frac{\mathcal{E}}{c}$, and $p^\mu = \left(\frac{\mathcal{E}}{c}, \vec{p}\right)$.

Recall $x^\mu = (ct, \vec{x})$. Then $x^\mu p_\mu = \mathcal{E} - \vec{x}\cdot\vec{p}$, a Lorentz scalar. Another scalar is:

$$\begin{aligned}
p^\mu p_\mu &= m\gamma(c, \vec{v}) \cdot m\gamma(c, \vec{v}) \\
&= m^2\gamma^2(c^2 - \vec{v}^2) \\
&= m^2\frac{c^2(1 - \vec{v}^2/c^2)}{1 - \vec{v}^2/c^2} \\
p^\mu p_\mu &= m^2c^2.
\end{aligned}$$

which means that for massless particles, $p^\mu p_\mu = 0$. In 1+3 dimensions this describes a **light cone**:

$$p^\mu p_\mu = (p^0)^2 - (p^1)^2 - (p^2)^2 - (p^3)^2 = 0.$$

For $m > 0$, we have $c^2 p^\mu p_\mu = \mathcal{E}^2 - c^2p^2 = m^2c^4$, giving:

$$\mathcal{E} = \pm c\sqrt{p^2 + m^2c^4}$$

where $p^2 = (p^1)^2 + (p^2)^2 + \dots$

2.2.8 Volume element

The volume element $d^4x = cdt\, dx^1\, dx^2\, dx^3$ in 1+3 dimensions is invariant under Lorentz transformation:

$$\int d^4x' = \int |J|\, d^4x$$

where $J^\alpha_\beta = \frac{\partial x'^\alpha}{\partial x^\beta}$. For a LT, $|J| = \text{ch}^2\zeta - \text{sh}^2\zeta = 1$, so $d^4x' = d^4x$.

Chapter 3

Field theories

3.1 Covariant field theory

3.1.1 Formulation

Formulation in terms of a variational principle is useful; to do this we need a **Lorentz scalar action**:

$$S = \frac{1}{c}\mathcal{L}\, d^4x.$$

If $\mathcal{L}$ is scalar, this ensures S is scalar.

We consider a scalar field $\phi(x^\mu)$ and a scalar Lagrangian density $\mathcal{L}(\phi, \partial_\mu\phi)$, e.g. ϕ, ϕ^2, $\partial_\mu\partial^\mu\phi$, $\partial_\mu\phi\partial^\nu\phi$, etc. (These are all scalars since indices are saturated.)

The EL equations for a scalar field are:

$$\partial_\mu\left[\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\right] - \frac{\partial\mathcal{L}}{\partial\phi} = 0.$$

3.1.2 Example

$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + f(\phi)$, then first term in EL equations reads:

$$\begin{aligned}
\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} &= \frac{\partial}{\partial_\mu\phi}\left[\frac{1}{2}\partial_\nu\phi\partial^\nu\phi + f(\phi)\right] \\
&= \frac{\partial}{\partial_\mu\phi}\left(\frac{1}{2}\partial_\nu\phi\partial^\nu\phi\right) \\
&= \frac{1}{2}\frac{\partial(\partial_\nu\phi)}{\partial(\partial_\mu\phi)}\partial^\nu\phi + \frac{1}{2}\partial_\nu\phi\frac{\partial(\partial^\nu\phi)}{\partial(\partial_\mu\phi)} \\
&= \frac{1}{2}\delta^\mu_\nu\partial^\nu\phi + \frac{1}{2}\partial^\nu\phi\frac{\partial(\partial_\nu\phi)}{\partial(\partial_\mu\phi)} \\
&= \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\nu\phi\delta^\mu_\nu \\
&= \frac{1}{2}\partial^\mu\phi + \frac{1}{2}\partial^\mu\phi \\
&= \partial^\mu\phi.
\end{aligned}$$

so the EL equations give:

$$\partial_\mu(\partial^\mu \phi) - \frac{\partial f}{\partial \phi} = 0$$

or in other words,

$$\Box\phi - \frac{\partial f}{\partial q} = 0$$

If $f = -\frac{1}{2}\sigma\phi^2$, for some constant σ, then the EL equations give:

$$\begin{aligned}\Box\phi - \frac{\partial}{\partial\phi}\left(-\tfrac{1}{2}\sigma\phi^2\right) &= 0\\ \Box\phi + \sigma\phi &= 0\\ \text{or simply }\; (\Box + \sigma)\,\phi &= 0.\end{aligned}$$

Solutions are of the form $\phi = A\exp(-ik_\mu x^\mu)$. Noting that:

$$\begin{aligned}\partial_\mu\phi &= A\partial_\mu \exp(-ik_\nu x^\nu)\\ &= -iAk_\mu \exp(-ik_\nu x^\nu)\end{aligned}$$

this gives:

$$\tfrac{1}{2}\partial_\mu\partial^\mu\phi = (-i)^2 Ak_\mu k^\mu \exp(-ik^\nu x_\nu)$$

Substituting into $\partial_\mu\partial^\mu\phi + \sigma\phi = 0$:

$$-A(k_\mu k^\mu - \sigma)\exp(-ik_\nu x^\nu) = 0.$$

$$k_\mu k^\mu = \sigma.$$

This means that k_μ is $\begin{cases}\text{time-like} & \text{if } \sigma > 0.\\ \text{light-like} & \text{if } \sigma = 0.\\ \text{space-like} & \text{if } \sigma < 0.\end{cases}$

For $k^\mu = (\frac{\omega}{c}, \vec{k})$, we write $\exp(-ik_\nu x^\nu) = e^{-i(\omega t - \vec{k}\cdot\vec{x})}$ where $k^0 = \omega/c$. ω is the angular frequency of a wave moving in the direction indicated by $\vec{k}$.

In quantum field theory, σ has a natural interpretation. $E = \hbar\omega = \hbar c k^0$ and $\vec{p} = \hbar\vec{k}$, so $p^\mu = \hbar k^\mu$.

$$\begin{aligned}p^\mu p_\mu = m^2c^2 &= \hbar^2 k^\mu k_\mu\\ m^2c^2 &= \hbar^2\sigma\\ \sigma &= \left(\frac{mc}{\hbar}\right)^2\\ \sigma^{-1/2} &= \frac{\hbar}{mc} = \lambdabar_C.\end{aligned}$$

where $\lambdabar_C = \lambda_C/2\pi = \hbar/mc$ is the *reduced Compton wavelength* of the particle.

So the equation of motion $\partial_\mu\partial^\mu\phi + \sigma\phi = 0$ in quantum mechanics reads:

$$\left[\Box + \frac{1}{\lambdabar_C^2}\right]\phi = 0$$

In more realistic theories, $f(\phi)$ would involve other terms, e.g.

$$f(\phi) = -\frac{1}{2}\left(\frac{c}{\hbar}\right)^2 m^2\phi^2 + \lambda\phi^4$$

for an interaction of the ϕ^4 form. Setting $m = 0$ for massless particles (e.g. photons) gives wave-like solutions with velocity of light.

3.2 Vector field theories

There are many more examples of vector field theories than scalars so far.

In a vector field theory, the field ϕ may have components ϕ^α, where $\alpha \in \{1, ..., N\}$. We need Lorentz scalars for $\mathcal{L}$ to provide a scalar action:

$$S = \frac{1}{c}\int \mathcal{L}\, d^4x.$$

An obvious scalar is $p^\mu p_\mu = m^2c^2$; and for a 4-vector A^μ, another scalar is $p^\mu A_\mu$, or $A_\mu A^\mu$.

3.3 Free field

3.3.1 Formulation

Relativistic action $= \int L\, dt = \int L\gamma\, d\tau$.

In order to have a covariant theory, we need $L\gamma$ to be a Lorentz scalar (since $d\tau$ is scalar). $L \propto 1/\gamma$ will work.

$$\frac{1}{\gamma} = \left(1 - \frac{\vec{v}^2}{c^2}\right)^{\frac{1}{2}} = 1 - \frac{1}{2}\frac{\vec{v}^2}{c^2} - \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}\left(\frac{\vec{v}^2}{c^2}^2\right)\dots$$

And in order for L to have dimension of energy, we multiply by mc^2:

$$mc^2\gamma^{-1} = mc^2 - \frac{1}{2}m\vec{v}^2 - \frac{1}{8}m\frac{\vec{v}^4}{c^2}\dots$$

and we flip the signs so that the $\frac{1}{2}m\vec{v}^2$ term is positive, as in the non-relativistic case.

$$L = -\frac{mc^2}{\gamma} = \frac{1}{2}m\vec{v}^2 - mc^2 + \frac{1}{8}m\frac{\vec{v}^4}{c^2}\dots$$

3.3.2 Interaction with a vector field

Lorentz scalars which can be added to L include $p_\mu A^\mu$, $\left(p_\mu A^\mu\right)^2$, $A_\mu A^\mu$, $\dots$ where A^μ is a 4-vector field which depends on x^μ.

To begin, we will use the simplest form, $p_\mu A^\mu$. To connect this with electromagnetism, we re call that $e\Phi$ and $e\vec{A}$ must have dimension of energy and so $p_\mu A^\mu$ should too. We choose mc as a denominator. So L should include $\frac{q}{mc}p_\mu A^\mu$. Again, we flip the sign to align this theory with known properties of EM. So we choose:

$$L = -\frac{m^2c^2}{\gamma} - \frac{q}{\gamma mc}p_\mu A^\mu.$$

The action for a particle of mass m located at $x^\mu(\tau)$ for invariant time τ with momentum $p^\mu = m\frac{dx^\mu}{dt}$ interacting with a 4-vector potential A^μ of strength q is:

$$S = -\int \left(mc^2 + \frac{q}{mc} A_\mu p^\mu \right) d\tau$$

But since $p_\mu p^\mu = m^2c^2$, this gives:

$$S = -\int \left(p_\mu + \frac{q}{c} A_\mu \right) dx^\mu.$$

We can use the variational principle:

$$\begin{aligned}
\delta S &= -\int \left(p_\mu + \frac{q}{c} A_\mu \right) \delta(dx^\mu) - \int \left(\delta p_\mu + \frac{q}{c} \delta A_\mu \right) dx^\mu \\
&= -\int \left(p_\mu + \frac{q}{c} A_\mu \right) d(\delta x^\mu) - \int \frac{q}{c} \delta A_\nu dx^\nu
\end{aligned}$$

Note: $p^\mu \delta p_\mu = 0$ since $p_\mu p^\mu = m^2c^2 = \text{constant}$.

Noting that $\delta A_\nu = \frac{\partial A_\nu}{\partial x^\mu} \delta x^\mu$, we can integrate the first term by parts:

$$\delta S = -\left[\left(p_\mu + \frac{q}{c} A_\mu \right) \delta x^\mu \right]_{\tau_1}^{\tau_2} + \int \left(dp_\mu + \frac{q}{c} dA_\mu \right) \delta x^\mu - \int \frac{q}{c} \delta A_\nu dx^\nu$$

But, $\frac{dA_\mu}{d\tau} = \frac{\partial A_\mu}{\partial x^\nu} \frac{\partial x^\nu}{\partial \tau}$, and $\delta x^\mu(\tau_1) = \delta x^\mu(\tau_2) = 0$. So:

$$\begin{aligned}
\delta S &= \int \left(\frac{dp_\mu}{dt} + \frac{q}{c} \frac{dA_\mu}{dt} \right) \delta\tau \, \delta x^\mu - \int \frac{q}{c} \frac{\partial A_\nu}{\partial x^\mu} \delta x^\mu \, dx^\nu \\
&= \int \left(\frac{dp^\mu}{d\tau} + \frac{q}{c} \frac{\partial A_\mu}{\partial x^\nu} \frac{dx^\nu}{d\tau} - \frac{q}{c} \frac{\partial A_\nu}{\partial x^\mu} \frac{dx^\nu}{d\tau} \right) dt \, \delta x^\mu
\end{aligned}$$

We want $\delta S = 0$ for all possible variations of the path, so we want integrand $= 0$, so:

$$\frac{dp_\mu}{d\tau} + \frac{q}{c} \left(\frac{\partial A_\mu}{\partial x^\nu} - \frac{\partial A_\nu}{\partial x^\mu} \right) \frac{dx^\nu}{d\tau} = 0$$

or

$$\begin{aligned}
\frac{dp_\mu}{d\tau} &= \frac{q}{c} \left(\frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} \right) \frac{dx^\nu}{d\tau} \\
&= \frac{q}{c} F_{\mu\nu} \frac{dx^\nu}{dt}
\end{aligned}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{3.1}$$

is the **antisymmetric field tensor.**

Another form of the equation of motion is:

$$\frac{dp^\mu}{d\tau} = \frac{q}{mc} F^{\mu\nu} p_\nu,$$

the Lorentz force in 4D.

3.4 The electromagnetic tensor

To link this with $\vec{E}$ and $\vec{B}$, we arrange $F^{i0} = E^i \quad i = 1, 2, 3$ and $F^{ij} = -\epsilon^{ijk}B^k$. The sum over ν is replaced by a sum over i:

$$\frac{dp_0}{d\tau} = \frac{q}{c}F_{0i}\frac{dx^i}{d\tau}.$$

But noting that raising a *spatial* index changes the sign, we know that $F_{0i} = -F_{i0} = F^{i0}$, giving us:

$$\begin{aligned}\frac{dp^0}{d\tau} &= \frac{q}{c}E^i\frac{dx^i}{d\tau}\\ \frac{dp^0}{\gamma dt} &= \frac{q}{c}E^i\frac{dx^i}{\gamma dt}\\ \frac{dp^0}{dt} &= \frac{q}{c}E^i\frac{dx^i}{dt}.\end{aligned}$$

Recalling that $p^\mu = (\frac{\mathcal{E}}{c}, \vec{p}) \Rightarrow p^0 = \frac{\mathcal{E}}{c}$, we have:

$$\frac{d\mathcal{E}}{dt} = q\vec{E}\cdot\vec{v}$$

where $\vec{v} = \frac{d\vec{x}}{dt}$ is the 3-velocity, $q\vec{E}$ is the electric force and $q\vec{E}\cdot\vec{v}$ is the rate of work done.

The spatial components of the Lorentz equation of motion are:

$$\begin{aligned}\frac{dp_i}{dt} &= \frac{q}{c}F_{i\nu}\frac{dx^\nu}{d\tau}\\ &= \frac{q}{c}\left(F_{i0}\frac{dx^0}{dt} + F_{ij}\frac{dx^j}{d\tau}\right)\\ -\frac{dp^i}{dt} &= \frac{q}{c}\left(-F^{i0}\frac{dx^0}{d\tau} - (-1)^2F^{ij}\frac{dx^j}{dt}\right)\\ \frac{dp^i}{dt} &= \frac{q}{c}\left(E^i\frac{d(ct)}{dt} + \epsilon^{ijk}B^k\frac{dx^j}{dt}\right)\\ \frac{dp^i}{dt} &= qE^i + \frac{q}{c}\epsilon^{ijk}v^jB^k.\end{aligned}$$

or, in 3-vector form:

$$\begin{aligned}\frac{d\vec{p}}{dt} &= q\vec{E} + \frac{q}{c}\vec{v}\times\vec{B}\\ \frac{d\mathcal{E}}{dt} &= q\vec{E}\cdot\vec{v}.\end{aligned}$$

3.4.1 3-dimensional equations

We assumed:

$$\begin{aligned}E^i = F^{i0} &= \partial^i A^0 - \partial^0 A^i\\ &= -\partial_0 A^i - \partial_i A^0\\ E^i &= \frac{1}{c}\frac{\partial}{\partial t}A^i - \frac{\partial}{\partial x^i}A^0\end{aligned}$$

In other words, $\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \vec{\nabla}\Phi$, where $A^\mu = (\Phi, \vec{A})$.

For a magnetic induction field $\vec{B}$, if ijk is an even permutation, $B^k = -F^{ij}$. Then:

$$\begin{aligned} B^k &= -\partial^i A^j + \partial^j A^i \\ &= \partial_i A^j - \partial_j A^i && \text{(lowering spatial index)} \\ &= (\vec{\nabla} \times \vec{A})^k && (k^{\text{th}} \text{ component.}) \end{aligned}$$

So we have

$$\vec{E} = -\partial_0 \vec{A} - \vec{\nabla}\Phi, \quad \vec{B} = \vec{\nabla} \times \vec{A},$$

the 3-dimensional versions of (3.1), given that $F^{i0} = E^i$, and $F^{ij} = -\epsilon^{ijk}B^k$.

3.4.2 Gauge invariance

If another potential, $A'_\mu = A_\mu + \partial_\mu \phi$ is introduced, the equations of motion are unaltered:

$$\begin{aligned} F'_{\mu\nu} &= \partial_\mu A'_\nu - \partial_\nu A'_\mu \\ &= \partial_\mu (A_\nu + \partial_\nu \phi) - \partial_\nu (A_\mu + \partial_\mu \phi) \\ &= \partial_\mu A_\nu - \partial_\nu A_\mu \\ &= F_{\mu\nu}. \end{aligned}$$

Adding a 4-gradient of some $\phi(x^\lambda)$ does not alter $F_{\mu\nu}$, so obviously it does not alter $\vec{E}$ or $\vec{B}$. This is analogous to a change in potential $V' = V + C$ for constant C not altering the force in 1-dimensional classical mechanics.

Gauge invariance is very closely linked to conservation of charge.

3.5 Equation of motion for a free vector field

We seek a Lagrangian density for a **free field** (i.e. no interaction with charge/current) which is a Lorentz scalar.

Remember that we used $p_\mu p^\mu$ for the free particle case without interaction (where $q = 0$). Recall also that for a scalar field $\phi(x^\mu)$ we used $\mathcal{L} = \frac{1}{2}(\partial_\mu \phi \, \partial^\mu \phi - \lambda\!\!\!^{-2} \phi^2)$.

Derivatives appear quadratically to yield linear equations of motion. For a field $F^{\mu\nu}$, the obvious suggestion is:

$$\mathcal{L} = C \, F_{\mu\nu} F^{\mu\nu}.$$

Also, we could include a term $m^2 A_\mu A^\mu$ which, like $F_{\mu\nu}F^{\mu\nu}$, is a Lorentz scalar. In fact, $m^2 A_\mu A^\mu$ corresponds to a vector field with mass m (in appropriate units, using $\hbar/c$).

For simplicity's sake, we assume $m = 0$, and use $\mathcal{L}$ with a constant $C = -1/4$, to agree with the equation of motion for EM fields in Heaviside-Lorentz units, so that;

$$\mathcal{L} = \frac{1}{4} F^\beta_\alpha \, F^\alpha_\beta \quad \left(= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right)$$

So the Euler–Lagrange equations of motion are:

$$\frac{\partial}{\partial x^\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} \right] - \frac{\partial \mathcal{L}}{\partial A_\nu} = 0.$$

where we replace ϕ from before with A_ν.

There are four EL equations, $\nu \in \{0, 1, 2, 3\}$.

$$\begin{aligned}
4\mathcal{L} &= -F_{\mu\nu}F^{\mu\nu} \\
&= -(\partial_\mu A_\nu - \partial_\nu A_\mu)(\partial^\mu A^\nu - \partial^\nu A^\mu)
\end{aligned}$$

Use $\mu \to \rho$ and $\nu \to \sigma$, since μ and ν already feature in EL equations:

$$4\mathcal{L} = -(\partial_\rho A_\sigma)(\partial^\rho A^\sigma) - (\partial_\sigma A_\rho)(\partial^\sigma A^\rho) + (\partial_\sigma A_\rho)(\partial^\rho A^\sigma) + (\partial_\rho A_\sigma)(\partial^\sigma A^\rho)$$

or, exchanging $\sigma \longleftrightarrow \rho$ in second and third terms:

$$\begin{aligned}
\mathcal{L} &= -2(\partial_\rho A_\sigma)(\partial^\rho A^\sigma) + 2(\partial_\rho A_\sigma)(\partial^\sigma A^\rho) \\
&= -2(\partial_\rho A_\sigma)(\partial^\rho A^\sigma - \partial^\sigma A^\rho);
\end{aligned}$$

so in EL equations we have:

$$2\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} = -\frac{\partial(\partial_\rho A_\sigma}{\partial(\partial_\mu A_\nu)}(\partial^\rho A^\sigma - \partial^\sigma A^\rho) - \partial_\rho A_\sigma \frac{\partial(\partial^\rho A^\sigma - \partial^\sigma A^\rho)}{\partial(\partial_\mu A_\nu)}.$$

In the second term, we raise ρ and σ and lower ρ and σ to leave the value unchanged. In the first term, there is a nonzero contribution only when $\rho = \mu$ *and* $\sigma = \nu$:

$$\begin{aligned}
2\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -\delta^\mu_\rho \delta^\nu_\sigma(\partial^\rho A^\sigma - \partial^\sigma A^\rho) - \partial^\rho A^\sigma \frac{\partial(\partial_\rho A_\sigma - \partial_\sigma A_\rho)}{\partial(\partial_\mu A_\nu)} \\
&= -(\partial^\mu A^\nu - \partial^\nu A^\mu) - \partial^\rho A^\sigma(\delta^\mu_\rho \delta^\nu_\sigma - \delta^\mu_\sigma \delta^\nu_\rho) \\
&= -F^{\mu\nu} - (\partial^\mu A^\nu - \partial^\nu A^\mu) \\
&= -F^{\mu\nu} - F^{\mu\nu} \\
&= -2F^{\mu\nu} \\
\frac{\partial\mathcal{L}}{\partial(\partial_\mu A_\nu)} &= -F^{\mu\nu}.
\end{aligned}$$

Since $\mathcal{L}$ does not depend on A_μ explicitly, $\frac{\partial\mathcal{L}}{\partial A_\nu} = 0$, as in the case of a free field.

So EL equations read:

$$\partial_\mu \left[-F^{\mu\nu}\right] - 0 = 0,$$

or:

$$\partial_\mu F^{\mu\nu} = 0 \tag{3.2}$$

the **dynamical equation** of a free vector field.

In 3D, space these are the Maxwell equations for an electromagnetic field with no charge and no 3-current:

$\nu = 0$: $\partial_\mu F^{\mu 0} = 0$. And since $F^{00} = 0$,

$$\begin{aligned}
&\partial_i F^{i0} = 0 \\
\Leftrightarrow\ &\partial_i E^i = 0 \\
\Leftrightarrow\ &\vec{\nabla} \cdot \vec{E} = 0,
\end{aligned}$$

Maxwell's first equation for free space.

$\nu = j$:

$$\begin{aligned}
\partial_0 F^{0j} + \partial_i F^{ij} &= 0 \\
-\frac{1}{c}\frac{\partial}{\partial t}E^j - \partial_i \epsilon^{ijk} B^k &= 0 \\
-\frac{1}{c}\frac{\partial}{\partial t}E^j - \epsilon^{ijk}\partial_i B^k &= 0 \\
-\frac{1}{c}\frac{\partial \vec{E}}{\partial t} - \vec{\nabla}\times\vec{B} &= 0,
\end{aligned}$$

Maxwell's second equation for free space.

3.5.1 Particle and field interaction

The Lagrangian density requires an interaction term $\mathcal{L}_{\text{int}}$:

$$\mathcal{L} = \frac{1}{4}F_\mu^\nu F_\nu^\mu + \mathcal{L}_{\text{int}}.$$

(Note that $F_\mu^\nu F_\nu^\mu = -F_{\mu\nu}F^{\mu\nu}$)

$\mathcal{L}_{\text{int}}$ should be a Lorentz scalar. Recall for particle case,

$$S_{\text{int}} = -\frac{q}{c}\int A_\mu \frac{dx^\mu}{dt} d\tau,$$

but we need $\int d^3x\, d\tau = \int d^4x$.

3.5.2 Four-current for particles

Four-current density is:

$$J^\mu(t,\vec{x}) = q\frac{dx^\mu}{dt}\delta^3(\vec{x} - \vec{x}_q(t))$$

Current $=$ charge $\times$ velocity. 3D Dirac delta function has dimension volume^{-1}, and provides current density.

In J^μ, $\vec{x}_q(t)$ is the 3-location of the charge q at time t. The zero-component $J^0(t,\vec{x}) = c\rho$, where ρ is the charge density in a region. For a volume Ω, the total charge is:

$$\begin{aligned}
\frac{1}{c}\int_\Omega J^0\, d^3x &= \frac{q}{c}\int_\Omega \frac{dx^0}{dt}\delta^3(\vec{x} - \vec{x}_q(t))\, d^3x \\
&= \frac{q}{c}\int_\Omega \delta^3(\vec{x} - \vec{x}_q(t))\, d^3x \\
&= q.
\end{aligned}$$

$J^i(t,\vec{x})$ is the 3-current associated with the moving charge q.

Since $-\frac{q}{c}A_\mu\frac{dx^\mu}{d\tau}$ was the interaction for the Lagrangian L, and action $S_{\text{int}} = \int L\, dt$ the generalisation for density $\mathcal{L}$,

$$\mathcal{L}_{\text{int}} = \frac{1}{c}A_\mu J^\mu$$

so that spatial integration gives a term like L_{int}. $\mathcal{L}_{\text{int}}$ is a Lorentz scalar as μ is summed, and $\mathcal{L}_{\text{int}}$ is reference frame-independent.

So,

$$\mathcal{L} = \frac{1}{4} F^{\nu}_{\mu} F^{\mu}_{\nu} - \frac{1}{c} A_{\mu} J^{\mu}.$$

The Euler–Lagrange equations of motion are:

$$\partial_{\mu} \left[\frac{\partial \mathcal{L}}{\partial(\partial_{\mu} A_{\nu})} \right] - \frac{\partial \mathcal{L}}{\partial A_{\nu}} = 0.$$

In the first term, $A_{\mu} J^{\mu}$ does not contribute; only $\partial_{\mu} A_{\nu}$ terms do. For the second term, only $A_{\mu} J^{\mu}$ contributes, and so:

$$\begin{aligned} -\frac{\partial \mathcal{L}}{\partial A_{\nu}} &= \frac{\partial}{\partial A_{\nu}} \left[\frac{1}{c} A_{\mu} J^{\mu} \right] \\ &= \frac{1}{c} J^{\nu}, \end{aligned}$$

or simply:

$$\partial_{\mu} F^{\mu\nu} = c^{-1} J^{\nu}.$$

$\nu = 0$: $\quad \partial_0 F^{00} + \partial_i F^{i0} = \frac{1}{c} J^0$. But $F^0 0 = 0$, so we have $\partial_i E^i = \frac{1}{c}(c\rho)$:

$$\vec{\nabla} \cdot \vec{E} = \rho,$$

Maxwell's first equation.

$\nu = i$:

$$\begin{aligned} \partial_0 F^{0i} + \partial_j F^{ji} &= \frac{1}{c} J^i \\ -\partial_0 F^{i0} - \partial_j \epsilon^{jik} B^k &= \frac{1}{c} J^i \\ -\frac{1}{c} \frac{\partial}{\partial t} E^i + \epsilon^{ijk} \partial_j B^k &= \frac{1}{c} J^i, \end{aligned}$$

giving:

$$\frac{\partial E}{\partial t} + c \vec{\nabla} \times \vec{B} = \vec{J},$$

Maxwell's second equation (using Heaviside–Lorentz units).

The remaining two Maxwell equations follow from $\partial_{\mu} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} = 0$.

3.6 Dual tensor

We saw that $\epsilon^{\mu\nu\rho\sigma}$ was Lorentz invariant. The dual tensor of $F^{\mu\nu} = \partial^{\mu} A^{\nu} - \partial^{\nu} A^{\mu}$ is:

$$\tilde{F}^{\mu\nu} = \tfrac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}.$$

For example,

$$\begin{aligned} \tilde{F}^{01} &= \tfrac{1}{2} \epsilon^{0123} F_{23} + \tfrac{1}{2} \epsilon^{0132} F_{32} \\ &= \tfrac{1}{2} \epsilon^{0123} F_{23} - \tfrac{1}{2} \epsilon^{0123} F_{23} \\ &= \tfrac{1}{2} \epsilon^{0123} F_{23} + \tfrac{1}{2} \epsilon^{0123} F_{23} \\ &= \epsilon^{0123} F_{23} \\ &= F_{23} \quad \left[= F^{23} \right]. \end{aligned}$$

Recall that Maxwell's first and second equations involved $\partial_\mu F^{\mu\nu}$. The 4-divergence of $\tilde{F}^{\mu\nu}$ is:

$$\begin{aligned}
\partial_\mu \tilde{F}^{\mu\nu} &= \partial_\mu(\tfrac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}) \\
&= \tfrac{1}{2}\partial_\mu \epsilon^{\mu\nu\rho\sigma}(\partial_\rho A_\sigma - \partial_\sigma A_\rho).
\end{aligned}$$

Swapping $\rho \leftrightarrow \sigma$ in negative term:

$$\begin{aligned}
\partial_\mu \tilde{F}^{\mu\nu} &= \tfrac{1}{2}\partial_\mu(\epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma - \epsilon^{\mu\nu\sigma\rho}\partial_\rho A_\sigma) \\
&= \tfrac{1}{2}\partial_\mu(\epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma + \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma) \\
&= \epsilon^{\mu\nu\rho\sigma}\partial_\mu(\partial_\rho A_\sigma).
\end{aligned}$$

But for functions like A_σ, second derivatives are symmetric: $\partial_\mu\partial_\rho = \partial_\rho\partial_\mu$. So, swapping $\mu \leftrightarrow \sigma$,

$$\begin{aligned}
\partial_\mu \tilde{F}^{\mu\nu} &= \epsilon^{\rho\nu\mu\sigma}\partial_\rho(\partial_\mu A_\sigma) \\
&= \epsilon^{\rho\nu\mu\sigma}\partial_\mu\partial_\rho A_\sigma \\
&= -\epsilon^{\mu\nu\rho\sigma}\partial_\mu\partial_\rho A_\sigma \\
&= -\partial_\mu \tilde{F}^{\mu\nu} \\
\Rightarrow \partial_\mu \tilde{F}^{\mu\nu} &= 0. \qquad (3.3)
\end{aligned}$$

Recall that $F^{i0} = E^i$ and $F^{ij} = -\epsilon^{ijk}B^k$. We want the dual elements in terms of $\vec{E}$ and $\vec{B}$.

$\tilde{F}^{i0}$:

$$\begin{aligned}
\tilde{F}^{i0} &= \tfrac{1}{2}\epsilon^{i0jk}F_{jk} \\
&= -\tfrac{1}{2}\epsilon^{0ijk}(-1)^2F^{jk} \\
&= -\tfrac{1}{2}\epsilon^{ijk}(-\epsilon^{jkl}B^l) \\
&= \tfrac{1}{2}\epsilon^{ijk}\epsilon^{jkl}B^l.
\end{aligned}$$

But $\epsilon^{ijk}\epsilon^{jkl} = 2\delta^{il}$. Thus,

$$\begin{aligned}
\tilde{F}^{i0} &= \delta^{il}B^l \\
\tilde{F}^{i0} &= B^i.
\end{aligned}$$

$\tilde{F}^{ij}$:

$$\begin{aligned}
\tilde{F}^{ij} &= \tfrac{1}{2}\epsilon^{ij\rho\sigma}F_{\rho\sigma} \\
&= \tfrac{1}{2}(\epsilon^{ijk0}F_{k0} + \epsilon^{ij0k}F_{0k}) \\
&= \tfrac{1}{2}(\epsilon^{ijk0}F_{k0} + (-1)^2\epsilon^{ijk0}F_{k0}) \\
&= \epsilon^{ijk0}F_{k0} \\
&= (-1)^3\epsilon^{ijk}(-F^{k0}) \\
&= \epsilon^{ijk}F^{k0}
\end{aligned}$$

$$\tilde{F}^{ij} = \epsilon^{ijk}E^k.$$

So, to summarise, we have:

$$\begin{aligned}
F^{i0} &= E^i & F^{ij} &= -\epsilon^{ijk}B^k \\
\tilde{F}^{i0} &= B^i & \tilde{F}^{ij} &= -\epsilon^{ijk}E^k
\end{aligned}$$

Now consider the components of $\tilde{F}^{\mu\nu}$ in (3.3).

$\nu = 0$: We have $\partial_\mu \tilde{F}^{\mu 0} = 0$. Like $F_{\rho\sigma}$, $\tilde{F}^{\mu\nu}$ is antisymmetric, i.e. $\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu}$. Thus, the diagonal elements of $\tilde{F}$ are 0, so the sum over $\mu \in \{0,1,2,3\}$ can be replaced by a sum over $i \in \{1,2,3\}$.

$$\begin{aligned}
\partial_i \tilde{F}^{i0} &= 0 \\
\partial_i B^i &= 0 \\
\vec{\nabla} \cdot \vec{B} &= 0,
\end{aligned}$$

Maxwell's third equation.

$\nu = i$: We have $\partial_\mu \tilde{F}^{\mu i} = 0$, i.e.

$$\begin{aligned}
\partial_0 \tilde{F}^{0i} + \partial_j \tilde{F}^{ji} &= 0 \\
-\partial_0 \tilde{F}^{i0} + \partial_j \epsilon^{jik} E^k &= 0 \\
-\partial_0 B^i - \epsilon^{ijk} \partial_j E^k &= 0 \\
\partial_0 B^i + \left(\vec{\nabla} \times \vec{E}\right)^i &= 0 \\
\partial_0 \vec{B} + \vec{\nabla} \times \vec{E} &= 0,
\end{aligned}$$

Maxwell's fourth equation.

3.6.1 Maxwell's equations

We have now derived all four of Maxwell's equations:

1. $\vec{\nabla} \cdot \vec{E} = \rho$
2. $-\frac{\partial}{\partial t}\vec{E} + c\vec{\nabla} \times \vec{B} = \vec{J}$
3. $\vec{\nabla} \cdot \vec{B} = 0$
4. $\frac{\partial}{\partial t}\vec{B} + c\vec{\nabla} \times \vec{E} = 0$

In other words,

$$\begin{aligned}
c\partial_\mu F^{\mu\nu} &= J^\nu \\
\partial_\mu \tilde{F}^{\mu\nu} &= 0
\end{aligned}$$

Here, J^μ is the four-current density with components $(c\rho, \vec{J})$.

3.6.2 Charge conservation

For N charges q_l, we have

$$J^\mu(t, \vec{x}) = \sum_{l=1}^{N} q_l \frac{dx^\mu}{dt} \delta^3\left(\vec{x} - \vec{x}_l(t)\right).$$

Integrating over a spatial region Ω containing all the charges,

$$\int_\Omega J^\mu(t, \vec{x})\, d^3x = \sum_{l=1}^{N} q_l \frac{dx_l^\mu}{dt},$$

the sum of (charge $\times$ 4-velocity) for the N particles.

Note that for $\mu = 0$, $J^0 = c\rho$:

$$\int\limits_{\Omega} c\rho\, d^3x = \sum_{l=1}^{N} q_l c$$

since $x_l^0 = ct$.

Recall that $c\partial_\mu F^{\mu\nu} = J^\nu$. Observe that ∂_ν acting here reveals $c\partial_\nu\partial_\mu F^{\mu\nu} = \partial_\nu F^\nu$. But $\partial_\nu\partial_\mu$ is symmetric and $F^{\mu\nu}$ is antisymmetric, so:

$$\partial_\mu J^\mu = 0,$$

i.e. conservation of charge.

$$\begin{aligned}
\partial_0 J^0 + \partial_i J^i &= 0\\
\frac{1}{c}\frac{\partial}{\partial t}(c\rho) + \vec{\nabla}\cdot\vec{J} &= 0\\
\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J} &= 0.
\end{aligned}$$

Consider a 3-volume Ω bounded by surface $\partial\Omega$ with charge density ρ and 3-current $\vec{J}$.

$$\begin{aligned}
\int\limits_{\Omega}\left(\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\vec{J}\right) d^3x &= 0\\
\int\limits_{\Omega}\frac{\partial\rho}{\partial t}\, d^3x + \int\limits_{\partial\Omega}\vec{J}\cdot d\vec{A} &= 0
\end{aligned}$$

(The 3D integral of a divergence $\vec{\nabla}\cdot\vec{J}$ is a surface integral of the dot product of $\vec{J}$ with directed element $d\vec{A}$ over $\partial\Omega$.)

So,

$$\frac{\partial}{\partial t}\left(\int\limits_{\Omega}\rho\, d^3x\right) = -\int\limits_{\partial\Omega}\vec{J}\cdot d\vec{A}.$$

If current flows *out* of region Ω according to $\vec{J}\cdot d\vec{A}$, then the total carge inside Ω decreases. Charge is conserved: $\partial_\mu J^\mu = 0$; the idea generalises: $\partial_\mu T^{\mu\nu} = 0$ means $T^{\mu\nu}$ is conserved.

3.6.3 Bianchi identity

An alternate form of the equation $\partial_\mu \tilde{F}^{\mu\nu} = 0$ is $\partial_\nu \tilde{F}^{\mu\nu} = 0$, since $\tilde{F}$ is symmetric. Suppose $\mu = 0$. Then $\partial_\nu \tilde{F}^{0\nu} = \partial_\nu \frac{1}{2}\epsilon^{0\nu\rho\sigma}F_{\rho\sigma} = 0$, so we have $\epsilon^{0\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$, i.e.

$$\left(\epsilon^{0123}\partial_1 F_{23} + \epsilon^{0132}\partial_1 F_{32}\right) + \left(\epsilon^{0231}\partial_2 F_{31} + \epsilon^{0213}\partial_2 F_{13}\right) + \left(\epsilon^{0312}\partial_3 F_{12} + \epsilon^{0321}\partial_3 F_{21}\right) = 0.$$

But the two terms in each bracket are equal, and $\epsilon^{0123} = \epsilon^{0231} = \epsilon^{0312} = 1$. Thus:

$$2\partial_1 F_{23} + 2\partial_2 F_{31} + 2\partial_3 F_{12} = 0.$$

In general,

$$\partial_\lambda F_{\mu\nu} + \partial_\mu F_{\nu\lambda} + \partial_\nu F_{\lambda\mu} = 0 \quad (\lambda, \mu, \nu \textit{ not} \text{ summed.})$$

This is called the **Bianchi identity** and is meaningful for $\lambda \neq \mu \neq \nu$.

3.6.4 Parity of electromagnetic tensor and dual tensor

Recall the parity transformation $P : (x, y, z)^{\mathrm{T}} \mapsto (-x, -y, -z)^{\mathrm{T}}$.

The field at location $\vec{r}$ of a charge q at origin is:

$$\vec{E} = \frac{q}{4\pi r^2}\frac{\vec{r}}{r}.$$

Under P, $\vec{r} \to -\vec{r}$, but $r^2 = \vec{r}\cdot\vec{r}$ is unchanged. So $\vec{E} \to -\vec{E}$ under parity; $\vec{E}$ is a **polar vector.**

The magnetic field for a charge q moving at a low velocity $\vec{v}$ is:

$$\vec{B} = \frac{1}{4\pi c}\frac{q}{r^3}\vec{v}\times\vec{r}.$$

As $\vec{v} = \frac{d\vec{r}}{dt}$, $\vec{v}$ changes sign under P. But $\vec{B} \to (-1)^2\vec{B}$, since $\vec{v}\times\vec{r} \to (-\vec{v})\times(-\vec{r}) = \vec{v}\times\vec{r}$. So $\vec{B}$ is unchanged under parity; $\vec{B}$ is an **axial vector.**

How do $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\tilde{F}^{\mu\nu}$ and $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ behave under P? These are Lorentz scalars and could be in $\mathcal{L}$.

$$\begin{aligned}
F^{\mu\nu}F_{\mu\nu} &= F^{0j}F_{0j} + F^{ij}F_{ij} + F^{i0}F_{i0} + F^{00}F_{00} \\
&= (-f^{j0})((-1)^2F^{j0}) + (-1)^2F^{ij}F^{ij} - F^{i0}F^{i0} + 0 \\
&= -F^{j0}F^{j0} + F^{ij}F^{ij} - F^{i0}F^{i0} \\
&= -F^{j0}F^{j0} - F^{i0}F^{i0} + \epsilon^{ijk}B^k\epsilon^{ijl}B^l \\
&= -2E^iE^i + 2\delta^{kl}B^kB^l \\
&= -2\vec{E}^2 + 2B^kB^k \\
F^{\mu\nu}F_{\mu\nu} &= -2\vec{E}^2 + 2\vec{B}^2.
\end{aligned}$$

$$\begin{aligned}
F_{\mu\nu}\tilde{F}^{\mu\nu} &= F_{0j}\tilde{F}^{0j} + F_{i0}\tilde{F}^{i0} + F_{ij}\tilde{F}^{ij} \\
&= -2F^{i0}\tilde{F}^{i0} + F^{ij}\tilde{F}^{ij} \\
&= -2E^iB^i - \epsilon^{ijk}B^k\epsilon^{ijl}E^l \\
&= -2\vec{E}\cdot\vec{B} - 2\delta^{kl}B^kE^l \\
&= -2\vec{E}\cdot\vec{B} - 2\vec{B}\cdot\vec{E} \\
F_{\mu\nu}\tilde{F}^{\mu\nu} &= -4\vec{E}\cdot\vec{B}.
\end{aligned}$$

So $F^{\mu\nu}F_{\mu\nu}$ is *even* under parity, and $F_{\mu\nu}\tilde{F}^{\mu\nu}$ is *odd* under parity. A $F_{\mu\nu}\tilde{F}^{\mu\nu}$ term in $\mathcal{L}$ violates parity; there is no evidence that electromagnetic interaction can violate parity (although weak interaction does.)

Note: The form $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu}$ is not used, since $\tilde{F}_{\mu\nu}\tilde{F}^{\mu\nu} = -F_{\mu\nu}F^{\mu\nu}$.

3.7 Canonical stress tensor

3.7.1 Construction

Recall in classical mechanics,

$$H = \frac{\partial l}{\partial \dot{q}_i}\dot{q}_i - L.$$

Generalise this to:

$$T^{\nu}_{\mu} = \frac{\partial \mathcal{L}}{\partial(\partial^{\mu}\phi)}\partial^{\nu}\phi - \delta^{\nu}_{\mu}\mathcal{L}.$$

Here $q_i \to \phi(x^{\rho})$, and $\frac{d}{dt} \to \frac{\partial}{\partial x^{\mu}}$ for a scalar field $\phi(x^{\rho})$.

For a vector field $A_{\lambda}(x^{\rho})$, we generalise again, $\phi \to A_{\lambda}$.

$$T^{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}A_{\lambda})}\partial^{\nu}A_{\lambda} - g^{\mu\nu}\mathcal{L},$$

the **canonical stress tensor.**

For a free electromagnetic field with $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$,

$$T^{\mu\nu} = -\frac{1}{4}\frac{\partial(F_{\rho\sigma}F^{\rho\sigma})}{\partial(\partial_{\mu}A_{\lambda})}\partial^{\nu}A_{\lambda} + \frac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

From an earlier calculation, we have:

$$\begin{aligned} T^{\mu\nu} &= -\tfrac{1}{4}(4F^{\mu\lambda})\partial^{\nu}A_{\lambda} + \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} \\ &= -F^{\mu\lambda}\partial^{\nu}A_{\lambda} + \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}, \end{aligned}$$

the canonical stress tensor for a free electromagnetic field.

There are a number of difficulties with this $T^{\mu\nu}$:

- It is not gauge invariant as A_{λ} appears explicitly. Recall, $F_{\rho\sigma}$ is gauge invariant.
- $T^{\mu\nu} \neq T^{\nu\mu}$, i.e. it is not symmetric.

Rewrite:

$$T^{\mu\nu} = -g^{\mu\rho}F_{\rho\lambda}\partial^{\nu}A^{\lambda} + \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

(obtained by lowering and raising λ and using $g^{\mu\rho}$ to lower ρ in first term)

Recall $F^{\nu\lambda} = \partial^{\nu}A^{\lambda} - \partial^{\lambda}A^{\nu}$, so that $\partial^{\nu}A^{\lambda} = F^{\nu\lambda} + \partial^{\lambda}A^{\nu}$. Substitute in $T^{\mu\nu}$:

$$T^{\mu\nu} = -g^{\mu\rho}F_{\rho\lambda}F^{\nu\lambda} - g^{\mu\rho}F_{\rho\lambda}\partial^{\lambda}A^{\nu} + \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}$$

Write $F^{\nu\lambda}$ as $-F^{\lambda\nu}$ and change λ to σ, and omit the second term to define a new tensor:

$$\Theta^{\mu\nu} = g^{\mu\rho}F_{\rho\sigma}F^{\sigma\nu} + \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma},$$

the **symmetric stress tensor** (now manifestly gauge invariant since it only involves $F_{\mu\nu}$ and not A_{μ} explicitly.)

3.7.2 Properties

Θ is symmetric ($\Theta^{\mu\nu} = \Theta^{\nu\mu}$).

The second term is in Θ is clearly symmetric, so to prove symmetry we will start by subtracting it from $\Theta^{\nu\mu}$:

$$\begin{aligned}
\Theta^{\nu\mu} - \tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} &= g^{\nu\rho}F_{\rho\sigma}F^{\rho\sigma} \\
&= F^{\nu}_{\ \sigma}F^{\sigma\mu} \\
&= F^{\nu\sigma}F_{\sigma}^{\ \mu} \\
&= (-F^{\mu}_{\ \sigma})(-F^{\sigma\nu}) \\
&= F^{\mu}_{\ \rho}F^{\sigma\nu} \\
&= g^{\mu\rho}F_{\rho\sigma}F^{\sigma\nu} \\
\Rightarrow \Theta^{\mu\nu} &= \Theta^{\nu\mu}.
\end{aligned}$$

$\Theta^{\mu\nu}$ is traceless:

$$\begin{aligned}
\Theta_{\mu}^{\nu} &= F_{\mu}^{\ \sigma}F_{\sigma}^{\ \nu} + \tfrac{1}{4}\delta_{\mu}^{\nu}F_{\rho\sigma}F^{\rho\sigma} \\
&= F_{\mu}^{\ \sigma}F_{\sigma}^{\ \mu} - \tfrac{1}{4}\delta_{\mu}^{\nu}F_{\rho}^{\ \sigma}F_{\sigma}^{\ \rho}. \\
\operatorname{tr}\Theta = \Theta_{\mu}^{\mu} &= F_{\mu}^{\ \sigma}F_{\sigma}^{\ \mu} - \tfrac{1}{4}\delta_{\mu}^{\mu}F_{\rho}^{\ \sigma}F_{\sigma}^{\ \rho} \\
&= F_{\mu}^{\ \sigma}F_{\sigma}^{\ \mu} - F_{\rho}^{\ \sigma}F_{\sigma}^{\ \rho},
\end{aligned}$$

since $\delta_{\mu}^{\mu} = 4$. Notice that these are all dummy indicies, so:

$$\begin{aligned}
\operatorname{tr}\Theta = \Theta_{\mu}^{\mu} &= F_{\mu}^{\ \sigma}F_{\sigma}^{\ \mu} - F_{\mu}^{\ \sigma}F_{\sigma}^{\ \mu} \\
&= 0.
\end{aligned}$$

The trace relates to the mass of the vector field (see Proca equation later).

Also, $\partial_{\mu}\Theta^{\mu\nu} = 0$; the tensor is conserved.

Note: Θ^{00} is energy density and Θ^{0i} is momentum density. These will be discussed in detail later.

3.7.3 Angular momentum of a field

Recall clasically, $\vec{L} = \vec{x} \times \vec{p}$. Generalise to 1+3 dimensions:

$$M^{\mu\nu\sigma} = T^{\mu\nu}x^{\sigma} - T^{\mu\sigma}x^{\nu}.$$

Is M conserved?

$$\begin{aligned}
\partial_{\mu}M^{\mu\nu\sigma} &= \partial_{\mu}(T^{\mu\nu}x^{\sigma} - \partial_{\mu}(T^{\mu\sigma}x^{\nu}) \\
&= (\partial_{\mu}T^{\mu\nu})x^{\sigma} + T^{\mu\nu}(\partial_{\mu}x^{\sigma}) - (\partial_{\mu}T^{\mu\sigma})x^{\nu} - T^{\mu\sigma}(\partial_{\mu}x^{\nu}) \\
&= 0 + T^{\mu\nu}\delta_{\mu}^{\sigma} - 0 - T^{\mu\sigma}\delta_{\mu}^{\nu} \\
&= T^{\sigma\nu} - T^{\nu\sigma} \\
&\neq 0, \quad \text{since } T^{\sigma\nu} \neq T^{\nu\sigma}.
\end{aligned}$$

But $\Theta^{\sigma\nu} = \Theta^{\nu\sigma}$, so we define field angular momentum as:

$$M^{\mu\nu\sigma} = \Theta^{\mu\nu}x\sigma - \Theta^{\mu\sigma}x^{\nu}.$$

This is conserved: $\partial_{\mu}M^{\mu\nu\sigma} = 0$. Many conservation laws are implied here.

3.7.4 Relation to electromagnetic field

Θ has 10 independent components: 4 of these relate to energy-momentum density. Recall:

$$\Theta^{\mu\nu} = g^{\mu\rho} F_{\rho\sigma} F^{\rho\nu} + \tfrac{1}{4} g^{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}$$

and:

$$F_{\rho\sigma} F^{\rho\sigma} = -2(\vec{E}^2 - \vec{B}^2)$$

So when $\mu = \nu = 0$:

$$\begin{aligned}
\Theta^{00} &= g^{0\rho} F_{\rho\sigma} F^{\rho\sigma} - \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) \\
&= g^{00} F_{0j} F^{j0} - \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) \\
&= (-1)^2 F^{j0} F^{j0} - \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) \\
&= E^j E^j - \tfrac{1}{2}\vec{E}^2 + \vec{B}^2 \\
&= \tfrac{1}{2}(\vec{E}^2 + \vec{B}^2) \qquad \equiv U.
\end{aligned}$$

U is the **energy density** of the field (in Heaviside–Lorentz units).

$\mu = i, \nu = 0$:

$$\begin{aligned}
\Theta^{i0} &= g^{i\rho} F_{\rho\sigma} F^{\rho\sigma} + 0 \\
&= (-1)^2 F_{i\sigma} F^{0\sigma} \\
&= -F_{i\sigma} F^{\sigma 0} \\
&= -F_{ij} F^{j0} \\
&= \epsilon^{ijk} B^k E^j \\
&= \epsilon^{ijk} E^j B^k \\
&= (\vec{E} \times \vec{B})^i, \quad \equiv c p^i.
\end{aligned}$$

$\vec{p}$ is the **momentum density** of the field.

$\mu = i, \nu = j$:

$$\begin{aligned}
\Theta^{ij} + \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) &= g^{i\rho} F_{\rho\sigma} F^{\sigma j} \\
&= (-1) F_{i\sigma} F^{\sigma j} \\
&= -F_{i0} F^{0j} - F_{ik} F^{kj} \\
&= -F^{i0} F^{j0} - F^{ik} F^{kj} \\
&= -E^i E^j - \epsilon^{ikl} B^l \epsilon^{kjm} B^m \\
&= -E^i E^j + \epsilon^{kil} \epsilon^{kjm} B^l B^m.
\end{aligned}$$

But $\epsilon^{kil}\epsilon^{kjm} = \delta^{ij}\delta^{lm} - \delta^{im}\delta^{jl}$, so:

$$\begin{aligned}
\Theta^{ij} + \tfrac{1}{2}(\vec{E}^2 - \vec{B}^2) &= -E^i E^j + (\delta^{ij}\delta^{lm} - \delta^{im}\delta^{jl}) B^l B^m \\
&= -E^i E^j + \delta^{ij} B^l B^l - B^j B^i \\
&= -E^i E^j - B^j B^i + \delta^{ij} \vec{B}^2
\end{aligned}$$

Noting that $g^{ij} = -\delta^{ij}$,

$$\begin{aligned}\Theta^{ij} &= \tfrac{1}{2}\delta^{ij}(\vec{E}^2 - \vec{B}^2) + \delta^{ij}\vec{B}^2 - E^iE^j - B^iB^j \\ &= \tfrac{1}{2}\delta^{ij}(\vec{E}^2 + \vec{B}^2) - E^iE^j - B^iB^j\end{aligned}$$

This is a symmetric tensor, and is equal to $-\mathcal{M}^{ij}$, where $\mathcal{M}^{ij}$ is the **Maxwell stress tensor** (in Heaviside–Lorentz units).

3.8 Interacting vector field

Introduce an external charge and current density $J^\mu = (c\rho, \vec{J})$. The Lorentz force is:

$$\partial_\mu F^{\mu\nu} = \frac{1}{c}J^\nu.$$

For a **free field**, translational invariance leads to $\partial_\mu \Theta^{\mu\nu} = 0$ (as we will show.)

Consider:

$$\begin{aligned}\partial_\mu\Theta^{\mu\nu} &= \partial_\mu(\tfrac{1}{4}g^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma} + g^{\mu\rho}F_{\rho\sigma}F^{\sigma\nu}) \\ &= \tfrac{1}{4}\partial^\mu(F_{\rho\sigma}F^{\rho\sigma}) + \partial^\rho(F_{\rho\sigma}F^{\sigma\nu}) \\ &= \tfrac{1}{4}(\partial^\nu F_{\rho\sigma})F^{\rho\sigma} + \tfrac{1}{4}F_{\rho\sigma}(\partial^\nu F^{\rho\sigma}) + (\partial^\rho F_{\rho\sigma})F^{\sigma\nu} + F_{\rho\sigma}(\partial^\rho F^{\sigma\nu}) \\ &= \tfrac{1}{4}F_{\rho\sigma}(\partial^\nu F^{\rho\sigma}) + \tfrac{1}{4}F_{\rho\sigma}(\partial^\nu F^{\rho\sigma}) + \tfrac{1}{c}J_\sigma F^{\sigma\nu} + F_{\rho\sigma}\partial^\rho F^{\rho\sigma} \\ 2(\partial_\mu\Theta^{\mu\nu} - \tfrac{1}{c}J_\sigma F^{\sigma\nu}) &= F_{\rho\sigma}(\partial^\nu F^{\rho\sigma} + 2\partial^\rho F^{\rho\sigma})\end{aligned}$$

One can write Fs in terms of As, or write one of the $2F_{\rho\sigma}\partial^\rho F^{\sigma\nu}$ as:

$$F_{\rho\sigma}\partial^\rho F^{\sigma\nu} = -F_{\sigma\rho}\partial^\rho F^{\sigma\nu} = -F_{\rho\sigma}\partial^\sigma F^{\rho\nu} = F_{\rho\sigma}\partial^\sigma F^{\nu\rho}.$$

So:

$$\begin{aligned}2(\partial_\mu\Theta^{\mu\nu} - \tfrac{1}{c}J_\sigma F^{\sigma\nu}) &= F_{\rho\sigma}(\partial^\nu F^{\rho\sigma} + \partial^\rho F^{\sigma\nu} + \partial^\sigma F^{\nu\rho}) \\ &= 0,\end{aligned}$$

by Bianchi identity.

Hence,

$$\partial_\mu\Theta^{\mu\nu} = \tfrac{1}{c}J_\sigma F^{\sigma\nu},$$

Energy-momentum conservation for particles and fields in combination.

$\nu = 0$:

$$\begin{aligned}\partial_\mu\Theta^{\mu 0} &= \tfrac{1}{c}J_\mu F^{\mu 0} \\ \partial_0\Theta^{00} + \partial_i\Theta^{i0} &= \tfrac{1}{c}J_iF^{i0} \\ \frac{1}{c}\frac{\partial}{\partial t}U + \partial_i\left(\tfrac{1}{c}S^i\right) &= -\tfrac{1}{c}J^iE^i \\ \frac{\partial U}{\partial t} + \vec{\nabla}\cdot\vec{S} &= -\vec{J}\cdot\vec{E}.\end{aligned}$$

$\nu = j$:

$$\begin{aligned}
\partial_\mu \Theta^{\mu j} &= \tfrac{1}{c}(J_0 F^{0j} + J_i F^{ij}) \\
&= -\tfrac{1}{c} c\rho F^{j0} - \tfrac{1}{c} J^i F^{ij} \\
&= -\rho E^j + \tfrac{1}{c} J^i \epsilon^{ijk} B^k \\
&= -\rho E^j - \tfrac{1}{c} \epsilon^{jik} J^i B^k \\
\partial_\mu \Theta^{\mu j} &= \rho E^j - \tfrac{1}{c}(\vec{J} \times \vec{B}).
\end{aligned}$$

Compare this to:

$$\frac{d\vec{p}}{dt} = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B},$$

the Lorentz force of an electromagnetic field $\vec{E}$ and $\vec{B}$ on a charge q.

Define a force density f^μ:

$$\begin{aligned}
f^\mu &= \tfrac{1}{c} F^{\mu\sigma} J_\sigma \\
&= -\tfrac{1}{c} J_\sigma F^{\sigma\mu}.
\end{aligned}$$

A spatial integral of f^μ provides the force on the particles:

$$\int\limits_\Omega f^\nu \, d^3x = \sum_{n=1}^{N} \frac{dp_n^\nu}{dt},$$

where p_n^ν is the 4-momentum of the n^{th} particle.

The space integral of the energy-momentum density for fields and particles is:

$$\int\limits_\Omega \left(\partial_\mu \Theta^{\mu\nu} + f^\nu\right) d^3x = \frac{d}{dt}\left(p^\nu_{\text{fields}} + p^\nu_{\text{particles}}\right) = 0$$

Force $q(\vec{E} + \frac{1}{c}\vec{v} \times \vec{B})$ acts on a charge q. Work is done at a rate $\vec{F} \cdot \vec{v} = q\vec{E} \cdot \vec{v}$ [since $\vec{v} \cdot (\vec{v} \times \vec{B}) = 0$]

For a charge q with path $\vec{x}_n(t)$,

$$\vec{J} = \sum_n q_n \vec{v}_n \delta^3(\vec{x} - \vec{x}_n)$$

Rate of work done is the power:

$$\begin{aligned}
\int\limits_\Omega \vec{E} \cdot \vec{J} \, d^3x &= \sum_n \int\limits_\Omega \vec{E} \cdot q_n \vec{v}_n \delta^3(\vec{x} - \vec{x}_n) \, d^3x \\
&= \sum_n q_n \vec{E}(\vec{x}_n) \cdot \vec{v}_n
\end{aligned}$$

Where particles convert energy to the fields, one can show from Maxwell's equations (in 3D or 4D):

$$-\vec{E} \cdot \vec{J} = \frac{1}{2}\frac{\partial}{\partial t}(\vec{E}^2 + \vec{B}^2) + \vec{\nabla} \cdot (\vec{E} \times \vec{B}) = \frac{\partial U}{\partial t} + \vec{\nabla} \cdot \vec{S}.$$

Energy is imparted by particles (q_m) to the fields in a volume Ω at rate:

$$\begin{aligned}
-\int\limits_\Omega \vec{E} \cdot \vec{J} \, d^3x &= \int\limits_\Omega \frac{\partial U}{\partial t} d^3x + \int\limits_\Omega \vec{\nabla} \cdot \vec{S} \, d^3x \\
&= \frac{\partial}{\partial t} \int\limits_\Omega U \, d^3x + \int\limits_{\partial\Omega} \vec{S} \cdot d\vec{a}.
\end{aligned}$$