MA3432: Classical Electrodynamics

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Chapter 1

Solving electrodynamic equations

1.1 Aim

We seek solutions to the equation:

$$\partial_\mu F^{\mu\nu} = \tfrac{1}{c} J^\nu,$$

where $J^\nu(x^\rho) = (c\rho, \vec{J})$ is a given four-current as a function of spacetime. We need to solve this in terms of A_μ, where $F_{\mu\nu} = \partial_\mu A'_\nu - \partial_\nu A'_\mu$, using a gauge change: $A'_\mu = A_\mu + \partial^\mu \phi$ for some scalar field ϕ, to ensure that $\partial^\mu A_\mu = 0$. So we need $\partial^\mu A'_\mu - \partial^\mu \partial_\mu \phi = 0$, or in other words, $= \partial^\mu A'_\mu$.

$\Box\phi = \partial^\mu A'_\mu$ can be solved when given some function $A'_\mu(x^\rho)$. Solving $\Box\phi = f(x) = \partial^\mu A_\mu$ gives the required function $\phi(x^\rho)$.

Using $\phi(x^\rho)$, we have $\partial^\mu A_\mu = 0$, the Lorenz gauge condition. So:

$$\begin{aligned}
\partial^\mu F_{\mu\nu} &= \tfrac{1}{c} J_\nu \\
\partial^\mu(\partial_\mu A_\nu - \partial_\nu A_\mu) &= \tfrac{1}{c} J_\nu \\
\partial^\nu \partial_\mu A_\nu - \partial_\nu \partial^\mu A_\mu &= \tfrac{1}{c} J_\nu.
\end{aligned}$$

But $\partial^\mu A_\mu = 0$, so:

$$\Box A_\nu = \tfrac{1}{c} J_\nu.$$

Solving this d'Alembertian equation for $A_\nu(x^\rho)$ given $J_\nu(x^\rho)$ eventually provides the electromagnetic field $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

1.2 Green function

1.2.1 Definition

George Green introduced a method for solving equations of the form $\Box f = g$ for given g.

Define a Green function:

$$\begin{aligned}
\Box G &= \delta^4(x^\mu - x'^\mu) \\
&= \delta(x_0 - x'_0)\delta^3(\vec{x} - \vec{x}')
\end{aligned} \tag{1.1}$$

Write $z_\mu = x_\mu - x'_\mu$ so that $\frac{\partial}{\partial x^i} = \frac{\partial}{\partial z^i}$, treating x'_μ as constant.

Finding a function $G(z_\mu)$ which obeys $\Box_z G = \delta^4(z)$ enables a solution to be written as:

$$f(x_\mu) = \int d^4x'\, G(x_\mu - x'_\mu) g(x'_\mu).$$

The reason is that:

$$\Box f(x_\mu) = \int d^4x'\, \Box G(x_\mu - x'_\mu) g(x'_\mu).$$

But G has been constructed to obey (1.1), so:

$$\begin{aligned}\Box f(x_\mu) &= \int d^4x'\, \delta^4(x_\mu - x'_\mu) g(x'_\mu) \\ &= g(x_\mu),\end{aligned}$$

the equation we wished to solve.

We need to evaluate a Green function obeying $\Box G = \delta^4(x^\mu - x'^\mu)$. Using $g \to \frac{1}{c} J_\mu$ and $f \to A_\mu$, the four-potential is:

$$A_\mu(x^\rho) = \frac{1}{c} \int d^4x'\, G(x_\nu - x'_\nu) J_\mu(x'_\nu)$$

1.2.2 Calculation

We will study Laplace's equation $\nabla^2 G = \delta^3(\vec{x} - \vec{x}')$ first, and $\Box G$ later. We will use these definitions of the Fourier transform and its inverse:

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ikx}\, dx, \quad f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k) e^{ikx}\, dk$$

The FT of the δ function is:

$$\tilde{\delta}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(x) e^{-ikx}\, dx = \frac{1}{\sqrt{2\pi}},$$

which means that the δ function can be represented as:

$$\delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk,$$

generalising this to 3 dimensions:

$$\delta^3(\vec{x}) = \frac{1}{(2\pi)^3} \int d^3k\, e^{i\vec{k}\cdot\vec{x}} \tag{1.2}$$

$G(\vec{x})$ obeys:

$$\vec{\nabla}^2 G(\vec{x}) = \delta^3(\vec{x}) \tag{1.3}$$

and has FT:

$$\tilde{G}(\vec{k}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3x\, G(\vec{x}) e^{-i\vec{k}\cdot\vec{x}}, \quad G(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3k\, \tilde{G}(\vec{k}) e^{i\vec{k}\cdot\vec{x}}.$$

Substituting the above expression for $G(\vec{x})$ and (1.2) into (1.3) yields:

$$\vec{\nabla}^2 \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3k\, \tilde{G}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3k\, e^{i\vec{k}\cdot\vec{x}}.$$

Taking everything inside the integrals and noting that $\vec{\nabla}^2 e^{i\vec{k}\cdot\vec{x}} = -e^{i\vec{k}\cdot\vec{x}}$, this becomes:

$$\int_{-\infty}^{\infty} -d^3k\, \tilde{G}(\vec{k}) \vec{k}^2 e^{i\vec{k}\cdot\vec{x}} = \int_{-\infty}^{\infty} \frac{1}{2\pi} d^3k\, e^{i\vec{k}\cdot\vec{x}}$$

This is only true for all x if the integrands agree, thus:

$$\tilde{G}(\vec{k}) = -\frac{1}{(2\pi)^{3/2}} \frac{1}{\vec{k}^2}.$$

Hence,

$$\begin{aligned} G(\vec{x}) &= \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} d^3k\, \tilde{G}(\vec{k}) e^{i\vec{k}\cdot\vec{x}} \\ &= \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} d^3k \frac{1}{\vec{k}^2} e^{i\vec{k}\cdot\vec{x}}. \end{aligned}$$

To evaluate this integral, first note that:

$$\begin{aligned} \int_{-\infty}^{\infty} d^3k\, \frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2} = I &= -\int_{-\infty}^{\infty} d^3k \int_0^{\infty} d\alpha\, e^{-\alpha\vec{k}^2} e^{i\vec{k}\cdot\vec{x}} \\ &= \int_0^{\infty} \int_{-\infty}^{\infty} d^3k\, e^{-\vec{p}^2} e^{i\vec{x}^2/4\alpha} \end{aligned}$$

where $\vec{p} \equiv \sqrt{\alpha}\vec{k} - \frac{i}{2\sqrt{\alpha}}\vec{x}$. Since $dp^i = \sqrt{\alpha}\, dk^i$, this gives:

$$\begin{aligned} I &= \int_0^{\infty} d\alpha\, \left(\frac{1}{\sqrt{\alpha}}\right)^3 \int_{-\infty}^{\infty} d^3p\, e^{-\vec{p}^2} e^{-\vec{x}^2/4\alpha} \\ &= \int_0^{\infty} d\alpha\, \alpha^{-3/2} (\sqrt{\pi})^3 e^{-4\vec{x}^2/4\alpha} \end{aligned}$$

Letting $\beta^2 = \frac{1}{4\alpha}$, this gives:

$$\begin{aligned} I &= \int_{\infty}^{0} (2\beta)^3 \pi^{3/2} e^{-\beta^2\vec{x}^2} (-\tfrac{1}{2}\beta^{-3}\, d\beta) \\ &= \int_0^{\infty} 4\pi^{3/2} e^{-\beta^2\vec{x}^2} d\beta \\ &= 4\pi^{3/2} \left(\frac{1}{2}\sqrt{\pi}\right) \sqrt{\frac{1}{\vec{x}^2}} \\ &= \frac{2\pi^2}{|\vec{x}|} \end{aligned}$$

Finally,

$$G(\vec{x}) = -\frac{1}{(2\pi)^3}\int d^3k\,\frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2} = -\frac{1}{4\pi\,|\vec{x}|}.$$

1.2.3 Example

For a static system of charges, ME1 states that $\vec{\nabla}\cdot\vec{E} = \rho$. But recall:

$$\begin{aligned}\vec{E} &= -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\\ &= -\vec{\nabla}\Phi \text{ for time-independent fields.}\end{aligned}$$

So ME1 reads $-\vec{\nabla}(\vec{\nabla}\Phi) = \rho$, i.e. $\vec{\nabla}^2\Phi = -\rho$.

As we have discovered, this has solution

$$\begin{aligned}\Phi(\vec{x}) &= -\int G(\vec{x}-\vec{x}')\rho(\vec{x}')\,d^3x'\\ &= \frac{1}{4\pi}\int\frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'.\end{aligned}$$

For a single charge, $\rho(\vec{x}') = q\delta^3(\vec{x}'-\vec{x}_q)$, so:

$$\Phi(\vec{x}) = \frac{q}{4\pi}\int\frac{\delta^3(\vec{x}'-\vec{x}_q)}{|\vec{x}-\vec{x}'|}d^3x' = \frac{q}{4\pi}\frac{1}{|\vec{x}-\vec{x}_q|},$$

the expected potential function.

For many charges q_i, $\Phi(\vec{x})$ is simply a sum over i of $\vec{x}_{q_i}$ terms.

1.3 Time dependence in electromagnetic equations

In 4D, Maxwell's equations read $\partial_\mu F^{\mu\nu} = \frac{1}{c}J^\nu$, or, using the definition of $F^{\mu\nu}$, $\partial_\mu\partial^\mu A^\nu - \partial^\nu\partial_\mu A^\mu = \frac{1}{c}J^\nu$. In the Lorenz gauge, $\partial_\mu A^\mu = 0$, so this simplifies to:

$$\Box A^\nu = \tfrac{1}{c}J^\nu.$$

We seek a 4D Green function $D(z^\rho)$, where $z^\rho = x^\rho - x'^\rho$, such that:

$$\Box_z D = \delta^4(z^\rho). \tag{1.4}$$

Using Fourier transforms yields:

$$D(z) = \left(\frac{1}{\sqrt{2\pi}}\right)^4\int d^4k\,\tilde{D}(k)e^{-ik_\nu z^\nu},$$

denoting z^ρ by z and k^ρ by k. Substitute this expression in (1.4):

$$\partial_\mu\partial^\mu\frac{1}{4\pi}\int_{-\infty}^{\infty} d^4k\,\tilde{D}(k)e^{-ik_\nu z^\nu} = \frac{1}{(2\pi)^4}\int_{-\infty}^{\infty} d^4k\,e^{-ik_\nu z^\nu}$$

Noting that $\frac{\partial}{\partial z^\mu} = \partial_\mu$, we move the $\partial_\mu \partial^\mu$ inside the integral and multiply by 2π, giving:

$$-\int d^4k\, \tilde{D}(k) k_\mu k^\mu e^{-ik_\nu z^\nu} = \frac{1}{(2\pi)^2} \int d^4k\, e^{-ik_\nu z^\nu}$$

For this to be true for all z, the integrands must match:

$$\tilde{D}(k) = -\frac{1}{4\pi^2} \frac{1}{k_\mu k^\mu}.$$

So our 4D Green function is:

$$D(z) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} d^4k \tilde{D}(k) e^{ik\cdot z}.$$

Here $k \cdot z = k_0 z^0 - \vec{k} \cdot \vec{z}$, the 4D scalar product, so:

$$\begin{aligned} D(z) &= -\frac{1}{(2\pi)^4} \int d^4k \frac{e^{-ik_\nu z^\nu}}{k_\mu k^\mu} \\ &= -\frac{1}{(2\pi)^4} \int d^3k\, e^{i\vec{k}\cdot\vec{z}} \int dk_0 \frac{e^{-ik_0 z^0}}{k_0^2 - k^2} \end{aligned}$$

where k now refers to $|\vec{k}|$.

1.3.1 First integral

The dk_0 integral can be performed using a countour method, treating k_0 as a complex variable. To arrange a decreasing exponential, we need $z^0 < 0$ if $\operatorname{Im} k_0 > 0$, and vice-versa. We want $t' \leqslant t$, i.e. cause before effect, and since $z_0 = c(t - t')$, this means $z_0 \geqslant 0 \Rightarrow \operatorname{Im} k_0 \leqslant 0$.

Since the integrand has poles at $\pm k$, we will evaluate the integral

$$\int_{-\infty}^{\infty} dk_0 \frac{e^{-ik_0 z^0}}{(k_0 + k)(k_0 - k)}$$

using this contour:

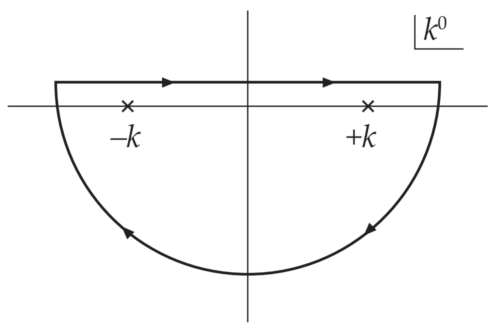

Figure 1.1: The contour used to evaluate the integral

The residues are:

$$\operatorname*{Res}_{k_0 = k} \frac{e^{-ik_0 z^0}}{(k_0 + k)(k_0 - k)} = \frac{e^{-kz^0}}{2k}, \quad \operatorname*{Res}_{k_0 = -k} \frac{e^{-ik_0 z^0}}{(k_0 + k)(k_0 - k)} = -\frac{e^{ikz_0}}{2k}$$

Ignoring the semicircle for now, Cauchy's theorem gives:

$$\int_{-\infty}^{\infty} dk_0 \, \frac{e^{-ik_0 z^0}}{k_0^2 - \vec{k}^2} = -2\pi i \left(\frac{e^{-ikz^0}}{2k} - \frac{e^{ikz_0}}{2k} \right)$$
$$= \frac{\pi i}{k}(e^{ikz^0} - e^{-ikz^0})$$
$$= -\frac{2\pi}{k}\theta(z^0)\sin(kz^0)$$

where θ is the Heaviside step function, which distinguishes the retarded and advanced Green functions.

As for the semicircle, we can write:

$$k_0 = Re^{-i\phi} = R(\cos\phi - i\sin\phi), \quad dk_0 = -iRe^{-i\phi}d\phi$$

So the integral over the semicircle is:

$$\int_S dk_0 \frac{e^{-ik_0 z^0}}{k_0^2 - \vec{k}^2} = -i\int_0^{\pi} d\phi \frac{Re^{-Rz^0 \sin\phi}e^{i\alpha}}{R^2 e^{-2\pi} - k^2}.$$

This tends to 0 as $R \to \infty$, since $\sin\phi \geqslant 0$ on $[0, \pi]$, $\frac{R}{R^2} \to 0$, and $e^{i\alpha}$ is a phase factor of unit length. So the contribution from the semicircle can indeed be ignored.

1.3.2 Second integral

We now have:

$$D_{\text{ret}}(z) = -\frac{1}{(2\pi)^4}\int d^3k \, e^{i\vec{k}\cdot\vec{z}}\left(-\frac{2\pi}{k}\theta(z^0)\sin(kz^0)\right)$$

We use spherical coordinates, and align the 3$^{\text{rd}}$ axis of $\vec{k}$ along $\vec{z}$, so that $\vec{k}\cdot\vec{z} = kz\cos\theta$, where θ is the angle between $\vec{k}$ and $\vec{z}$.

Since $d^3k = k^2\,dk\,\sin\theta\,d\theta\,d\phi$,

$$D_{\text{ret}}(z^\rho) = \frac{\theta(z^0)}{(2\pi)^3}\int_0^{\infty} k^2\,dk \int_0^{\pi}\sin\theta\,d\theta \int_0^{2\pi} d\phi \frac{e^{ikz\cos\theta}}{k}\sin(kz^0).$$

Letting $y = \cos\theta$,

$$\begin{aligned}
D_{\text{ret}}(z^\rho) &= \frac{\theta(z^0)}{(2\pi)^2}\int_0^\infty k\,dk \int_1^{-1}(-dy)e^{ikzy}\sin(kz^0) \\
&= \frac{\theta(z^0)}{(2\pi)^2}\int_0^\infty k\,dk\,\sin(kz^0)\int_{-1}^{1}dy\,e^{ikzy} \\
&= \frac{\theta(z^0)}{(2\pi)^2}\int_0^\infty dk\left(\frac{e^{ikz^0}-e^{-ikz^0}}{2i}\right)\frac{1}{iz}(e^{ikz}-e^{-ikz}) \\
&= -\frac{\theta(z^0)}{8\pi^2 z}\int_0^\infty dk\left(e^{ik(z^0+z)}+e^{-ik(z^0+z)}-e^{ik(z^0-z)}-e^{-ik(z^0-z)}\right) \\
&= -\frac{\theta(z^0)}{8\pi^2 z}\int_{-\infty}^\infty dk\left(e^{ik(z^0+z)}-e^{-ik(z^0-z)}\right) \\
&= -\frac{\theta(z^0)}{8\pi^2 z}\left(2\pi\delta(z^0+z)-2\pi\delta(z^0-z)\right),
\end{aligned}$$

by the integral definition of the δ function.

Recall that $z = |\vec{z}| \geqslant 0$, and $z^0 > 0$, so $\delta(z^0 + z) = 0$. Therefore:

$$D_{\text{ret}}(z^\rho) = \frac{\theta(z^0)}{4\pi z}\delta(z^0 - z)$$

1.3.3 Covariant form

Note that $\delta(z^0 - z)$ changes from frame to frame, so D_{ret} does too. Is there a Lorentz scalar form? We need the fact that:

$$\delta(ab) = \frac{\delta(a)}{|b|} + \frac{\delta(b)}{|a|}$$

Recalling that $z^\mu z_\mu = z_0^2 - z^2 = (z_0 - z)(z_0 + z)$, this means that:

$$\delta(z_\mu z^\mu) = \frac{\delta(z_0 - z)}{|z_0 + z|} + \frac{\delta(z_0 + z)}{|z_0 - z|}$$

But as above, $\delta(z^0 + z) = 0$, so:

$$\delta(z_\mu z^\mu) = \frac{\delta(z^0 - z)}{|z^0 + z|}.$$

Since $\delta(z^0 - z) = 0$ except at $z = z^0$, so this can be written:

$$\delta(z_\mu z^\mu) = \frac{\delta(z^0 - z)}{2z}.$$

Note that $z_\mu z^\mu$ is a covariant Lorentz scalar.

So a covariant expression for D_{ret} is:

$$D_{\text{ret}}(z^\rho) = \frac{\theta(z^0)}{2\pi}\delta(z_\mu z^\mu)$$

1.4 Solving Maxwell's equations

Recall that we initially defined D so that:

$$A_\nu(x_\mu) = \frac{1}{c}\int d^4x'\, D(x_\mu - x'_\mu) J_\nu(x'_\mu)$$

and that $J_\nu = (c\rho, \vec{J})$ for a single charge q located at $\vec{x}_q(t)$ is:

$$J_\nu(x'_\mu) = q\frac{dx'_\nu}{dt'}\delta^3(\vec{x}' - \vec{x}_q(t)).$$

Taking a single charge q and separating A_ν into time and space integrals,

$$\begin{aligned} A_\nu(x_0, \vec{x}) &= \int dt' \int d^3x'\, D(x_0 - x'_0, \vec{x} - \vec{x}')q\frac{dx'_\nu}{dt'}\delta^3(\vec{x}' - \vec{x}_q(t')) \\ &= q\int^t dt'\, D(x_0 - x'_0, \vec{x} - \vec{x}_q)\frac{dx_q}{dt'} \end{aligned}$$

Here,

$$D = \frac{\theta(t-t')}{4\pi|\vec{x} - \vec{x}_q(t')|}\delta\left(ct - ct' - |\vec{x} - \vec{x}_q(t')|\right)$$

Introduce $\vec{R}(t')$, the location of the observation point $\vec{x}$ relative to $\vec{x}_q(t')$:

$$\begin{aligned} \vec{R}(t') &= \vec{x} - \vec{x}_q(t') \\ R^\mu(t') &= x^\mu - x^\mu_q(t'). \end{aligned}$$

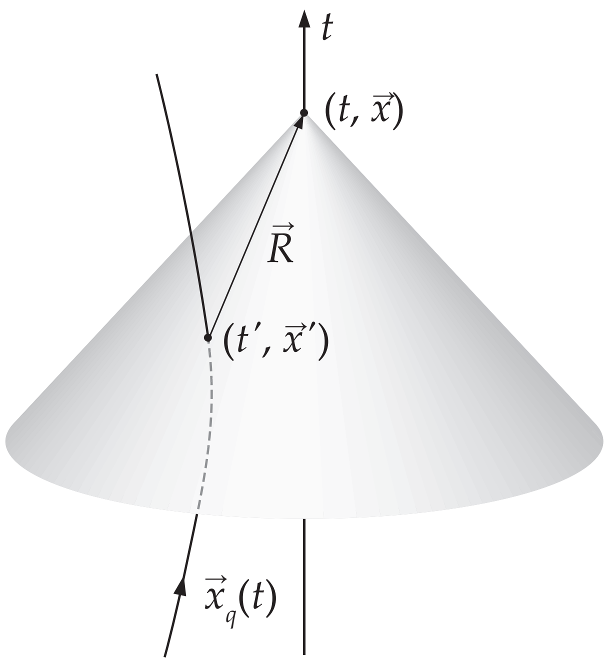

Figure 1.2: The past light-cone of $(t, \vec{x})$

The four-potential can now be written:

$$A_\nu(x_0, \vec{x}) = \frac{q}{4\pi}\int dt' \frac{\theta(t-t')}{R(t')}\delta\left(ct' - ct - R(t')\right)\frac{dx^\nu_q}{dt'}$$

We want to find $\delta\left(ct' - ct - R(t')\right)$, and for this we need:

$$\delta(f(x)) = \sum_i \frac{\delta(x - x_i)}{|f'(x_i)|},$$

where x_i are the simple zeroes of $f(x)$, i.e. where $f(x) = 0$ and $f'(x) \neq 0$.

There is only one root of $f(t')$, and

$$\frac{df}{dt'} = c - 0 - \frac{dR}{dt'}.$$

We also have:

$$\begin{aligned} R^2 = \vec{R}^2 \Rightarrow 2R\frac{dR}{dt'} &= 2\vec{R}\frac{d\vec{R}}{dt'} \\ \Rightarrow \frac{dR}{dt'} &= \frac{\vec{R}}{R} \cdot \frac{d\vec{R}}{dt'}, \end{aligned}$$

giving:

$$\frac{df}{dt'} = c + \frac{\vec{R}}{R} \cdot \vec{v},$$

where $\vec{v} = \frac{d\vec{x}_q}{dt}$, the velocity of q. Defining $\hat{n}$ as $\frac{\vec{R}}{R}$, we have:

$$\frac{df}{dt'} = c - \hat{n} \cdot \vec{v}.$$

$|\hat{n}| = 1$ and $|\vec{v}| < c$, so $\frac{df}{dt'} > 0$. Hence:

$$\left|\frac{df}{dt'}\right| = \frac{df}{dt'}$$

So finally,

$$\delta(f(t')) = \frac{\delta(t - t_0)}{c - \hat{n} \cdot \vec{v}},$$

where t_0 is the location of the root of f. This means that:

$$\begin{aligned} ct - ct_0 - R(t_0) &= 0 \\ \Rightarrow c^2(t_0 - t)^2 &= (\vec{x} - \vec{x}_q(t_0))^2 \end{aligned}$$

Or in other words,

$$R^\mu R_\mu = 0.$$

This equation describes events on the light-cone with apex $(t, \vec{x})$ as seen in Fig. 1.2.

So we can now write the four-potential as:

$$\begin{aligned} A^\nu(t, \vec{x}) &= \frac{q}{4\pi} \int dt' \frac{dx_q^\mu}{dt'} \frac{\theta(t - t')}{R(t')} \frac{\delta(t' - t_0)}{c - \hat{n} \cdot \vec{v}} \\ &= \frac{q}{4\pi} \frac{dx_q^\nu}{dt} \frac{1}{R(t_0)(c - \hat{n} \cdot \vec{v})} \end{aligned}$$

Splitting this into components, we obtain:

$$\Phi(t,\vec{x}) = \frac{q}{4\pi}\frac{c}{cR - \vec{v}\cdot\vec{R}}\bigg|_{t_0}$$
$$\vec{A}(t,\vec{x}) = \frac{q}{4\pi}\frac{\vec{v}}{cR - \vec{v}\cdot\vec{R}}\bigg|_{t_0}$$

These are called the **Linard–Wiechert potentials.**

In covariant form, where four-velocity $V^\mu = \gamma(c,\vec{v})$,

$$V_\sigma(x^\sigma - x_q^\sigma) = \gamma(V_0 R^0 - \vec{V}\cdot\vec{R}).$$

Here $R^0 = x^0 - x_q^0 = R$. We can now write:

$$A^\mu(t,\vec{x}) = \frac{q}{4\pi}\frac{\gamma(c,\vec{v})}{\gamma(cR - \vec{v}\cdot\vec{R})}\bigg|_{t_0} = \frac{q}{4\pi}\frac{V^\mu}{V_\sigma R^\sigma}\bigg|_{\tau_p}$$

where τ_p refers to the *past* light-cone. Note that τ_0 obeys $R^\mu R_\mu = 0$, i.e.

$$(x^\sigma - x_q^\sigma(\tau_0))(x_\sigma - x_{q\,\sigma}(\tau_0)) = 0.$$

All this means that the electromagnetic potential $A^\mu(x^\rho)$ receives a contribution from one event only: $x_q^\mu(\tau_0)$, the spacetime location of the charge q as it passes through the past light-cone of the event $x^\mu = (ct,\vec{x})$ where the radiation is observed.

The cause at $(ct',\vec{x}')$ produces an effect at x^μ at the limiting speed c.

1.4.1 Static limit

We have:

$$4\pi\Phi = \frac{q}{R - \vec{\beta}\cdot\vec{R}}\bigg|_{\tau_p}$$

As $\vec{v} = c\vec{\beta} \to 0$,

$$\Phi \to \frac{q}{4\pi R},$$

the expected electrostatic potential.

We also have:

$$4\pi\vec{A} = \frac{q\vec{v}}{cR - \vec{v}\cdot\vec{R}}\bigg|_{\tau_p}$$

As $\vec{v} \to 0$, $A \to 0$. So a charge at rest produces no magnetic field.

Chapter 2

Radiated fields

2.1 Introduction

In order to find the radiated fields from a charge q, we need to find $\partial^\mu A^\nu$. This invites consideration of x^μ and $x^\mu + \Delta x^\mu$, as in Fig. 2.1.

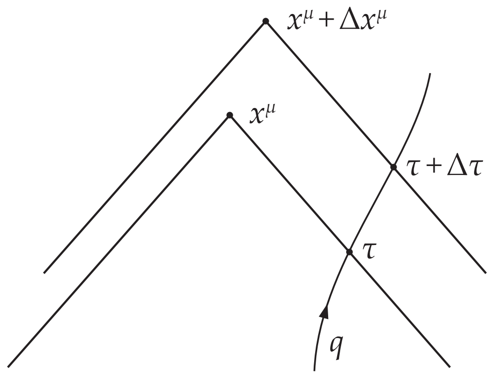

Figure 2.1: Light-cones of x^μ and $x^\mu + \Delta x^\mu$

Using the expression for A^ν which we calculated, we have:

$$\begin{aligned}\frac{4\pi}{q}\partial^\mu A^\nu &= \partial^\mu\left(\frac{V^\nu}{R^\sigma V_\sigma}\right) \\ &= \frac{\partial^\mu V^\nu}{R^\sigma V_\sigma} - \frac{V^\nu}{(R_\sigma V_\sigma)^2}(R_\rho \partial^\mu V^\rho + V^\rho \partial^\mu R_\rho). \end{aligned} \tag{2.1}$$

So we need to determine $\partial^\mu V^\nu$, and $\partial^\mu R^\sigma$.

Recall first that:

$$\frac{4\pi}{q}A^\nu = \frac{V^\nu}{R^\sigma V_\sigma}, \quad \text{and} \quad R^\sigma R_\sigma = 0.$$

Consider R^σ as a function of x^ρ and τ:

$$R^\sigma(x^\rho, \tau) = x^\sigma - x_q^\sigma(\tau) \tag{2.2}$$

Since q's path intersects the past light-cone of x^ρ *once*, we can regard the invariant time τ as $\tau(x^\rho)$.

From (2.2) we have:

$$\partial_\mu R^\sigma = \frac{\partial R^\sigma}{\partial x^\mu} = \frac{\partial x^\sigma}{\partial x^\mu} - \frac{\partial x_q^\sigma}{\partial \tau}\frac{\partial \tau}{\partial x^\mu}$$
$$= \delta^\sigma_\mu - V^\sigma \partial_\mu \tau. \tag{2.3}$$

We know that $R^\sigma R_\sigma = 0$, therefore:

$$\partial_\mu (R^\sigma R_\sigma) = (\partial_\mu R^\sigma) R_\sigma + R^\sigma \partial_\mu R_\sigma$$
$$= 2R_\sigma \partial_\mu R^\sigma = 0.$$

So (2.3) gives:

$$R_\sigma(\delta^\sigma_\mu - V^\sigma \partial_\mu \tau) = 0.$$

Hence,

$$\partial_\mu \tau = \frac{R_\mu}{R^\sigma V_\sigma}.$$

We can now find $\partial^\mu V^\nu$:

$$\partial^\mu V^\nu = \frac{\partial V^\nu}{\partial x_\mu} = \frac{\partial V^\nu}{\partial \tau}\frac{\partial \tau}{\partial x_\mu}$$
$$= \dot{V}^\nu \frac{R^\mu}{R^\sigma V_\sigma}$$
$$= \frac{R^\mu \dot{V}^\nu}{R^\sigma V_\sigma} \tag{2.4}$$

Substituting (2.4) into (2.1) yields:

$$\frac{4\pi}{q}\partial^\mu A^\nu = \frac{R^\mu \dot{V}^\nu}{(R^\sigma V_\sigma)^2} - \frac{V^\nu}{(R^\sigma V_\sigma)^2}\left[R_\rho \frac{R^\mu \dot{V}^\rho}{R^\lambda V_\lambda} + \left(\delta^\mu_\rho - V_\rho \frac{R^\mu}{R^\lambda V_\lambda}\right) V^\rho\right]$$
$$= \frac{R_\mu \dot{V}^\nu}{(R^\sigma V_\sigma)^2} - \frac{R_\rho \dot{V}^\nu}{(R^\sigma V_\sigma)^3} R^\mu V^\nu - \frac{V^\mu V^\nu}{(R^\sigma V_\sigma)^2} + \frac{V_\rho V^\rho}{(R^\sigma V_\sigma)^3} R^\mu V^\nu.$$

The third term is symmetric in $\mu \leftrightarrow \nu$ and so disappears in the expression for $F^{\mu\nu}$. Also, $V_\rho V^\rho = \gamma^2(c^2 - \vec{v}^2) = \gamma^2 c^2 \gamma^{-2} = c^2$.

Splitting $F^{\mu\nu}$ into components, we have:

$$4\pi F^{\mu\nu}_{\text{acc}} = \frac{q}{(R^\sigma V_\sigma)^2}(R^\mu \dot{V}^\nu - R^\nu \dot{V}^\mu) - \frac{q R^\rho \dot{V}_\rho}{(R^\sigma V_\sigma)^3}(R^\mu V^\nu - R^\nu V^\mu)$$
$$4\pi F^{\mu\nu}_{\text{vel}} = \frac{qc^2}{(R^\sigma V_\sigma)^3}(R^\mu V^\nu - R^\nu V^\mu).$$

Both terms in $F^{\mu\nu}_{\text{acc}}$ include $\dot{V}^\mu$, the four-acceleration.

2.1.1 Comments

- $F^{\mu\nu}_{\text{acc}}$ has $\dot{V}^\mu$ in every term, so if the charge moves at constant velocity, $F^{\mu\nu}_{\text{acc}} = 0$. A charge at constant velocity does not radiate.

- By contrast, $F^{\mu\nu}_{\text{vel}}$ has no terms with $\dot{V}^\mu$ and so is non-zero in general. For $\vec{v} = 0$, we see:

$$E^i_{\text{vel}} = \frac{qn^i}{4\pi R^2}, \quad \text{i.e.} \quad \vec{E}_{\text{vel}} = \frac{q\vec{R}}{4\pi R^3},$$

  the expected inverse square electric field. In fact, it is the static field boosted to a constant velocity $\vec{v}$.

- Recall that $R^\mu = (R, R\vec{n})$ where $R^0 = R$ due to $R_\mu R^\mu = 0$, and $\vec{R} = R\vec{n}$. So $F^{\mu\nu}_{\text{acc}}$ has the form $\frac{1}{r}$ as $R \to \infty$ (the typical R-dependence of a radiative long-range field.

- By contrast, $F^{\mu\nu}_{\text{vel}}$ has R-dependence $\frac{1}{r^2}$.

2.1.2 Orthogonality properties

Consider the dual field $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$ and the contraction $2R_\mu\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}R_\mu F_{\rho\sigma}$. (Note that every term in $F^{\mu\nu}$ has an R^μ four-vector. So $2R_\mu\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma}R_\mu R_\rho[\ldots]$.)

Since $\epsilon^{\mu\nu\rho\sigma} = -\epsilon^{\rho\nu\mu\sigma}$ but $R_\mu R_\rho = R_\rho R_\mu$, summing over μ and ρ yields zero. So $R_\mu\tilde{F}^{\mu\nu} = 0$ for both 'acc' and 'vel' elements.

With $R_\mu = (R_0, -R\vec{n})$, we have:

$$R_\mu\tilde{F}^{\mu\nu} = R_0\tilde{F}^{0i} + R_j\tilde{F}^{ji} \quad \text{for } \nu = i.$$

So $0 = -R\tilde{F}^{i0} + R_j\epsilon^{ijk}E^k$ or:

$$RB^i = \epsilon^{ijk}Rn^jE^k$$

and since $R \neq 0$,

$$B^i = \epsilon^{ijk}n^jE^k.$$

In vector form,

$$\vec{B} = \vec{n} \times \vec{E}$$

(in Heaviside–Lorentz units.) This is true for $\vec{B}_{\text{rel}}$ *and* $\vec{B}_{\text{acc}}$.

Is $\vec{n} \perp \vec{E}$? Study $R_\mu F_{\text{rad}}\mu\nu$.

Recall:

$$\begin{aligned} F^{i0} &= E^i, & F^{ij} &= -\epsilon^{ijk}B^k \\ \tilde{F}^{i0} &= B^i, & \tilde{F}^{ij} &= \epsilon^{ijk}E^k \end{aligned}$$

So:

$$\begin{aligned} R_\mu 4\pi(R^\sigma V_\sigma)^3 q^{-1}F^{\mu\nu}_{\text{rad}} &= (R^\sigma V_\sigma)(R_\mu R^\mu\dot{V}^\nu - R_\mu R^\nu\dot{V}^\mu) - (R^\rho\dot{V}_\rho)(R_\mu R^\mu V^\nu - R_\mu R^\nu V^\mu) \\ &= (R^\sigma V_\sigma)R_\mu R^\nu\dot{V}^\mu + (R^\rho\dot{V}_\rho)R_\mu R^\nu V^\mu \qquad = 0 \end{aligned}$$

So $R_\mu F^{\mu\nu}_{\text{rad}}$.

$\nu = 0:$

$$\begin{aligned} R_i F^{i0}_{\text{rad}} &= 0 \\ -R_i E^i_{\text{rad}} &= 0 \\ Rn^i E^i_{\text{rad}} &= 0 \\ n^i E^i_{\text{rad}} &= 0 \end{aligned}$$

or

$$\vec{n} \cdot \vec{E}_{\text{rad}} = 0.$$

In other words, $\vec{n} = \frac{\vec{R}}{R}$ is $\perp$ $\vec{E}_{\text{rad}}$.

$\nu = j$:

$$R_\mu F^{\mu j}_{\text{rad}} = 0$$

leads to:

$$\vec{E}_{\text{rad}} = \vec{B}_{\text{rad}} \times \vec{n}.$$

Summary:

$$R^\mu \tilde{F}_{\mu\nu} = 0 : \begin{cases} \nu = 0 : & \vec{n} \cdot \vec{B} = 0 \\ \nu = i : & \vec{B} = \vec{n} \times \vec{E} \end{cases}$$

$$R_\mu F^{\mu\nu}_{\text{rad}} = 0 : \begin{cases} \nu = 0 : & \vec{n} \cdot \vec{E}_{\text{rad}} = 0 \\ \nu = i : & \vec{E}_{\text{rad}} = \vec{B}_{\text{rad}} \times \vec{n} \end{cases}$$

2.2 Wave-like solutions

$\Box A^\mu = \frac{1}{c} J^\mu$ has homogeneous solutions where $\Box A^\nu = 0$, namely:

$$A^\nu_{\text{hom}} = A^\nu_0 e^{-ik_\mu x^\mu}. \tag{2.5}$$

Substitute:

$$A^\nu_0 (-ik_\mu)(-ik_\mu) e^{-ik_\mu x^\mu} = 0,$$

where A^ν_0 are four constants.

For (2.5) to obey $\Box A^\nu = 0$, we must have $k_\mu k^\mu = 0$, or in other words $(k_0)^2 = (\vec{k})^2$. Homogeneous solutions are waves:

$$A^\nu_{\text{hom}} = A^\nu_0 e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

Such A^ν_{hom} may be added to a non-homogeneous solution of $\Box A^\nu = \frac{1}{c} J^\nu$:

$$A^\nu = A^\nu_{\text{hom}} + A^\nu_{\text{non-hom}}.$$

Does $A^\nu_{\text{non-hom}}$ obey the Lorenz condition ($\partial_\nu A^\nu = 0$)? Recall:

$$A^\nu(x_\rho) = \frac{1}{c} \int D_{\text{ret}}(x_\rho - x'_\rho) J^\nu(x'_\rho) \, d^4 x'$$

and note that:

$$\partial_\nu D_{\text{ret}}(x - x') = -\partial'_\nu D_{\text{ret}}(x - x').$$

So:

$$\begin{aligned} \partial_\nu A^\nu &= \frac{1}{c} \int d^4 x' \, \partial_\nu D(x - x') J^\nu(x') \\ &= -\frac{1}{c} \int d^4 x' \, \left[\partial'_\nu (x - x') \right] J^\nu(x'). \end{aligned}$$

Using integration by parts:

$$\partial_\nu A^\nu = \frac{1}{c}\int d^4x'\,\partial'_\nu J^\nu(x') D_{\text{ret}}(x-x') - \frac{1}{c}\int_\Omega d^4x'\,\partial_\nu[D_{\text{ret}} J_\nu(x)].$$

Recall that $\partial_\nu J^\nu(x') = 0$:

$$\partial_\nu A^\nu = 0 - \frac{1}{c}\int_{\partial\Omega} D_{\text{ret}}(x-x') J^\nu(x')\,d^3\Sigma_\nu$$

where Σ_ν is the directed 'area' element on the boundary $\partial\Omega$ of the region Ω. For large Ω, the boundary integral $= 0$. So $\partial_\nu A^\nu = 0$ for the solution obtained, recalling that $\partial_\mu F^{\mu\nu} = 0$ becomes:

$$\begin{aligned}
\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) &= \tfrac{1}{c} J^\nu \\
\Box A^\nu - \partial^\nu(\partial_\mu A^\mu) &= \tfrac{1}{c} J^\nu \\
\Box A^\nu &= \tfrac{1}{c} J^\nu
\end{aligned}$$

if $\partial_\mu A^\mu = 0$.

2.2.1 Static solution

We let $c \to \infty$. Recall $R_\mu R^\mu = 0$, therefore $R_0^2 = R^2$, $\Rightarrow R_0 = R$.

$$R^\mu = (R^0, \vec{R}) = (R, R\vec{n}) = R(1, \vec{n}).$$

Recall

$$A^\nu(x) = \frac{1}{c}\int d^4x'\,D_{\text{ret}}(x-x') J^\nu(x'),$$

$$D_{\text{ret}}(x-x') = \frac{\theta(x_0 - x_0')}{4\pi R}\delta(x_0 - x_0' - R)$$

$t - t' = \frac{1}{c}R$ from δ. As $c \to \infty$, $t - t' \to 0$, i.e. $t \to t'$. So the solution reverts to 'action at a distance':

$$A^i = \frac{1}{c}\int d^3x'\,\frac{J^i(x^0, \vec{x})}{|\vec{x} - \vec{x}'|}$$

2.2.2 Velocity field with $\vec{E}$ and $\vec{B}$

$$4\pi F^{\mu\nu}_{\text{rel}} = \frac{qc^2(R^\mu V^\nu - R^\nu V^\mu)}{(R^\sigma V_\sigma)^3}$$

Recall $R^\sigma = R(1, \vec{n})$.

$$V^\sigma = c\gamma(1, \vec{\beta}) = \gamma(c, \vec{v}).$$

We need:

$$\begin{aligned}
R^\sigma V_\sigma &= R^0 V^0 - R^i V^i \\
&= c\gamma R - c\gamma R n^i \beta^i \\
&= c\gamma R(1 - \vec{n}\cdot\vec{\beta}).
\end{aligned}$$

For the electric velocity field, we need $\mu = i$ and $\nu = 0$.

$$\begin{aligned}\frac{4\pi}{qc^2}F^{i0}_{\text{vel}} &= \frac{R^i V^0 - R^0 V^i}{c^3\gamma^3 R^3(1-\vec{n}\cdot\vec{\beta})^3} \\ &= \frac{c\gamma R(n^i - \beta^i)}{c^3\gamma^3 R^3(1-\vec{n}\cdot\vec{\beta})^3}\end{aligned}$$

$$E^i_{\text{vel}} = \frac{q}{4\pi R^2\gamma^2}\frac{n^i-\beta^i}{(1-\vec{n}\cdot\vec{\beta})^3}, \quad \text{i.e.} \quad \vec{E}_{\text{vel}} = \frac{q(\vec{n}-\vec{\beta})}{4\pi R^2\gamma^2(1-\vec{n}\cdot\vec{\beta})^3}$$

At rest, when $\vec{\beta} = 0$, this reverts to:

$$\vec{E}_{\text{vel}} = \frac{q\vec{n}}{4\pi R^2},$$

the expected inverse square law.

We know that $\vec{B} = \vec{n}\times\vec{E}$ for both 'vel' and 'rad', so:

$$\begin{aligned}\vec{B}_{\text{vel}} &= \vec{n}\times\left(\frac{q}{4\pi R^2\gamma^2}\frac{\vec{n}-\vec{\beta}}{(1-\vec{n}\cdot\vec{\beta})^3}\right) \\ &= -\frac{q}{4\pi R^2\gamma^2}\left.\frac{\vec{n}\times\vec{\beta}}{(1-\vec{n}\cdot\vec{\beta})^3}\right|_{t_0}\end{aligned}$$

For $\vec{\beta} = 0$, $\vec{B}_{\text{vel}} = 0$ as expected.

Now,

$$\frac{4\pi}{q}F^{\mu\nu}_{\text{rad}} = \frac{R^\mu\dot{V}^\nu - R^\nu\dot{V}^\mu}{(R^\sigma V_\sigma)^2} - \frac{R^\rho\dot{V}_\rho(R^\mu V^\nu - R^\nu V^\mu)}{(R^\sigma V_\sigma)^3}$$

We need the acceleration terms $\dot{V}^\mu = \frac{dV^\mu}{d\tau} = c\frac{d}{d\tau}(\gamma, \gamma\vec{\beta})$, oberving that $\vec{\beta}$ and γ depend on τ.

$$\dot{V}^0 = c\frac{d\gamma}{d\tau} = c\dot{\gamma}.$$

$$\begin{aligned}\dot{V}^i = \frac{dV^i}{d\tau} &= c\frac{d}{dt}(\gamma\beta^i) \\ &= c\frac{d\gamma}{d\tau}\beta^i + c\gamma\frac{d\beta^i}{dt} \\ &= c\dot{\gamma}\beta^i + c\gamma^2\frac{d\beta^i}{dt}\end{aligned}$$

In a local frame, $\frac{d}{d\tau} = \gamma\frac{d}{dt}$. We introduce $\vec{\alpha} = \frac{d\vec{\beta}}{dt}$, so that $c\vec{\alpha}$ is acceleration of q (analagous to how $c\vec{\beta}$ is its velocity).

We use $\mu = i$ and $\nu = 0$:

$$\frac{4\pi}{q}(R^\sigma V_\sigma)^3 F^{\mu\nu}_{\text{rad}} = R^\sigma V_\sigma(R^\mu\dot{V}^\nu - R^\nu\dot{V}^\mu) - R^\sigma\dot{V}_\sigma(R^\mu V^\nu - R^\nu V^\mu)$$

$$\begin{aligned}\frac{4\pi}{q}(R^\sigma V_\sigma)^3 F^{i0}_{\text{rad}} &= c\gamma R(1-\vec{n}\cdot\vec{\beta})(R^i\dot{V}^0 - R^0\dot{V}^i) - (R^0\dot{V}^0 - R^j\dot{V}^j)(R^iV^0 - R^0V^i) \\ &= c\gamma R(1-\vec{n}\cdot\vec{\beta})cR(n^i\dot{\gamma} - \dot{\gamma}\beta^i - \gamma^2\alpha^i) - cR(\dot{\gamma} - n^j\dot{\gamma}\beta^j - n^j\gamma^2\alpha^j)c\gamma R(n^i - \beta^i)\end{aligned}$$

All $\dot{\gamma}$ terms cancel, leaving:

$$\frac{4\pi}{q}(c\gamma R)^3(1-\vec{n}\cdot\vec{\beta})^3 E^i_{\text{rad}} = -c^2\gamma R^2(1-\vec{n}\cdot\vec{\beta})\gamma^2\alpha^i + c^2\gamma R^2 n^j\alpha^j\gamma^2(n^i-\beta^i)$$

$$\frac{4\pi cR}{q}(1-\vec{n}\cdot\vec{\beta})^3\vec{E}_{\text{rad}} = \vec{n}\cdot\vec{\alpha}(\vec{n}-\vec{\beta}) - (1-\vec{n}\cdot\vec{\beta})\vec{\alpha}$$

So:

$$\vec{E}_{\text{rad}} = \frac{q}{4\pi cR}\frac{\vec{n}\cdot\vec{\alpha}(\vec{n}-\vec{\beta}) - (1-\vec{n}\cdot\vec{\beta})\alpha}{(1-\vec{n}\cdot\vec{\beta})^3}\Bigg|_{t_0}$$

and

$$\vec{B}_{\text{rad}} = \vec{n}\times\vec{E}_{\text{rad}}$$

Using vector relations allows $\vec{n}\times[(\vec{n}-\vec{\beta})\times\alpha]$ to be rewritten as $\vec{n}\cdot\vec{\alpha}(\vec{n}-\vec{\beta}) - (1-\vec{\beta}\cdot\vec{n})\vec{\alpha}$. Hence,

$$\vec{E}_{\text{rad}} = q\frac{\vec{n}\times[(\vec{n}-\vec{\beta})\times\alpha]}{4\pi cR(1-\vec{n}\cdot\vec{\beta})^3}\Bigg|_{\text{ret}}$$

It follows directly that $\vec{n}\cdot\vec{E}_{\text{rad}}$ (because of "$\vec{n}\times$" in the numerator). As $R\to\infty$, $\vec{E}_{\text{rad}}$ and $\vec{B}_{\text{rad}} \to \frac{1}{R}$.

2.3 Larmor's power formula

It is possible to arrange $\vec{\beta}=0$ at a particular instant using a Lorentz transformation, to obtain non-relativistic expressions.

For $\vec{\beta}=0$, $1-\vec{\beta}\cdot\vec{n}=1$.

$$E^{\text{NR}}_{\text{rad}} = q\frac{\vec{n}\times(\vec{n}\times\vec{\alpha})}{4\pi cR}$$

Since $\vec{n}\times\vec{\alpha}\perp\vec{n}$,

$$|\vec{n}\times(\vec{n}\times\vec{\alpha})| = |\vec{\alpha}|\sin\theta.$$

Power is given by Poynting's formula (in HL units):

$$\vec{S} = c\vec{E}_{\text{rad}}\times\vec{B}_{\text{rad}}$$

Recall that $c\Theta^{i0} = S^i$ where $\Theta^{\mu\nu}$ has dimension energy density.

Energy radiated per second into solid angle $d\Omega = \sin\theta\,d\theta\,d\phi$ is the element of power:

$$dP = |\vec{S}|R^2\,d\Omega$$
$$\frac{dP}{d\Omega} = R^2|\vec{S}|$$

For low velocities of q, $|\vec{E}^{\text{NR}}_{\text{rad}}| = |q\alpha\sin\theta|$ where $\alpha=|\vec{\alpha}|$. Note that since $\vec{B}^{\text{NR}}_{\text{rad}} = \vec{n}\times\vec{E}^{\text{NR}}_{\text{rad}}$,

$$|\vec{E}_{\text{rad}}\times\vec{B}_{\text{rad}}| = |\vec{E}_{\text{rad}}||\vec{B}_{\text{rad}}| = |\vec{E}_{\text{rad}}|^2.$$

So $|\vec{S}| = c|\vec{E}_{\text{rad}}|^2$. In fact,

$$\vec{S} = c|\vec{E}_{\text{rad}}|^2\vec{n}.$$

So,

$$\frac{dP^{\text{NR}}}{d\Omega} = cR^2|\vec{E}^{\text{NR}}_{\text{rad}}|^2 = \frac{q^2\alpha^2\sin^2\theta}{16\pi^2 c},$$

Larmor's power formula.

Note that along the direction $\vec{a} = c\vec{\alpha}$, no radiation is emitted, forwards *or* backwards. There is also no azimuthal dependence.

2.4 Total power emitted

Integrating over all angles,

$$\begin{aligned}
P_{\text{NR}} &= \int_0^{2\pi}\int_0^{\pi} \frac{dP_{\text{NR}}}{d\Omega}\sin\theta\, d\theta\, d\phi \\
&= 2\pi\int_{-1}^{1} dz\, \frac{dP_{\text{NR}}}{d\Omega} \\
\frac{8\pi c}{q^2\alpha^2}P_{\text{NR}} &= \int_{-1}^{1}(1-z^2)\, dz \\
&= \tfrac{4}{3}.
\end{aligned}$$

So,

$$P_{\text{NR}} = \frac{q^2\alpha^2}{6\pi c} = \frac{q^2|\vec{a}|^2}{6\pi c^3}$$

where $\vec{a}$ is the acceleration of the charge q at low velocity.

2.5 Relativistic radiation formula

Power is a Lorentz scalar. The generalisation of $\vec{a}$ is $\frac{dV^\mu}{d\tau}$, and $|\vec{a}|^2 = \frac{dv^i}{dt}\frac{dv^i}{dt}$ generalises to:

$$-\frac{dV^\mu}{d\tau}\frac{dV_\mu}{d\tau} = -\left(\frac{dV^0}{d\tau}\right) + \left(\frac{dV^i}{d\tau}\right)^2 \tag{2.6}$$

Recall V^μ has components $c\gamma(1,\vec{\beta})$ and that $\gamma^{-2} = 1-\beta^2$, where $\beta = |\vec{\beta}|$. Use $\frac{d}{d\tau}$:

$$\begin{aligned}
-2\gamma^{-3}\frac{d\gamma}{d\tau} &= -2\vec{\beta}\cdot\frac{d\beta}{d\tau} \\
\Rightarrow c\frac{d\gamma}{d\tau} &= \dot{V}^0 = c\gamma^3\beta\frac{d\beta}{dt}\gamma \\
\dot{V}^0 &= c\gamma^4\vec{\beta}\cdot\vec{\alpha}
\end{aligned}$$

Also we need $\frac{dV^i}{d\tau}$ in terms of $\vec{\alpha}$ and $\vec{\beta}$ in (2.6):

$$\begin{aligned}
\frac{dV^i}{d\tau} = \frac{d(c\gamma\beta^i)}{d\tau} &= c\frac{d\gamma}{d\tau}\beta^i + c\gamma\frac{d\beta^i}{d\tau} \\
\frac{dV^i}{d\tau} &= \dot{V}^0\beta^i + c\gamma^2\alpha^i
\end{aligned}$$

So,

$$\frac{dV^\mu}{d\tau}\frac{dV_\mu}{d\tau} = -(c\gamma^4\vec{\alpha}\cdot\vec{\beta})^2 + (c\gamma^4\vec{\alpha}\cdot\vec{\beta}\beta^i + c\gamma^2\alpha^i)^2$$

$$\frac{\dot{V}^\mu\dot{V}_\mu}{c^2\gamma^4} = -\gamma^4(\vec{\alpha}\cdot\vec{\beta})^2 + \gamma^4(\vec{\alpha}\cdot\vec{\beta})^2\beta^i\beta^i + 2\gamma^2\vec{\alpha}\cdot\vec{\beta}\beta^i\alpha^i + \alpha^i\alpha^i$$

$$= \gamma^4(\vec{\alpha}\cdot\vec{\beta})^2(\vec{\beta}^2 - 1) + 2\gamma^2(\vec{\alpha}\cdot\vec{\beta})^2 + \alpha^2.$$

Using, $\gamma^{-2} = 1 - \beta^2$,

$$P = -\frac{q^2}{6\pi c^3}\dot{V}_\mu\dot{V}^\mu = \frac{q^2c^2\gamma^4}{6\pi c^3}\left[\gamma^2(\vec{\alpha}\cdot\vec{\beta})^2 + \vec{\alpha}^2\right]$$

$$P = \frac{q^2\gamma^4}{6\pi^2 c}\left[\alpha^2 + (\vec{\alpha}\cdot\vec{\beta})\gamma^2\right],$$

Liénard's formula. Observe that when $\vec{\alpha}\perp\vec{\beta}$, as in circular motion,

$$P_\odot = -\frac{q^2\gamma^4}{6\pi c}\alpha^2.$$

Also note that for $\vec{\beta}\approx 0$, $\gamma\approx 1$ and $P\approx P_{\text{NR}}$.

Another form for P, useful for linear acceleration (i.e. $\vec{\alpha}\parallel\vec{\beta}$) uses:

$$(\vec{\alpha}\cdot\vec{\beta})^2 + \gamma^{-2}\vec{\alpha}^2 = \vec{\alpha}^2 - (\vec{\alpha}\times\vec{\beta})^2$$

So,

$$P = \frac{q^2\gamma^6}{6\pi c}\left[\vec{\alpha}^2 - (\vec{\alpha}\times\vec{\beta})^2\right].$$

When $\vec{\alpha}\parallel\vec{\beta}$, $\vec{\alpha}\times\vec{\beta} = 0$, so

$$P = \frac{q^2\gamma^6}{6\pi c^3}\vec{\alpha}^2.$$

As $|\vec{v}|$ increases, $\gamma = (1-\beta^2)^{-1/2}$ becomes large, and the radiation becomes substantial. For accelerators, radiation loss needs to be minimised. For synchrotrons, radiation is enhanced to arrange useful determination of chemical structures.

2.6 Angular distribution

Energy flux for the radiative part is given by:

$$\vec{S} = c\vec{E}_{\text{rad}}\times\vec{B}_{\text{rad}}.$$

So

$$\vec{S} = \frac{q^2}{4\pi c R^2}\frac{\left|\vec{n}\times[(\vec{n}-\vec{\beta})\times\vec{\alpha}]\right|^2}{(1-\vec{n}\cdot\vec{\beta})^6}\vec{n}.$$

This component of energy flux in the direction $\vec{R} = R\vec{n}$ is the energy per area per second observed at time t. The radiation was emitted by a charge q at the retarded time $t' = t - \frac{1}{c}R(t')$.

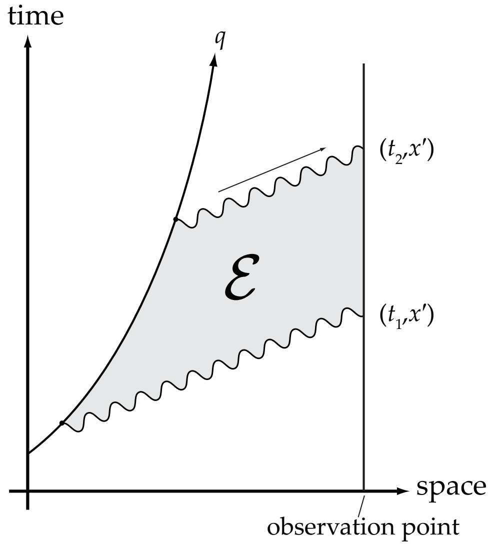

Figure 2.2: The total energy radiated

The energy radiated from time t_1 to time t_2 as q accelerates is:

$$\mathcal{E} = \int_{t_1}^{t_2} \vec{S} \cdot \vec{n}\, dt,$$

the energy/area at position $\vec{x}$ between times t_1 and t_2, where $t_i = t_i' + \frac{1}{c} R(t_i')$. (see Fig. ref-fig:rad)

In terms of the charge's coordinates,

$$\mathcal{E} = \int_{t_1'}^{t_2'} \vec{S} \cdot \vec{n} \left(\frac{dt}{dt'} \right) dt'$$

where $\vec{S} \cdot \vec{n}$ is evaluated at t'.

We need $\frac{dt}{dt'}$. Recall that $t = t' + \frac{1}{c} R(t')$:

$$\frac{dt}{dt'} = 1 + \frac{1}{c} \frac{dR}{dt'}, \tag{2.7}$$

so we need $\frac{dR}{dt'}$. But $R^2 = \vec{R}^2 \Rightarrow 2R\frac{dR}{dt'} = 2\vec{R} \cdot \frac{d\vec{R}}{dt'}$. So $\frac{dR}{dt'} = -\vec{n} \cdot \vec{v}$, where $\vec{v} = -\frac{dR}{dt'}$. (2.7) becomes:

$$\begin{aligned} \frac{dt}{dt'} &= 1 - \frac{1}{c} \vec{n} \cdot (c\vec{\beta}) \\ &= 1 - \vec{n} \cdot \vec{\beta} \end{aligned}$$

So,

$$\vec{S} \cdot \vec{n} \left(\frac{dt}{dt'} \right) = \frac{q^2}{4\pi c R^2} \frac{\left| \vec{n} \times [(\vec{n} - \vec{\beta}) \times \vec{\alpha}] \right|^2}{(1 - \vec{n} \cdot \vec{\beta})^6} (1 - \vec{n} \cdot \vec{\beta})$$

The power radiated by q in its coordinates at t' is:

$$\begin{aligned}\frac{dP'}{d\Omega} &= R^2 \vec{S}\cdot\vec{n}\left(\frac{dt}{dt'}\right)\\ &= \frac{q^2}{16\pi^2 c}\frac{\left|\vec{n}\times[(\vec{n}-\vec{\beta})\times\vec{\alpha}]\right|^2}{(1-\vec{n}\cdot\vec{\beta})^5}.\end{aligned}$$

When $\vec{v} = c\vec{\beta} \parallel \vec{a} = c\vec{\alpha}$,

$$\begin{aligned}(\vec{n}-\vec{\beta})\times\vec{\alpha} &= \vec{n}\times\vec{\alpha}\\ |\vec{n}\times\vec{\alpha}| &= \alpha\sin\theta.\end{aligned}$$

So,

$$\frac{16\pi^2 c}{q^2}\frac{dP'}{d\Omega} = \frac{\alpha^2\sin^2\theta}{(1-\beta\cos\theta)^5},$$

as $1-\vec{n}\cdot\vec{\beta} = 1-|\vec{n}||\vec{\beta}|\cos\theta$, and $\vec{\beta}$ makes angle θ with $\vec{n}$.

2.6.1 Angle for maximum radiation

Set $x = 1-\beta\cos\theta$, i.e.

$$\cos\theta = \frac{1-x}{\beta}, \tag{2.8}$$

giving $\sin^2\theta = 1-\beta^{-2}(1-x^2)$. Write:

$$\frac{4\pi c}{q^2\alpha^2}\frac{dP'}{d\Omega} = f(x) = \frac{1-\beta^{-2}(1-x)^2}{x^5}$$

We wish to find the maximum:

$$\begin{aligned}\beta^2 f(x) &= \frac{\beta^2-1+2x-x^2}{x^5}\\ &= \frac{\beta^2-1}{x^5}+\frac{2}{x^4}-\frac{1}{x^3}\\ \beta^2\frac{df}{dx} &= \frac{5(1-\beta^2)}{x^6}-\frac{8}{x^5}+\frac{3}{x^4}=0\end{aligned}$$

$$3x^2-8x+5(1-\beta^2)=0$$

$$\Rightarrow x_{\max} = \frac{4-\sqrt{1+15\beta^2}}{3}.$$

From (2.8),

$$\cos\theta_{\max} = \frac{\sqrt{1+15\beta^2}-1}{3\beta}.$$

so as $\beta\to 0$, $\cos\theta_{\max}\to 0$. (see Fig. 2.3) $\frac{dP(t')}{d\Omega} = 0$ when $\theta = 0$, and increases to a maximum at $\theta_{\max}$. Power is $\propto q^2$ and $\propto \alpha^2$.

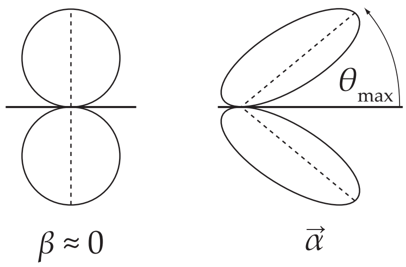

Figure 2.3: The dependence of θ_{max} on β.

For $\beta \to 1$, $\gamma \to \infty$. We relate θ_{max} to γ when γ is large. Write $\cos\theta_{\text{max}}$ in terms of γ, a useful relation. With $\gamma^{-2} = 1 - \beta^2$, and $\beta^2 = 1 - \gamma^{-2}$,

$$
\begin{aligned}
\cos\theta_{\text{max}} &= \frac{\sqrt{15 - 15\gamma^{-2} + 1} - 1}{3(1-\gamma^{-2})^{1/2}} \\
&= \frac{1}{3}\left[\sqrt{16(1 - \frac{15}{16\gamma^2})} - 1\right](1-\gamma^2)^{-1/2} \\
&= \frac{1}{3}\left[4\left(1 - \frac{15}{32\gamma^2} + \ldots\right) - 1\right]\left(1 + \frac{1}{2\gamma^2}\right)
\end{aligned}
$$

When $\sqrt{\ }$s are expanded in series, it is sufficient to keep only the $1/\gamma^2$ term. So,

$$
\begin{aligned}
\cos\theta_{\text{max}} &\approx \frac{1}{3}\left[3 - \frac{15}{8\gamma^2} + \ldots\right]\left(1 + \frac{1}{2\gamma^2}\right) \\
&\approx \left(1 - \frac{5}{8\gamma^2}\right)\left(1 + \frac{4}{8\gamma^2}\right) \\
&\approx 1 - \frac{1}{8\gamma^2} + O\left(\frac{1}{\gamma^4}\right)
\end{aligned}
$$

But $\cos\theta_{\text{max}} \approx 1 - \frac{1}{2}\theta_{\text{max}}^2$. So for large γ, small θ_{max},

$$
-\frac{1}{2}\theta_{\text{max}}^2 = -\frac{1}{2}\left(\frac{1}{2\gamma}\right)^2.
$$

So for large γ,

$$
\theta_{\text{max}} \approx \frac{1}{2\gamma}.
$$

2.7 Computation

The cone of maximum radiation has an apical angle of about $1/\gamma$. An approximate formula for the radiation induced by a high-energy charge q at time t' is:

$$
\frac{dP(t')}{d\Omega} = \frac{q^2\vec{\alpha}^2}{16\pi^2 c}\frac{\sin^2\theta}{(1-\beta\cos\theta)^5}
$$

$$
\frac{16\pi^2 c}{q^2\vec{\alpha}^2}\frac{dP'}{d\Omega} = \frac{(\theta - \frac{1}{6}\theta^3 + \ldots)^2}{\left[1 - \beta(1 - \frac{1}{2}\theta^2) + \ldots\right]^5}, \tag{2.9}
$$

expanding sin and cos to $O(\theta^2)$.

$$\beta = 1 - \frac{1}{2\gamma^2} + O\left(\frac{1}{\gamma^4}\right)$$

So (2.9) becomes:

$$\begin{aligned}
\frac{16\pi^2 c}{q^2\vec{\alpha}^2}\frac{dP'}{d\Omega} &= \frac{\theta^2}{\left[1 - 1 + \frac{1}{2\gamma^2} + \frac{1}{2}\theta^2 + \dots\right]^5} \\
&= \frac{32\theta^2}{(\gamma^{-2} + \theta^2)^5} \\
&= \frac{32\gamma^{10}\theta^2}{(1 + \gamma^2\theta^2)^5} \\
&= \frac{32\gamma^8(\gamma\theta)^2}{(1 + \gamma^2\theta^2)^5} \\
\frac{dP'}{d\Omega} &= \frac{2q^2\vec{\alpha}^2\gamma^8(\gamma\theta)^2}{\pi^2 c(1 + \gamma^2\theta^2)^5}.
\end{aligned}$$

In HL units, the maximum is:

$$\frac{dP'_{\max}}{d\Omega} = \frac{\frac{1}{2}q^2\alpha^2\gamma^8}{\pi^2 c\left(\frac{5}{4}\right)^5}$$

The power generated is $\propto \gamma^8\vec{\alpha}^2$. So high γ and high acceleration provide considerable synchrotron radiation.

2.8 Total power

$$P' = \int_0^{2\pi}\int_0^{\pi} \frac{dP'}{d\Omega} \sin\theta \, d\theta \, d\phi.$$

This should match Liénard's formula:

$$P' = \frac{q^2\gamma^6}{6\pi c}\left[\vec{\alpha}^2 - (\vec{\alpha} \times \vec{\beta})^2\right],$$

which for $\vec{\alpha} \parallel \vec{\beta}$ gives:

$$P' = \frac{q^2\gamma^6}{6\pi c}\vec{\alpha}^2.$$

Note: In astrophysics, $\gamma \sim 10^3$, and in LHC, $\gamma \sim 4 \times 10^3$.