MA448: K-theory and Solitons
Problem Set 6

Due at 13 pm on Wednesday, 8 April 2009.

1 Hopf Bundle

Identifying the fibe S^3 with the group $SU(2)$ demonstrate that the Hopf fibration $S^7 \to S^4$ is the Instanton principal bundle with the instanton number equal to one.

2 Gauge Transformations

For a principal bundle $P(M,G)$ let U be an open contractible chart in M. Let σ_1 and σ_2 be local sections of $P(M,G)$ over U, then there is a group-valued function g on U such that $\sigma_2(p) = \sigma_1(p)g(p)$. Show that the corresponding local forms of a connection A_1 and A_2 are related by

$$A_{2\mu} = q^{-1}(p)A_{1\mu}(p)g(p) + g^{-1}(p)\partial_\mu g(p). \tag{1}$$

3 Adjoint and its covariant derivative

Under a gauge transformation g the connection one-form $A = A_\mu dx^\mu$ and an 'adjoint' field B transform as

$$B(x) \rightarrow g(x)Bg(x)^{-1}, \tag{2}$$

$$A \rightarrow g(x)Ag(x)^{-1} - (dg(x))g(x)^{-1}. \tag{3}$$

The covariant derivative ∇_μ of B is defined by

$$\nabla_\mu B = \frac{\partial B}{\partial x^\mu} + A_\mu B - bA_\mu. \tag{4}$$

Verify that under the gauge transformation the covariant derivative transforms as B does, i.e.

$$\nabla_\mu B(x) \rightarrow g(x)\nabla_\mu B(x)g(x)^{-1}. \tag{5}$$

4 Curvature

Find how the curvature two-form $F = F_{\mu\nu}dx^\mu \wedge dx^\nu$, defined by

$$F_{\mu\nu} = [\nabla_\mu, \nabla_\nu] = \frac{\partial A_\nu}{\partial x^\mu} - \frac{\partial A_\mu}{\partial x^\nu} + [A_\mu, A_\nu], \tag{6}$$

transforms under a gauge transformation of Eq.(3).