MA448: K-theory and Solitons
Problem Set 2

Due at 1 pm on Wednesday, 11 February 2009

1 Trivial Bundles

Prove that a k-dimensional vector bundle $E \to X$ is trivial if and only if it has k sections $s_1, s_2, \ldots, s_k$ such that the vectors $s_1(x), s_2(x), \ldots, s_k(x)$ are linearly independent for every $x \in X$.

2 Möbius Bundle and its Sections

a) Give an explicit description of the Möbius bundle with its local trivializations and clutching functions.

b) In the trivializations you have chosen give an example of a global section of the Möbius bundle.

c) Prove that there can be no nowhere vanishing sections and thus this is a nontrivial bundle.

3 A section of TS^{2n-1}

Show that there is a nowhere vanishing section of the tangent bundle TS^{2n-1}.

Hint 1: Find a unit tangent vector field to the sphere S^{2n-1} using the fact that S^{2n-1} can be viewed as a unit sphere in $\mathbb{C}^{2n}$:
$S^{2n-1} = \{(z_1, z_2, \ldots, z_n) | z_1\bar{z}_1 + z_2\bar{z}_2 + \ldots z_n\bar{z}_n = 1, z_j \in \mathbb{C}\}$.

Hint 2: Consider S^1 and S^3 first.

4 Restriction of a Bundle

For any two natural numbers n and m consider an 'equatorial sphere' $S^n \subset S^{n+m}$. Demonstrate that the restriction $TS^{n+m}|_{S^n}$ of the tangent bundle TS^{n+m} to the smaller sphere S^n is isomorphic to the direct sum of the tangent bundle of S^n and a trivial bundle $\epsilon^m \to S^n$, i.e.

$$TS^{n+m}|_{S^n} \approx TS^n \oplus \epsilon^m.$$