Recap: open string in lightcone gauge

$$X^+(t,\sigma) = x^+(t)$$

$$P_-(t,\sigma) = p_-(t)$$

solved $L_n = 0$ constraints for $n \neq 0$

$$\Rightarrow \quad \alpha_n^- = -\frac{\sqrt{\pi T}}{2p_-} \sum_{k\in\mathbb{Z}} \underline{\alpha}_k \cdot \underline{\alpha}_{n-k}$$

transverse oscillators

left with

$$S = \int dt \left(\dot{x}^\mu p_\mu + \sum \frac{i}{k} \dot{\alpha}_k^I \alpha_{-k}^J \delta_{IJ} - \frac{1}{2} e_0 L_0 \right)$$

$$L_0 = \frac{1}{2}\alpha_0^2 + \frac{1}{2}\sum_{k\neq 0} \alpha_k^I \alpha_{-k}^J \delta_{IJ} \qquad I,J = 2,\dots,D-1$$

transverse indices

Canonical quantisation

$$\Rightarrow \quad [\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu{}_\nu$$

$$[\hat{\alpha}_k^I, \hat{\alpha}_l^J] = k\,\delta_{k+l,0}\,\delta^{IJ} \qquad \hat{\alpha}_k^\dagger = \hat{\alpha}_{-k}$$

Consider the constraint: there is now a normal ordering ambiguity

$$L_0 = \frac{1}{2}\frac{\hat{p}^2}{\pi T} + \frac{1}{2}\sum_{k>0} \hat{\alpha}_k^I \hat{\alpha}_{-k}^J \delta_{IJ} + \frac{1}{2}\sum_{k<0} \hat{\alpha}_{+k}^I \hat{\alpha}_{-k}^J \delta_{IJ}$$

$$= \frac{\hat{p}^2}{2\pi T} + \frac{1}{2}\sum_{k>0}\left([\hat{\alpha}_k^I, \hat{\alpha}_{-k}^J] + \hat{\alpha}_{-k}^I \hat{\alpha}_k^J \right)\delta_{IJ}$$

$$+ \frac{1}{2}\sum_{k>0} \hat{\alpha}_{-k}^I \hat{\alpha}_k^J \delta_{IJ}$$

$$= \frac{p^2}{2\pi T} + \sum_{k>0} \alpha_{-k}^I \alpha_k^J \delta_{IJ} + \frac{1}{2}\sum_{k>0} k \underbrace{\delta^{IJ}\delta_{IJ}}_{D-2}$$

define $\hat{N} = \sum_{k>0} \hat{\alpha}_{-k}^I \hat{\alpha}_k^J \delta_{IJ}$

then

$$L_0 = \frac{1}{2\pi T}\hat{p}^2 + N - a \qquad \text{mass-shell constraint}$$

$$a \equiv -\frac{D-2}{2}\sum_{k>0} k \qquad \Rightarrow \quad M^2 = 2\pi T(N-a)$$

we will determine a below

let us first construct the states of the string

given string state has some momentum p (eigenvalue of $\hat{p}$)

and some oscillator structure

define vacuum $|0;p\rangle$

s.t. $\hat{p}|0;p\rangle = p|0;p\rangle$

$$\alpha_{+k}^I|0;p\rangle = 0 \quad \forall\, k>0$$

create states by acting with α_{-k}^I on $|0;p\rangle$

cf quantum harmonic oscillator $\Rightarrow$ $D-2$ commuting oscillators, further labelled by $D-2$ dirs can check that $\hat{N}|0;p\rangle = 0$

$$[\hat{N}, \hat{\alpha}_{-k}^{I}] = k\,\hat{\alpha}_{-k}^{I}$$

$\hat{N}$ - level ~~counter~~ number operator

acting with $\hat{\alpha}_{-k}^{I}$ adds k to the level ($\hat{N}$ eigenvalue)

$k>0$

order states by $\hat{N}$ eigenvalue / level

N	states	# states
0	$\|0;p\rangle$	1
1	$\hat{\alpha}_{-1}^{I}\|0;p\rangle$	$D-2$
2	$\hat{\alpha}_{-2}^{I}\|0;p\rangle$, $\hat{\alpha}_{-1}^{I}\hat{\alpha}_{-1}^{J}\|0;p\rangle$	$D-2+\frac{1}{2}(D-2)(D-1)$
3	$\hat{\alpha}_{-3}^{I}\|0;p\rangle$, $\hat{\alpha}_{-2}^{I}\hat{\alpha}_{-1}^{J}\|0;p\rangle$, $\hat{\alpha}_{-1}^{I}\hat{\alpha}_{-1}^{J}\hat{\alpha}_{-1}^{K}\|0;p\rangle$	$D-2+(D-2)^2+\frac{1}{6}(D-2)(D-1)(D)$
$\vdots$		

let $2\pi T = \alpha'$ (conventional)

$N=0$ $\quad M^2 = -\dfrac{a}{\alpha'}$

$N=1$ $\quad M^2 = \dfrac{1-a}{\alpha'}$ $\quad (D-2)$ states

Fact: a massless vector field A_μ in D-dimensions has $(D-2)$ degrees of freedom (polarisations)

Lorentz invariance requires that the $N=1$ states correspond to such a state

$\Rightarrow$ need $a=1$ $\Rightarrow$ $M^2 = 0$

also ~~require~~ implies $N=0$ state has $M^2 = -\frac{1}{\alpha'} < 0$

$\Rightarrow$ tachyon, bad

continuing

$N=2$ $\quad M^2 = \frac{a}{\alpha'} > 0$ $\quad$ massive spin 2 rep $\quad t_{\mu\nu} \quad t_{(\mu\nu)} = t_{\mu\nu}$

(consider little gp classification

$\square\!\square + \square$ of $SO(D-2)$

$\longrightarrow \square\!\square$ of $SO(D-1)$)

Why $a=1$?

$$a = -\frac{D-2}{2} \sum_{k>0} k$$

consider Riemann zeta $\quad \zeta(s) = \sum_{k=1}^{\infty} k^{-s} \qquad Re(s) > 1$

has an analytical continuation $\forall s$ s.t. $\zeta(-1) = -\frac{1}{12}$

$$\Rightarrow \sum_{k=1}^{\infty} k = -\frac{1}{12}$$

$$\Rightarrow \quad a = \frac{D-2}{24}$$

Alternatively, regulate

$$\sum_{k=1}^{\infty} k \to \sum_{k=1}^{\infty} e^{-\varepsilon k} k = -\frac{\partial}{\partial \varepsilon} \sum_{k=1}^{\infty} e^{-\varepsilon k}$$

now $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$ so $\sum_{k=1}^{\infty} x^k = \frac{1}{1-x} - 1 = \frac{x}{1-x}$

$\Rightarrow$ have

$$-\frac{\partial}{\partial \varepsilon} \frac{e^{-\varepsilon}}{1-e^{-\varepsilon}} = -\frac{\partial}{\partial \varepsilon} \frac{1}{(e^{\varepsilon}-1)}$$

$$= \frac{1}{(e^{\varepsilon}-1)^2} e^{\varepsilon}$$

$$= \frac{\left(1+\varepsilon+\frac{1}{2}\varepsilon^2+O(\varepsilon^3)\right)}{\left(\varepsilon+\frac{1}{2}\varepsilon^2+\frac{1}{6}\varepsilon^3+O(\varepsilon^4)\right)^2}$$

$$= \frac{1}{\varepsilon^2} \frac{\left(1+\varepsilon+\frac{1}{2}\varepsilon^2+O(\varepsilon^3)\right)}{\left(1+\varepsilon\frac{1}{2}+\frac{1}{6}\varepsilon^2 O(\varepsilon^3)\right)^2}$$

$$= \frac{1}{\varepsilon^2} \frac{\left(1+\varepsilon+\frac{1}{2}\varepsilon^2+O(\varepsilon^3)\right)}{\left(1+\varepsilon+\frac{7}{12}\varepsilon^2+O(\varepsilon^3)\right)}$$

use $\frac{1}{1-x} \approx 1+x+x^2$

$$\simeq \frac{1}{\varepsilon^2}\left(1+\varepsilon+\frac{1}{2}\varepsilon^2+O(\varepsilon^3)\right)\left(1-\varepsilon-\frac{7}{12}\varepsilon^2+\varepsilon^2 + O(\varepsilon^3)\right)$$

$$= \frac{1}{\varepsilon^2}\left(1-\frac{1}{12}\varepsilon^2\right) + O(\varepsilon^3)$$

$$= \frac{1}{\varepsilon^2} - \underbrace{\frac{1}{12}}_{\text{finite part}}$$

note that $a = 1 \Rightarrow D = 26$ 'critical dimension'

spacetime dimension fixed by consistency of the string worldsheet theory!

Lorentz invariance also provides the same check

quantum Lorentz charges $\hat{J}^{\mu\nu} = \hat{L}^{\mu\nu} + \hat{S}^{\mu\nu}$ $\quad [\hat{L}^{\mu\nu}, \hat{S}^{\rho\sigma}] = 0$

$$\hat{S}^{\mu\nu} = -2 \sum \frac{i}{k} \hat{\alpha}_{-k}^{[\mu} \alpha_k^{\nu]}$$

for $D \geq 3$ one finds

$$0 \stackrel{?}{=} [\hat{S}^{-I}, \hat{S}^{-J}] = \frac{4\pi T}{p_-^2} \sum_{k=1}^{\infty} \left[\left(\frac{D-2}{12} - 2 \right) k + \frac{1}{k} \left(2a - \frac{D-2}{12} \right) \right] \times$$
$$\times \hat{\alpha}_{-k}^{[I} \hat{\alpha}_k^{J]}$$

$$\Rightarrow \begin{array}{l} D = 26 \\ a = 1 \end{array} \quad \text{again}$$

Closed string: behaves like two copies of open string

$$[x^\mu, p_\nu] = i\delta^\mu_\nu$$

and

$$[\hat{\alpha}_{+k}^I, \hat{\alpha}_{+l}^J] = [\hat{\tilde{\alpha}}_k^I, \tilde{\alpha}_l^J] = k\delta_{k+l,0}\delta^{IJ}$$

in lightcone gauge

define $$\hat{N} = \sum_{k>0} \hat{\alpha}_{-k}^I \hat{\alpha}_k^J \delta_{IJ}$$

$$\hat{\tilde{N}} = \sum_{k>0} \hat{\tilde{\alpha}}_{-k}^I \hat{\tilde{\alpha}}_k^J \delta_{IJ}$$

remaining constraints $L_0 = \tilde{L}_0 = 0$ imply

$$\hat{p}^2 + 4\pi T(\hat{N} + \hat{\tilde{N}} - a_L - a_R) = 0$$

and $$\hat{N} - a_R = \hat{\tilde{N}} - a_L$$

oscillator vacuum $|0\rangle = |0\rangle_R \otimes |0\rangle_L$

$\hat{\alpha}_{-k}^I$ acts on $|0\rangle_R$, $\tilde{\alpha}_{-k}^I$ acts on $|0\rangle_L$

define $$\hat{\alpha}_{+k}|0\rangle_R = 0 \quad k>0$$

$$\hat{\tilde{\alpha}}_k |0\rangle_L = 0 \quad k>0$$

$L_0, \tilde{L}_0 = 0$ on $|0\rangle \Rightarrow a_R = a_L$

$\Rightarrow$ write $$p^2 + 8\pi T(N-a) = 0 \quad \text{mass-shell constraint}$$

$$N = \tilde{N} \quad \text{level matching}$$

states

$N=0 \qquad |0\rangle \qquad M^2 = -\frac{4a}{\alpha'}$

$N=1 \qquad \alpha_{-1}^{I}|0\rangle_R \otimes \tilde{\alpha}_{-1}^{J}|0_L\rangle \qquad M^2 = \frac{4(1-a)}{\alpha'}$

$(D-2)^2$ states

generic $N=1$ state

$$|\Psi\rangle = \left(h_{IJ} + \delta_{IJ}\phi + b_{IJ} \right) \alpha_{-1}^{I}|0\rangle_R \otimes \tilde{\alpha}_{-1}^{J}|0\rangle_L$$

$h_{[IJ]}=0$ symm traceless

$h_I^{\ I}=0$

$b_{(IJ)}=0$ antisymmetric

Lorentz inv $\Rightarrow$ massless spin 2 $h_{\mu\nu} \to$ graviton

2-form $b_{\mu\nu} \to$ 'B-field'

scalar $\phi \to$ dilaton

$\Downarrow$

$D=26$

$a=1$

so string theory has a graviton state in its spectrum $\to$ describes (quantum) gravity

Open string (ND): consider now a string with mixed boundary conditions

split $\mu = (a, i)$

$a = 0, 1, \dots p \qquad X'^{a}\big|_{\sigma=0} = 0 = X'^{a}\big|_{\sigma=\pi} \quad N$

$i = p+1, \dots D-1 \qquad X^{\mu}\big|_{\sigma=0} = 0\ ,\ X^{\mu}\big|_{\sigma=\pi} = L^{\mu} \quad D$

Lorentz symmetry broken

$$SO(1, D-1) \to SO(1,p) \times SO(D-p-1)$$

Lorentz inv in $(p+1)$-dims plane Neumann

rotational invariance in Dirichlet dirs

work out mode expansions with doubling trick and

$$(P + TX')^{a}(\sigma) = (P - TX')^{a}(-\sigma)$$

$$(P + TX')^{i}(\sigma) = -(P - TX')^{i}(-\sigma) \Rightarrow P_{i}: \text{zero at endpoints}$$

$$\alpha_k{}^{i} = -\tilde{\alpha}_k{}^{i}$$

now

$$X^{i} = \frac{L^{i}\sigma}{\pi} - \frac{1}{\sqrt{\pi T}} \sum \frac{1}{k} \sin(k\sigma)\, \alpha_k{}^{i}$$

Quantise light cone gauge $(p \geq 3)$, set $L^{i} = 0$ (both ends same plane)

$$M^2 = 2\pi T(N - a) \qquad N = \sum_{k=1}^{\infty} \left(\alpha_{-k}{}^{I} \alpha_k{}^{J} \delta_{IJ} + \alpha_{-k}{}^{i} \alpha_k{}^{j} \delta_{ij} \right)$$

$I = 2, \dots p$

$p^2 = -M^2$ in $(p+1)$-dim Minkowski

$N=1$ states take the form

$$|\Psi\rangle = \left(A_I \, \alpha_{-1}^{I} + \phi_i \, \alpha_{-1}^{i} \right) |0\rangle$$

$D-p-1$ scalars in Mink $(p+1)$

$p-2$ phys cpts of $(p+1)$ massless vec field A_a

$a=1$, $D=26$

translational invariance broken by Dirichlet conds

$\Rightarrow$ string can gain / lose momentum in directions orthogonal to $(p+1)$-dim plane

interpret the scalars ϕ_i as fluctuations of brane

this is a 'D-brane'

string theory is a theory also containing dynamics of other extended objects like this!

~~theory that lives on a D-brane is a gauge theory~~