Turkey: NG String (Open) ← classical (6 pages)

Recall: closed string $\sigma \in [0, 2\pi]$

$$X = x + \frac{1}{\sqrt{4\pi T}} \sum \frac{i}{k} e^{ik\sigma} (\alpha_k - \tilde{\alpha}_{-k})$$

$$P = \frac{1}{2\pi} p + \sqrt{\frac{T}{4\pi}} \sum e^{ik\sigma} (\alpha_k + \tilde{\alpha}_{-k})$$

Open string

take $\sigma \in [0, \pi]$

note

$$S = \int dt\, d\sigma \left(\dot{X}^\mu P_\mu - \frac{1}{2} e \left(P^2 + (TX')^2 \right) - u X' \cdot P \right)$$

when we vary, we must now take into account the endpoints

the total derivative pieces are

$$\delta S = -\int dt \left((T^2 e X' + uP) \cdot \delta X \right]_{\sigma=0}^{\sigma=\pi} + \text{eom}$$

still ignore total time derivatives

δX^0 should not be fixed (moves in time always)

X^0 free $\Rightarrow$ $\dot{X}^0$ free $\Rightarrow$ P^0 free

$$\Rightarrow \quad u|_{\sigma=0} = u|_{\sigma=\pi} = 0 \qquad X^{0\prime}|_{\sigma=0} = X^{0\prime}|_{\sigma=\pi} = 0$$

then need also

$$\vec{X}' \cdot \delta \vec{X} |_{\sigma=0, \sigma=\pi} = 0$$

at each $\sigma = 0, \sigma = \pi$, for a given X^i can have $\begin{cases} X^{i\prime} = 0 & \text{Neumann} \\ \delta X^i = 0 & \text{Dirichlet} \end{cases}$

note $\delta X^i = 0$ breaks Lorentz inv

$X^{i\prime}|_{\text{endpts}} = 0 \quad \forall X^i \Rightarrow$ full Lorentz inv

$\Rightarrow X'^2$ zero at endpoints

$\Rightarrow P^2 = 0$ at endpoints $\quad$ (for $P^i \pm TX'^i = 0$)

as $P|_{\text{ends}} = e^{-1}(\dot{X} - uX')|_{\text{ends}} = e^{-1}\dot{X}|_{\text{ends}} = 0$

$\Rightarrow \dot{X}^2|_{\text{ends}} = 0 \Rightarrow$ string endpoints move at speed of light

Can have any bry conds (at cost of full Lorentz inv)

ie NN
DD
ND
DN

Fourier expansion for NN open string:

trick to let us use closed string results

formally allow X, P to depend on $\sigma \in [0, 2\pi]$

relate $[\pi, 2\pi)$ to $[0, \pi]$

if $\sigma \in [0, \pi]$ then $-\sigma \sim -\sigma + 2\pi \in [\pi, 2\pi)$

so relate σ to $-\sigma$ by:

$$(P + TX')(\sigma) = (P - TX')(-\sigma)$$

$\sigma = 0 \Rightarrow X'(0) = 0$

$\pi \sim -\pi \Rightarrow X'(\pi) = 0$

Also $\Rightarrow$ $\alpha_k = \tilde{\alpha}_k$

so for open string

$$P \pm TX' = \sqrt{\frac{T}{\pi}} \sum_k e^{\mp ik\sigma} \alpha_k \qquad \alpha_k^* = \alpha_{-k}$$

and

$$X = x + \frac{i}{\sqrt{\pi T}} \sum_{k \neq 0} \frac{1}{k} \cos k\sigma \, \alpha_k$$

$$P = \frac{1}{\pi} p + \sqrt{\frac{T}{\pi}} \sum_{k \neq 0} \cos k\sigma \, \alpha_k \qquad p = \int_0^\pi d\sigma \, P$$

require $\lambda^+(\sigma) = \lambda^-(-\sigma)$ $\Rightarrow$ $u|_{\text{ends}} = 0 = e^-|_{\text{ends}}$

as $\mathcal{H}_+(\sigma) = \mathcal{H}_-(-\sigma)$

then one can show the action is

$$S = \int dt \left(\dot{x}^\mu p_\mu + \sum \frac{i}{k} \dot{\alpha}_k \cdot \alpha_{-k} - \sum \lambda_{-k} L_k \right)$$

$$L_n = \frac{1}{2} \sum \alpha_k \cdot \alpha_{n-k}$$

and $\{x^\mu, p_\nu\} = \delta^\mu_\nu \qquad \{\alpha_k{}^\mu, \alpha_{-k}{}^\nu\} = -ik\eta^{\mu\nu}$

can check

$$\{L_n, \alpha_k{}^\mu\} = ik\, \alpha_{n+k}{}^\mu$$

$$\delta_\xi \alpha_k{}^\mu = -ik \sum \xi_{-n} \alpha_{n+k}{}^\mu \quad \left(= \sum_{n \in \mathbb{Z}} \xi_{-n} \{\alpha_k{}^\mu, L_n\} \right)$$

$$\delta_\xi x^\mu = \frac{1}{\sqrt{\pi T}} \sum \xi_{-n} \alpha_n$$

$$\delta_\xi p_\mu = 0$$

$$\delta_\xi \lambda_n = \dot{\xi}_n + i \sum_{k \in \mathbb{Z}} (2k-n)\, \xi_k \lambda_{n-k}$$

Analyse this in light cone gauge

define as before $\quad X^\pm = \frac{1}{\sqrt{2}}\left(X^1 \pm X^0 \right) \qquad \underline{X} = \left(X^2, \dots, X^{D-1} \right)$

$$P_\pm = \frac{1}{\sqrt{2}}\left(P_1 \pm P_0 \right) \qquad \underline{P} = \left(P_2, \dots, P_{D-1} \right)$$

gauge conds

$$X^+(t,\sigma) = x^+(t) \qquad \text{ie no oscillators}$$

$$P_-(t,\sigma) = p_-(t)$$

or $\quad (P \pm T X')^+ = p_-(t) \iff \alpha_k^+ = 0 \quad \forall\, k \neq 0$

check if gauge really fixed

$$\delta_\xi \alpha_k^+ = -ik \sum \xi_{-n} \alpha_{n+k}^+$$

$$= -ik\, \alpha_0 \xi_k$$

if $k \neq 0$, need $\xi_k = 0$ so fixed all gauge trafs

except those generated by L_0

now, $L_n = \frac{1}{2} \sum \alpha_k \cdot \alpha_{n-k}$

$$= \frac{1}{2} \sum_{k\in\mathbb{Z}} \left(\alpha_k^+ \alpha_{n-k}^- + \alpha_k^- \alpha_{n-k}^+ \right)$$

$$+ \frac{1}{2} \sum_{k\in\mathbb{Z}} \underline{\alpha}_k \cdot \underline{\alpha}_{n-k}$$

for $n \neq 0$, $L_n = \alpha_0^+ \alpha_n^- + \frac{1}{2} \sum_{k\in\mathbb{Z}} \underline{\alpha}_k \cdot \underline{\alpha}_{n-k}$

$$\Rightarrow \alpha_n^- = - \frac{\sqrt{\pi T}}{2p_-} \sum_{k\in\mathbb{Z}} \underline{\alpha}_k \cdot \underline{\alpha}_{n-k}$$

as $\alpha_0^+ = \left(\sqrt{\pi T}\right)^{-1} p^+$

$$L_0 = \frac{1}{2} \alpha_0^2 + \frac{1}{2} \sum_{k\neq 0} \underline{\alpha}_k \cdot \underline{\alpha}_{-k}$$

$$= \frac{1}{2} \alpha_0^2 + \sum_{k=1}^{\infty} \underline{\alpha}_{-k} \cdot \underline{\alpha}_k \qquad \text{not in quantum!}$$

$$\equiv \frac{1}{2\pi T} \left(p^2 + 2\pi T \underbrace{N}_{\text{level number}} \right)$$

$$S = \int dt \left(\dot{x}^\mu p_\mu + \sum \frac{i}{k} \dot{\underline{\alpha}}_k \cdot \underline{\alpha}_{-k} - \frac{1}{2} e_0 \left(p^2 + M^2 \right) \right)$$

$M^2 = 2\pi T N$

action ind of α_n^- $(n \neq 0)$ but it still appears in Lorentz charges!

$$J^{\mu\nu} = L^{\mu\nu} + S^{\mu\nu}$$

$$S^{\mu\nu} = -2 \sum_{k=1}^{\infty} \frac{i}{k} \alpha_{-k}^{[\mu} \alpha_k^{\nu]}$$

$$S^{-I} = -\sum_{k=1}^{\infty} \frac{i}{k} \left(\alpha_{-k}^{-} \alpha_k^{I} - \alpha_{-k}^{I} \alpha_k^{-} \right)$$

$$S^{IJ} = -2 \sum_{k=1}^{\infty} \frac{i}{k} \alpha_{-k}^{[I} \alpha_k^{J]}$$

PBs $\quad \{ x^{\mu}, p_{\nu} \} = \delta^{\mu}_{\nu}$

$$\{ \alpha_k^{I}, \alpha_{-k}^{J} \} = -ik\, \delta^{IJ}$$

$S^{\mu\nu}$ also obey Lorentz alg (because $\{ L^{\mu\nu}, S^{\rho\sigma} \} = 0$)

$$\downarrow$$
$$x^{\mu} p^{\nu} - x^{\nu} p^{\mu}$$

can check they do still

need eg $\quad \{ \alpha_k^{-}, \alpha_l^{-} \} = \frac{\pi T}{4 p_-^2} \sum_{m,n} \{ \alpha_m^{I} \alpha_{k-m\, I}, \alpha_n^{J} \alpha_{l-n\, J} \}$

$$= \ldots$$

$$= \frac{i \sqrt{\pi T}}{p_-} (k-l)\, \alpha_{k+l}^{-}$$

$$\{ \alpha_k^{-}, \alpha_l^{I} \} = -i \frac{\sqrt{\pi T}}{p_-}\, l\, \alpha_{k+l}^{I}$$