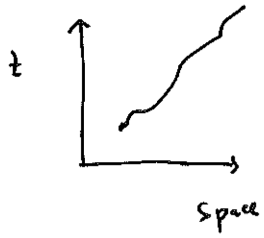
Turkey: NG String (classical) (10 pages)

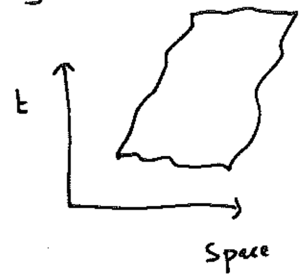
└ closed

Recall point particle



action ~ length of worldline

string



extends in space

→ traces out 2d surface in spacetime

worldsheet

Strings can be open 

closed 

parametrise ~~string~~ ~~length~~ position on string by σ

Closed strings $\sigma \in [0, 2\pi]$ periodic $\sigma \sim \sigma + 2\pi$

2d coords on world sheet $\sigma^\alpha = (t, \sigma)$

position in spacetime $X^\mu(t, \sigma)$

closed string $X^\mu(t, \sigma + 2\pi) = X^\mu(t, \sigma)$

pull back metric to worldsheet

$$\gamma_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X^\nu \eta_{\mu\nu}$$

ie $ds^2 = \eta_{\mu\nu} dX^\mu dX^\nu = \underbrace{\eta_{\mu\nu} \frac{\partial X^\mu}{\partial \sigma^\alpha} \frac{\partial X^\nu}{\partial \sigma^\beta}}_{\text{defines metric on ws}} d\sigma^\alpha d\sigma^\beta$

Action ~ area of ws

Nambu-Goto action

$$S_{NG} = -T\int dt\,d\sigma\,\sqrt{-\det\gamma}$$

T - string tension

$[T] = L^{-2}$

eom $\quad \partial_\alpha\left(\sqrt{-\det\gamma}\,\gamma^{\alpha\beta}\partial_\beta X^\mu\right) = 0$

wave eqn for scalar fields X^μ

note $\gamma_{\alpha\beta} = \begin{pmatrix} \dot{X}^2 & \dot{X}\cdot X' \\ \dot{X}\cdot X' & X'^2 \end{pmatrix}$ $\qquad \dot{X} \equiv \frac{\partial}{\partial t}X$, $\quad X' \equiv \frac{\partial}{\partial\sigma}X$

so

$$S_{NG} = -T\int dt\,d\sigma\,\sqrt{(\dot{X}\cdot X')^2 - \dot{X}^2 X'^2}$$

Reparametrisation inv $\quad (t',\sigma') = (t'(t,\sigma),\ \sigma'(t,\sigma))$

infinitesimally, $\quad \delta_\xi X^\mu = \xi^\alpha\partial_\alpha X^\mu$ $\qquad$ Diff_2

$$\Rightarrow \delta_\xi\sqrt{-\det\gamma} = \partial_\alpha\left(\xi^\alpha\sqrt{-\det\gamma}\right)$$

Hamiltonian form :

$$P_\mu = \frac{\partial L}{\partial\dot{X}^\mu} = \frac{T}{\sqrt{-\det\gamma}}\left(\dot{X}_\mu X'^2 - X'_\mu(\dot{X}\cdot X')\right)$$

$$\Rightarrow \quad P^2 + (TX')^2 = 0$$

$$X'^\mu P_\mu = 0$$

and Hamiltonian

$$H = \dot{X}^\mu P_\mu - L = 0$$

Now 2 constraints $\rightarrow$ 2 Lagrange multipliers e, u

$$S[X, P, e, u] = \int dt\, d\sigma \left(\dot{X}^\mu P_\mu - \frac{1}{2} e \left(P^2 + (TX')^2 \right) - u X'^\mu P_\mu \right)$$

check: P eom $\Rightarrow$ $P = e^{-1} D_t X$

$$D_t \equiv \partial_t - u \partial_\sigma$$

$$\Rightarrow S[X, P = e^{-1} D_t X, e, u] = \int dt\, d\sigma\, \frac{1}{2} \left(e^{-1} (D_t X)^2 - e (TX')^2 \right)$$

vary u $\Rightarrow$ $u = \dfrac{\dot{X} \cdot X'}{X'^2}$ $\Rightarrow$ $D_t X = \dot{X} - u X'$

$$= \frac{\det \gamma}{X'^2}$$

then

$$S = \int dt\, d\sigma\, \frac{1}{2} \left(e^{-1} \frac{\det \gamma}{X'^2} - e (TX')^2 \right)$$

vary e $\Rightarrow$ $Te = \dfrac{\sqrt{-\det \gamma}}{X'^2}$

$$\Rightarrow S = S_{NG}$$

n.b. assumed $e \neq 0$, $X'^2 \neq 0$

Alternative form:

define $\mathcal{H}_\pm = \frac{1}{4T}\left(P \pm TX' \right)^2$

$$\Rightarrow S = \int dt\, d\sigma \left(\dot{X}^\mu P_\mu - \lambda^- \mathcal{H}_- - \lambda^+ \mathcal{H}_+ \right)$$

$$\lambda^\pm = Te \pm u$$

The PBs are $\{ X^\mu(\sigma), P_\nu(\sigma') \} = \delta^\mu{}_\nu\, \delta(\sigma - \sigma')$

to check this fully, Fourier expand

let us write

$$P - TX' = \sqrt{\frac{T}{\pi}} \sum_{k\in\mathbb{Z}} e^{ik\sigma} \alpha_k(t) \qquad \alpha_k^* = \alpha_{-k}$$

$$P + TX' = \sqrt{\frac{T}{\pi}} \sum_{k\in\mathbb{Z}} e^{-ik\sigma} \tilde{\alpha}_k(t) \qquad \tilde{\alpha}_k^* = \tilde{\alpha}_{-k}$$

determine zero modes $(k=0)$

$$\int_0^{2\pi} d\sigma \left(P \pm TX' \right) = \int_0^{2\pi} d\sigma\, P \qquad \text{define } p \equiv \int_0^{2\pi} d\sigma\, P$$

$$\phantom{\int_0^{2\pi}} = \begin{cases} \sqrt{\frac{T}{\pi}}\, 2\pi\, \alpha_0 \\ \sqrt{\frac{T}{\pi}}\, 2\pi\, \tilde{\alpha}_0 \end{cases}$$

$$\Rightarrow \alpha_0 = \tilde{\alpha}_0 = \frac{p}{\sqrt{4\pi T}}$$

note $\int_0^{2\pi} e^{in\sigma}\, d\sigma = \begin{cases} 0 & n \neq 0 \\ 2\pi & n = 0 \end{cases}$

now find

$$P(t,\sigma) = \frac{1}{2\pi} p(t) + \sqrt{\frac{T}{4\pi}} \sum_{k\neq 0} e^{ik\sigma} \left(\alpha_k(t) + \tilde{\alpha}_{-k}(t) \right)$$

$$X(t,\sigma) = x(t) + \frac{1}{\sqrt{4\pi T}} \sum_{k\neq 0} \frac{i}{k} e^{ik\sigma} \left(\alpha_k(t) - \tilde{\alpha}_{-k}(t) \right)$$

so $$\dot{X}(t,\sigma) = \dot{x} + \frac{1}{\sqrt{4\pi T}} \sum_{k\neq 0} \frac{i}{k} e^{ik\sigma} \left(\dot{\alpha}_k - \dot{\tilde{\alpha}}_{-k} \right)$$

$$\int_0^{2\pi} d\sigma\, \dot{X}^\mu P_\mu = \int_0^{2\pi} d\sigma \left(\dot{x}^\mu p_\mu \frac{1}{2\pi} + \frac{1}{4\pi} \sum_{\substack{k,l \\ \neq 0}} e^{i(k+l)\sigma} \frac{i}{k} \left(\dot{\alpha}^\mu_k - \dot{\tilde{\alpha}}^\mu_{-k} \right) \left(\alpha_{l\mu} + \tilde{\alpha}_{-l\mu} \right) \right)$$

only contributes if $k+l=0$

$$= \dot{x}^\mu p_\mu + \frac{1}{2} \sum_{\substack{k \\ \neq 0}} \frac{i}{k} \Big(\dot{\alpha}_k \cdot \alpha_{-k} + \dot{\alpha}_k \cdot \tilde{\alpha}_k - \dot{\tilde{\alpha}}_{-k} \cdot \alpha_{-k} - \dot{\tilde{\alpha}}_{-k} \cdot \tilde{\alpha}_k \Big)$$

$$= \dot{x}^\mu p_\mu + \frac{1}{2} \sum_{k\neq 0} \frac{i}{k} \left(\dot{\alpha}_k \alpha_{-k} + \dot{\tilde{\alpha}}_k \cdot \alpha_{-k} \right)$$

$$+ \frac{1}{2} \sum_{k\neq 0} \frac{i}{k} \frac{d}{dt} \left(\alpha_k \cdot \tilde{\alpha}_k \right)$$

Also def

$$\mathcal{H}_\pm = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} e^{\mp in\sigma} L^\pm_k$$

s.t. $$L^\pm_k = \int_0^{2\pi} d\sigma\, e^{\pm ik\sigma} \mathcal{H}_\pm$$

One finds $$L^+_k = \frac{1}{2} \sum_{k\in\mathbb{Z}} \tilde{\alpha}_k \cdot \tilde{\alpha}_{n-k}$$

$$L^-_k = \frac{1}{2} \sum_{k\in\mathbb{Z}} \alpha_k \cdot \alpha_{n-k}$$

then

$$S = \int dt \left(\dot{x}^\mu p_\mu + \sum_{k \neq 0} \frac{i}{k} \left(\dot{\alpha}_k{}^\mu \alpha_{-k\mu} + \dot{\tilde{\alpha}}_k{}^\mu \tilde{\alpha}_{-k\mu} \right) - \sum_k \left(\lambda^-_{-k} L^-_k + \lambda^+_{-k} L^+_k \right) \right)$$

note if $S = \int \frac{i}{c} \dot{\alpha} \alpha^*$

let $\alpha = \sqrt{\frac{|c|}{2}} \left(q + i\, \mathrm{sign}(c)\, p \right)$

$\alpha^* = \sqrt{\frac{|c|}{2}} \left(q - i\, \mathrm{sign}(c)\, p \right)$

$$\dot{\alpha}\alpha^* = \frac{|c|}{2} \left(\dot{q} q + i\,\mathrm{sign}(c) \dot{p} q - i\,\mathrm{sign}(c)\, p \dot{q} + p \dot{p} \right)$$

$$= \frac{i\,\mathrm{sign}(c)|c|}{2} \left(\dot{p} q - p \dot{q} \right) + \frac{d}{dt}(\ldots)$$

$$= -\frac{ic}{2} 2 p \dot{q} + \frac{d}{dt}(\ldots)$$

$$\Rightarrow S = \int p \dot{q} \Rightarrow \{q, p\} = 1 \Rightarrow \{\alpha, \alpha^*\} = -ic$$

<u>so</u>:

$$\{ x^\mu, p_\nu \} = \delta^\mu{}_\nu$$

$$\{ \alpha_k{}^\mu, \alpha_l{}^\nu \} = -ik \eta^{\mu\nu} \delta_{k+l,0}$$

$$\{ \tilde{\alpha}_k{}^\mu, \tilde{\alpha}_l{}^\nu \} = -ik \eta^{\mu\nu} \delta_{k+l,0}$$

and alg of constraints

$$\{ L_k, L_l \} = -i(k-l) L_{k+l}$$

Gauge inv and symmetry:

These PBs do imply $\{ X^\mu(\sigma), P_\nu(\sigma') \} = \delta^\mu_\nu \delta(\sigma-\sigma')$

and

$$\{ \mathcal{H}_\pm(\sigma), \mathcal{H}_\pm(\sigma') \} = \pm \left(\mathcal{H}_\pm(\sigma) + \mathcal{H}_\pm(\sigma') \right) \delta'(\sigma-\sigma')$$

$$\{ \mathcal{H}_+(\sigma), \mathcal{H}_-(\sigma') \} = 0$$

$\Rightarrow$ first class, generate $\mathrm{Diff}_1 \oplus \mathrm{Diff}_1$

gauge transfs

$$\delta_\xi F = \{ F, \int d\sigma \, (\xi^- \mathcal{H}_- + \xi^+ \mathcal{H}_+) \}$$

$$\Rightarrow \quad \delta_\xi X = \frac{1}{2T} \xi^- (P - TX') + \frac{1}{2T} \xi^+ (P + TX')$$

$$\delta_\xi P = -\frac{1}{2} \left(\xi^- (P - TX') \right)' + \frac{1}{2} \left(\xi^+ (P + TX') \right)'$$

note $\delta_{\xi^\pm} (P \mp TX') = 0 \quad , \quad \delta_{\xi^\mp} \mathcal{H}_\pm = 0$

also $\delta \lambda^\pm = \dot{\xi}^\pm \mp \lambda^\pm (\xi^\pm)' \pm \xi^\pm (\lambda^\pm)'$

Poincaré inv: $X'^\mu = \omega^\mu{}_\nu X^\nu + a^\mu$

$\qquad P_\mu = \omega_\mu{}^\nu P_\nu$

Noether charges $\mathcal{P}_\mu = \int_0^{2\pi} d\sigma \, P_\mu$

$\qquad \mathcal{J}_{\mu\nu} = 2 \int_0^{2\pi} d\sigma \, X_{[\mu} P_{\nu]}$

one finds in terms of Fourier exp that

$$P_\mu = p_\mu$$

$$J^{\mu\nu} = 2\, x^{[\mu} p^{\nu]} + S^{\mu\nu}$$

'spin'

$$S^{\mu\nu} = -2 \sum \frac{i}{k} \left(\alpha_{-k}^{[\mu} \alpha_k^{\nu]} + \tilde{\alpha}_{-k}^{[\mu} \alpha_k^{\nu]} \right)$$

Gauge fixing:

i) Monge / static gauge

$X^0(t,\sigma) = t$ $\quad \Rightarrow \quad \underline{X}' \cdot \underline{P} = 0$ $\qquad \underline{X}'$ tangent to string

$X^1(t,\sigma) = \sigma$ $\qquad \underline{X} = (X^2, \dots X^{D-1})$

$\underline{P}$ orthogonal to tangent to string

$\Rightarrow$ no longitudinal fluctuations, only transverse

ii) Conformal gauge

$\lambda^+ = \lambda^- = 1 \quad \Rightarrow \quad e = \frac{1}{T} \quad \Rightarrow \quad X'^2 = \sqrt{-\det \gamma}$

$\qquad u = 0 \quad \Rightarrow \quad \dot{X} \cdot X' = 0$

$$\Rightarrow \gamma_{\alpha\beta} = X'^2 \, \eta_{\alpha\beta}$$ <u>conformal</u> to 2-dim Minkowski metric

after integrating out P_μ,

$$S = \frac{T}{2} \int dt\, d\sigma \left(\dot{X}^2 - X'^2 \right) = -\frac{T}{2} \int dt\, d\sigma\, \eta^{\alpha\beta}\, \partial_\alpha X \cdot \partial_\beta X$$

2D Mink sp free field theory, X^0 wrong kin sign have constraints $\dot{X}\cdot X' = 0$

$$\dot{X}^2 + X'^2 = 0$$

Related to Polyakov action

$$S_{Poly} = -\frac{T}{2}\int d^2\sigma \sqrt{-\det\gamma}\, \gamma^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X^\nu\, \eta_{\mu\nu}$$

now $\gamma_{\alpha\beta}$ dynamical ws metric

eom $\partial_\alpha X \cdot \partial_\beta X - \frac{1}{2}\gamma^{\gamma\delta}\partial_\gamma X \cdot \partial_\delta X \;\gamma_{\mu\nu} = 0$

$$\Rightarrow \gamma_{\alpha\beta} = \underset{\substack{\uparrow \\ \text{some conformal factor}}}{\Omega^2} (\gamma_{\alpha\beta})_{NG}$$

$$\Rightarrow S_{Poly} = S_{NG}$$

2d always possible locally to have $\gamma_{\kappa\beta} = \Omega^2\, \eta_{\kappa\beta}$ ie conf gauge

Residual gauge inv in conf gauge:

$$\delta\lambda^\pm = \dot{\xi}^\pm \mp (\xi^\pm)'$$

so $\delta\lambda^\pm = 0$ if

$$(\partial_\tau - \partial_\sigma)\xi^+ = 0$$

$$(\partial_\tau + \partial_\sigma)\xi^- = 0$$

so $\xi^\pm = \xi^\pm(\tau \pm \sigma)$

correspond to conf transfs of $\eta_{\alpha\beta}$, $\delta_\xi \eta_{\alpha\beta} = 2\partial_{(\alpha}\xi^\gamma \eta_{\beta)\gamma} = \underset{\substack{\text{conf factor} \\ \downarrow}}{\chi}\, \eta_{\kappa\beta}$

$d>2$ finite # vectors, $d=2$ ∞ conf symmetry gp eom for X is $\underbrace{(\partial_t^2 - \partial_\sigma^2)} X = 0$

$$= (\partial_\tau - \partial_\sigma)(\partial_\tau + \partial_\sigma) X$$

$$\Rightarrow \quad X^\mu = X_L^\mu(t+\sigma) + X_R^\mu(t-\sigma)$$

have $\delta_\xi X_L^\mu(t+\sigma) = \xi^+ \partial_+ X_L^\mu$

$$\delta_\xi X_R^\mu(t-\sigma) = \xi^- \partial_- X_R^\mu$$

$\sigma^\pm = t \pm \sigma$