Turkey: Intro and rel point particle (14 pages)

$$L = L(x, \dot{x})$$

$$S = \int dt \; L(x, \dot{x})$$

$$\delta S = \int dt \; \delta L(x, \dot{x})$$

$$= \int dt \left(\frac{\partial L}{\partial x} \delta x + \underbrace{\frac{\partial L}{\partial \dot{x}} \frac{d}{dt} \delta x}_{} \right)$$

$$= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \delta x \right) - \frac{d}{dt} \frac{\partial L}{\partial \dot{x}} \delta x$$

$$\Rightarrow \quad \delta S = \int_{t_A}^{t_B} dt \left[\delta x \left(\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) \right) + \underbrace{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \delta x \right)}_{} \right]$$

assume

$$\delta x|_{t_A} = 0$$

$$\delta x|_{t_B} = 0$$

eom $$\frac{\partial L}{\partial x} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}} \right) = 0$$

Hamiltonian

define momentum $$p = \frac{\partial L}{\partial \dot{x}}$$

Hamiltonian $$H = \dot{x} p - L$$

$$H = H(x, p)$$

$$L = \dot{x} p - H(x, p)$$

so $$S = \int dt \left(\dot{x} p - H(x, p) \right)$$

Hamilton's eom: $$\delta S = \int dt \left(\delta x (-\dot{p}) - \frac{\partial H}{\partial x} \delta x - \frac{\partial H}{\partial p} \delta p + \dot{x} \delta p \right)$$

$$\Rightarrow \quad \dot{p} = -\frac{\partial H}{\partial x}$$

$$\dot{x} = \frac{\partial H}{\partial p}$$ ← solve for p algebraically and eliminate from action

let x^I, p_I $\quad I = 1, \dots n$

$$S = \int dt \left(\dot{x}^I p_I - H(x,p) \right)$$

Poisson bracket $\{ f, g \} = \dfrac{\partial f}{\partial x^I} \dfrac{\partial g}{\partial p_I} - \dfrac{\partial g}{\partial x^I} \dfrac{\partial f}{\partial p_I}$

f, g any two fns. of x^I, p_I

antisymmetric

Jacobi: $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$

have $\{ x^I, p_J \} = \delta^I{}_J$

eom $\quad \dot{x}^I = \{ x^I, H \}$ $\qquad$ in general $\dot{f} = \{ f, H \}$

$\dot{p}_I = \{ p_I, H \}$

Quantisation: classical observables become operators acting on states in Hilbert space

$$x^I, p_I \longrightarrow \hat{x}^I, \hat{p}_I$$

obey commutation relations

$$\{\ ,\ \} \rightarrow -\frac{i}{\hbar}[\ ,\ ] \quad \Rightarrow \quad [\hat{x}^I, \hat{p}_J] = i\hbar\, \delta^I{}_J$$

Schrodinger eqn $\quad i\hbar \dfrac{\partial \Psi}{\partial t} = \hat{H} \Psi \qquad \Psi$ wave function realise via

$$\hat{p}_I = -i\hbar \partial_I$$

$\hat{H} = \hat{H}(\hat{x}, \hat{p})$ note operator ordering ambiguities can occur → string theory

So: Hamiltonian formalism v. useful for quantisation

also appropriate for systems with constraints

(Dirac quantisation)

constraints eg $p = \frac{\partial L}{\partial \dot{x}}$ can't invert to get $\dot{x} = \dot{x}(p)$

for example if $\frac{\partial L}{\partial \dot{x}} = 0$ electromagnetism $\frac{\partial L}{\partial(\partial_0 A_0)} = 0$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2$$

$$E_i = \partial_0 A_i - \partial_i A_0$$

We will apply it to:

i) relativistic point particle

ii) relativistic (bosonic) string

Relativity (special): space and time → spacetime

coordinates $x^\mu = (ct, x^1, \dots x^{D-1})$

Symmetry: Lorentz transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

preserve the spacetime interval

$$ds^2 = -c^2 dt^2 + (dx^1)^2 + \dots + (dx^{D-1})^2$$

$$\equiv \eta_{\mu\nu} dx^\mu dx^\nu$$

$\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \dots +1)$ Minkowski metric

defines distances in spacetime

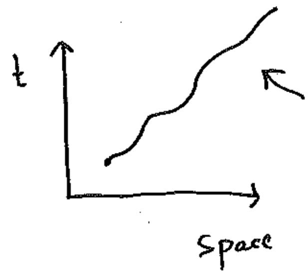
in fact $x'^\mu = \underbrace{\Lambda^\mu{}_\nu x^\nu}_{\text{Lorentz}} + \underbrace{a^\mu}_{\text{constant}}$

Poincaré transfs

Lorentz transfs: spatial rotations
boosts (mix time / space)

Poincaré: time + space translational invariance

Relativistic point particle:



action should involve some geometric quantity in spacetime - worldline

$$S = -mc^2 \int_A^B d\tau \qquad \tau \text{ proper time}$$

$$= -mc \int_A^B \sqrt{-ds^2}$$

$$= -mc \int_{t_A}^{t_B} \sqrt{-\dot{x}^2}\, dt$$

introduce t arbitrary worldline parameter

$$\dot{x} = \frac{dx}{dt}, \quad x^2 \equiv \eta_{\mu\nu} x^\mu x^\nu$$

$$x^\mu = x^\mu(t)$$

pick a differnt worldline time t'

$$x'(t') = x(t) \qquad \text{Scalar field on worldline}$$

say $t' = t - \xi(t)$ $\quad \xi$ infinitesimal

then $x(t) = x'(t-\xi) = x(t-\xi) + \delta_\xi x(t)$

$$= x(t) - \xi \dot{x}(t) + \delta_\xi x(t)$$

$\Rightarrow \quad \delta_\xi x(t) = \xi(t)\, \dot{x}(t) \qquad$ gauge 'symmetry' $\to$ redundancy in description

show $\delta_\xi \sqrt{-\dot{x}^2} = \frac{d}{dt}\left(\xi \sqrt{-\dot{x}^2}\right) \Rightarrow \delta_\xi S = 0$ if $\xi(t_A) = \xi(t_B) = 0$

say, ξ, ξ' 2 separate gauge trafs

$$\delta_\xi \delta_{\xi'} x = \delta_\xi (\xi' \dot{x})$$
$$= \xi (\dot{\xi}' \dot{x} + \xi' \ddot{x})$$

$$\Rightarrow [\delta_\xi, \delta_{\xi'}] x = (\xi \dot{\xi}' - \xi' \dot{\xi}) \dot{x}$$
$$\delta_{[\xi,\xi']} x$$

$$[\xi, \xi'] = \xi \dot{\xi}' - \xi' \dot{\xi}$$

$\Rightarrow$ form an algebra (Lie algebra)

called Diff_1

Gauge fixing: to remove redundancy

eg temporal gauge $\quad x^0 = ct, \quad x^i = x^i(t) \quad i = 1, \dots D-1$

$\delta_\xi x^0 = c\xi \quad \Rightarrow \quad \xi = 0$ is fixed

$$\Rightarrow \dot{x}^2 = -c^2 + \dot{\vec{x}}^2 = c^2\left(-1 + \frac{v^2}{c^2}\right) \qquad v^2 \equiv \dot{\vec{x}}^2$$

$$S = -mc^2 \int dt \sqrt{1 - \frac{v^2}{c^2}} = \int dt \left(-mc^2 + \tfrac{1}{2} m v^2 \left(1 + O\left(\tfrac{v^2}{c^2}\right)\right)\right)$$

non-rel limit $\Rightarrow$ $S = \int dt \, \frac{1}{2} m v^2$

(drop const pot energy $-mc^2$)

From now on, set $c = 1$. (choice of units)

Hamiltonian formalism of rel point particle

Lagrangian is $L = -m\sqrt{-\dot{x}^2}$

$$p_\mu = \frac{\partial L}{\partial \dot{x}^\mu} = \frac{m\dot{x}_\mu}{\sqrt{-\dot{x}^2}}$$

note $A_\mu \equiv \eta_{\mu\nu}A^\nu$
$A^\mu \equiv \eta^{\mu\nu}A_\nu$

but this $\Rightarrow$ $p^2 + m^2 = 0$

p is constrained, not all cpts ind

can't solve for $\dot{x}^\mu$ in terms of p_μ

Hamiltonian is $H = \dot{x}^\mu p_\mu - L$

$= 0$ vanishes!

Dirac's method applies

$p^2 + m^2 = 0$ 'primary' constraint (arises from defn of momentum)
(will not see secondary constraints)

impose using a Lagrange multiplier

Hamiltonian = (Lag mult) $\times$ (constraint)

then

$$S^{(x,p,e)} = \int \left(\dot{x}\cdot p - \tfrac{1}{2}e(p^2+m^2) \right) dt$$

e Lag mult

$$\frac{\delta S}{\delta e} = p^2+m^2 = 0$$ eom of e imposes constraint

Is this action the same?

check: p eom is $p = \frac{1}{e}\dot{x}$

can sub back in to $S = S[x, p, e]$

$$S[x, p = \tfrac{1}{e}\dot{x}, e] = \frac{1}{2}\int dt\,(e^{-1}\dot{x}^2 - em^2)$$

now solve e eom $\Rightarrow$ $me = \sqrt{-\dot{x}^2}$

$$\Rightarrow \quad S[x] = -m\int dt\sqrt{-\dot{x}^2} \qquad \text{yes} \checkmark$$

Gauge transfs of $S[x,p,e]$

$$\delta_\xi x = \xi\dot{x} \qquad \delta_\xi e = \frac{d}{dt}(e\xi)$$

$$\delta_\xi p = \xi\dot{p} \qquad\qquad \text{Diff}_1$$

equivalent to

$$\left.\begin{aligned} \delta_\alpha x &= \alpha(t)p \\ \delta_\alpha p &= 0 \\ \delta_\alpha e &= \dot{\alpha} \end{aligned}\right\} \text{ canonical transfs}$$

using eom, for $\alpha(t_A) = \alpha(t_B) = 0$

to see this, $$\delta_\alpha\begin{pmatrix} x \\ p \\ e\end{pmatrix} = \underbrace{\delta_{(\xi = \alpha e^{-1})}\begin{pmatrix} x \\ p \\ e\end{pmatrix}}_{\substack{\text{a Diff}_1 \text{ transf} \\ \text{with } \xi = \alpha e^{-1}}} + \underbrace{\begin{pmatrix} \alpha(p - e^{-1}\dot{x}) \\ -\frac{\alpha}{e}\dot{p} \\ 0\end{pmatrix}}_{\text{vanishes by eom}}$$

Gauge fixing of $S[x, p, e]$

i) temporal gauge $\quad x^0(t) = t$

then $\quad \dot{x}^\mu p_\mu = \dot{\vec{x}} \cdot \vec{p} - p^0$

$$p^2 + m^2 = 0 \Rightarrow p^0 = \pm \sqrt{|\vec{p}|^2 + m^2}$$

so $\quad S = \int (\dot{\vec{x}} \cdot \vec{p} - H) \quad$ gauge fixed action

$$H = p^0 = \pm \sqrt{|\vec{p}|^2 + m^2}$$

ii) light cone gauge

define $\quad x^\pm = \frac{1}{\sqrt{2}} (x^1 \pm x^0) \qquad \underline{x} = (x^2, \dots x^{D-1})$

$$p_\pm = \frac{1}{\sqrt{2}} (p_1 \pm p_0) \qquad \underline{p} = (p_2 \dots p_{D-1})$$

metric $\quad ds^2 = dx^+ dx^- + d\underline{x} \cdot d\underline{x} \quad \Rightarrow \eta_{+-} = \eta_{-+} = \eta^{+-} = \eta^{-+} = 1$

$$\dot{x}^\mu p_\mu = \dot{x}^+ p_+ + \dot{x}^- p_- + \dot{\underline{x}} \cdot \underline{p}$$

$$p^2 = 2 p_+ p_- + |\underline{p}|^2$$

gauge fixing is $\quad x^+(t) = t$

$\delta_\alpha x^+ = \alpha p^+ = \alpha p_- \quad \Rightarrow$ need $p_- \neq 0 \Rightarrow \alpha = 0$ to fix

then

$$S = \int dt \left(\dot{\underline{x}} \cdot \underline{p} + \dot{x}^- p_- - H \right)$$

$$H = -p_+ = \frac{1}{2p_-} \left(|\underline{p}|^2 + m^2 \right)$$

Constraints in Hamiltonian formalism:

Say $$S = \int dt \left(\dot{x}^I p_I - \lambda^i \varphi_i(x,p) \right)$$

$I = 1, \dots, N$ $\qquad\qquad$ $\varphi_i = 0$ constraints

$i = 1, \dots, n < N$

if $$\{ \varphi_i, \varphi_j \} = f_{ij}{}^k \varphi_k \qquad f_{[ij]}{}^k = f_{ij}{}^k$$

call them 1st class constraints

these generate gauge ~~invariances~~ transformations

any arb fn $Q = Q(x,p)$ generates a canonical transf

infinitesimal change of coords preserving Poisson bracket

$$\delta_\varepsilon f = \{ f, Q \} \varepsilon$$

so $$\delta_\varepsilon x^I = \varepsilon \frac{\partial Q}{\partial p_I}$$

$$\delta_\varepsilon p_I = -\varepsilon \frac{\partial Q}{\partial x^I}$$

$$\delta_\varepsilon (\dot{x}^I p_I) = \frac{d}{dt}\left(\varepsilon \frac{\partial Q}{\partial p_I} \right) p_I - \dot{x}^I \varepsilon \frac{\partial Q}{\partial x^I}$$

$$= \frac{d}{dt}\left(\varepsilon \frac{\partial Q}{\partial p_I} p_I \right) - \varepsilon \frac{\partial Q}{\partial x^I} \frac{\partial H}{\partial p_I} + \varepsilon \frac{\partial Q}{\partial p_I} \frac{\partial H}{\partial x^I}$$

$$= \frac{d}{dt}\left(\varepsilon \frac{\partial Q}{\partial p_I} p_I \right) \underbrace{- \varepsilon \{ Q, H \}}_{= -\varepsilon \dot{Q}}$$

$$= \frac{d}{dt}\left(\varepsilon \left[\frac{\partial Q}{\partial p_I} p_I - Q \right] \right) + \dot{\varepsilon} Q$$

if $\varepsilon Q = \varepsilon^i \varphi_i$

then $\delta_\varepsilon (\dot{x}^I p_I) = \dot{\varepsilon}^i \varphi_i + \frac{d}{dt} (\dots)$

while $\delta_\varepsilon (\lambda^i \varphi_i) = \delta_\varepsilon \lambda^i \varphi_i + \lambda^i \varepsilon^j \{ \varphi_i, \varphi_j \}$

$= \left(\delta_\varepsilon \lambda^k + \lambda^i \varepsilon^j f_{ij}{}^k \right) \varphi_k$

so

$\delta_\varepsilon S = \int dt \left(\left(\dot{\varepsilon}^k - \delta_\varepsilon \lambda^k - \lambda^i \varepsilon^j f_{ij}{}^k \right) \varphi_k + \frac{d}{dt} (\dots) \right)$

pick $\delta_\varepsilon \lambda^i = \dot{\varepsilon}^i + \varepsilon^j \lambda^k f_{jk}{}^i$

then $\delta_\varepsilon S = 0$ if $\varepsilon^i (t_A) = \varepsilon^i (t_B) = 0$

eg point particle

$\varphi = \frac{1}{2} (p^2 + m^2)$ trivially 1st class

$\delta_\alpha x = \frac{1}{2} \{ \alpha x, p^2 + m^2 \} = \alpha p$

$\delta_\alpha p = \frac{1}{2} \{ \alpha p, p^2 + m^2 \} = 0$

$\delta_\alpha e = \dot{\alpha}$

Gauge fixing in Hamiltonian formalism:

pick n gauge fixing conds $\chi^i (x, p) = 0$

$\delta_\varepsilon \chi^i = \{ \chi^i, \varphi_j \} \varepsilon^j = 0$

need $\det (\{ \chi^i, \varphi_j \}) \neq 0$ so $\varepsilon^j = 0$ $\rightarrow$ fixed

this also guarantees can find gauge transf to set $\chi^i = 0$ if $\chi^i = f^i$

say $\{\chi^i, \varphi_j\}$ not invertible

then i) gauge fixing conds don't fix the gauge

ii) can't always satisfy gauge fixing conds via gauge transf

Symmetries and Noether's theorem:

suppose $S[\phi]$ invariant under $\delta_\varepsilon \phi$ $\quad \varepsilon$ constant

then for arbitrary ε

$$\delta_\varepsilon S = \int dt \ \dot{\varepsilon}\, Q = -\int dt \ \varepsilon\, \dot{Q}$$

but $\delta_\varepsilon S = 0$ by eom

$\Rightarrow$ need $\dot{Q} = 0$ $\quad$ conserved

have also that $\delta_\varepsilon \phi = \{\phi, \varepsilon Q\}$

eg point particle

Poincare symmetry in spacetime

small x, p

$$\delta_\Lambda X^\mu = \Lambda^\mu{}_\nu X^\nu + a^\mu \qquad \Lambda_{\mu\nu} = -\Lambda_{\nu\mu} \quad \text{infinitesimal Lorentz rotation}$$

$$\delta_\Lambda P_\mu = \Lambda_\mu{}^\nu P_\nu \qquad a^\mu \text{ constant}$$

$$\delta_\Lambda S = \int dt \left(\dot{a}^\mu p_\mu + \tfrac{1}{2} \dot{\Lambda}^\mu{}_\nu \left(X^\nu p_\mu - p^\nu x_\mu \right) \right)$$

$$\Rightarrow \quad P_\mu = p_\mu \qquad \text{are conserved}$$

$$J_{\mu\nu} = x_\mu p_\nu - x_\nu p_\mu$$

note

$$\{ J_{\mu\nu}, J_{\rho\sigma} \} = \left(\{ x_\mu p_\nu , x_\rho p_\sigma \} - \mu\nu \right) - \rho\sigma$$

$$= \left(\left(x_\mu \{ p_\nu, x_\rho \} p_\sigma + p_\nu \{ x_\mu, p_\sigma \} x_\rho \right) - \mu\nu \right) - \rho\sigma$$

$$\begin{aligned} = & - x_\mu p_\sigma \eta_{\nu\rho} + x_\rho p_\nu \eta_{\mu\sigma} \\ & + x_\nu p_\sigma \eta_{\mu\rho} - x_\rho p_\mu \eta_{\nu\sigma} \\ & + x_\mu p_\rho \eta_{\nu\sigma} - x_\sigma p_\nu \eta_{\mu\rho} \\ & - x_\nu p_\rho \eta_{\mu\sigma} + x_\sigma p_\mu \eta_{\nu\rho} \end{aligned}$$

$$\begin{aligned} = & \ \eta_{\mu\rho} J_{\nu\sigma} - \eta_{\nu\rho} J_{\mu\sigma} \\ & - \eta_{\mu\sigma} J_{\nu\rho} + \eta_{\nu\sigma} J_{\mu\rho} \end{aligned}$$

defines Lorentz symmetry algebra

aside: let $\Lambda^T \eta \Lambda = \eta$ Lorentz gp

let $\Lambda^\mu{}_\nu = \delta^\mu{}_\nu + \varepsilon \omega^\mu{}_\nu$ ε infinitesimal

$$\Rightarrow \quad \varepsilon \left(\omega_{\mu\nu} + \omega_{\nu\mu} \right) + O(\varepsilon^2) = 0$$

$\Rightarrow$ Lorentz algebra generated by $\omega_{\mu\nu} = - \omega_{\nu\mu}$ antisymmetric

basis $\left(J_{\mu\nu} \right)_\rho{}^\sigma = \eta_{\mu\rho} \delta_\nu{}^\sigma - \eta_{\nu\rho} \delta_\mu{}^\sigma$

~~note~~ antisym

labelling the $\frac{1}{2} D(D-1)$ independent generators

has above commutation rels

Quantization of rel point particle :

i) in temporal gauge

let $i = 1, \ldots D-1$

$$S = \int \left(\dot{x}^i p_i - H \right) \qquad H = \pm \sqrt{|\vec{p}|^2 + m^2}$$

have $\{ \hat{x}^i , \hat{p}_j \} = i\hbar \delta^i_j$

$$\hat{p}_i = -i\hbar \partial_i$$

$$i\hbar \frac{\partial \Psi}{\partial t} = \pm \sqrt{-\hbar^2 \nabla^2 + m^2}$$

square $\Rightarrow$ $\left(-\nabla^2 + \partial_t^2 + \left(\frac{m}{\hbar}\right)^2 \right) \Psi = 0$

Klein-Gordon equation Lorentz inv

ii) Von Dirac methods

Lorentz invariant

$$[\hat{x}^\mu , \hat{p}_\nu] = i\hbar \delta^\mu_\nu$$

$$\hat{p}_\mu = -i\hbar \partial_\mu$$

to remove unphysical states (gauge dof only)

impose constraint $(\hat{p}^2 + m^2) | \Psi \rangle = 0$ KG eqn

generally $\hat{\varphi}_i | \Psi \rangle = 0$

need $[\hat{\varphi}_i , \hat{\varphi}_j] | \Psi \rangle = 0 \quad \forall i,j$

$\sim$

$= i\hbar f_{ij}{}^k \hat{\varphi}_k$ but may be op ordering ambiguities

Now let $\hbar = 1$.