Revision notes for Part III Supersymmetry

Even though the sound of it is something quite atrocious...
- from "Supercalifragilisticexpialidocious," *Mary Poppins*

Notes by Chris Blair, May 2011

1 Supersummary

Basic definitions/identities:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$$

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon^{\dot\alpha\dot\beta} \quad \varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta} \qquad \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon^{\dot\alpha\dot\beta}$$

$$\psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta \qquad \psi_\alpha \equiv \varepsilon_{\alpha\beta}\psi^\beta \quad \psi\chi \equiv \psi^\alpha\chi_\alpha \quad \bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}$$

$$\sigma^\mu_{\alpha\dot\alpha} = (\mathbb{I},\sigma^1,\sigma^2,\sigma^3) \quad (\bar\sigma^\mu)^{\dot\alpha\alpha} = (\mathbb{I},-\sigma^1,-\sigma^2,-\sigma^3)$$

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right) \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{i}{4}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)$$

$$\mathrm{tr}\left(\sigma^\mu\bar\sigma^\nu\right) = 2\eta^{\mu\nu} \quad \sigma^\mu_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\beta\beta} = 2\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha}$$

$$(\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha} \qquad (\psi^\alpha)^\dagger = \bar\psi^{\dot\alpha}$$

$$(\psi\chi)^\dagger = \bar\chi\bar\psi \quad (\psi\sigma^\mu\bar\chi)^\dagger = \chi\sigma^\mu\bar\psi$$

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}(\theta\theta) \quad \theta_\alpha\theta_\beta = +\frac{1}{2}\varepsilon_{\alpha\beta}(\theta\theta)$$

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = +\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}(\bar\theta\bar\theta) \quad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}(\bar\theta\bar\theta)$$

Some results:

$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta)$$

$$(\theta\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\psi\chi)$$

$$(\theta\psi)(\bar\chi\bar\eta) = -\frac{1}{2}(\theta\sigma^\mu\bar\eta)(\bar\chi\bar\sigma_\mu\psi)$$

$$\psi\sigma^\mu\bar\chi = -\bar\chi\bar\sigma^\mu\psi$$

$$\psi\sigma^\mu\bar\sigma^\nu\chi = \chi\sigma^\nu\bar\sigma^\mu\psi$$

Super-Poincare algebra for $\mathcal{N}=1$ SUSY:

$$[P^\mu,P^\nu] = 0 \quad [M^{\mu\nu},P^\sigma] = i(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma}) \quad [M^{\mu\nu},M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})$$

$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta \quad [\bar{Q}^{\dot\alpha}, M^{\mu\nu}] = (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} \bar{Q}^{\dot\beta} \quad [Q_\alpha, P_\mu] = [\bar{Q}^{\dot\alpha}, P_\mu] = 0$$

$$\{Q_\alpha, Q^\beta\} = \{\bar{Q}_{\dot\alpha}, \bar{Q}^{\dot\beta}\} = 0 \quad \{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

$$[Q_\alpha, T_i] = 0 \quad [Q_\alpha, R] = Q_\alpha \quad [\bar{Q}_{\dot\alpha}, R] = -\bar{Q}_{\dot\alpha}$$

Extended SUSY:

$$\{Q^A_\alpha, \bar{Q}_{\dot\beta B}\} = 2\sigma^\mu_{\alpha\dot\beta} P_\mu \delta^A_B \quad \{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta} Z^{AB} \quad \{\bar{Q}_{\dot\alpha A}, \bar{Q}_{\dot\beta B}\} = \varepsilon_{\dot\alpha\dot\beta} (Z^\dagger)_{AB}$$

Superfields:

$$S(x^\mu, \theta_\alpha, \bar\theta^{\dot\alpha}) = \varphi + \theta\psi + \bar\theta\bar\chi + (\theta\theta)M + (\bar\theta\bar\theta)N + (\theta\sigma^\nu\bar\theta)V_\nu + (\theta\theta)\bar\theta\bar\lambda + (\bar\theta\bar\theta)\theta\rho + (\theta\theta)(\bar\theta\bar\theta)D$$

$$\delta S = i(\varepsilon\mathcal{Q} + \bar\varepsilon\bar{\mathcal{Q}})S \quad \mathcal{Q}_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\frac{\partial}{\partial x^\mu} \quad \bar{\mathcal{Q}}_{\dot\alpha} = i\frac{\partial}{\partial\bar\theta^{\dot\alpha}} + \theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial x^\mu}$$

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \quad \bar{D}_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

$$\{D_\alpha, \bar{D}_{\dot\alpha}\} = -2i\sigma^\mu_{\alpha\dot\alpha}\partial_\mu \qquad \text{others zero}$$

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta \quad \bar{D}_{\dot\alpha} y^\mu = 0$$

Chiral superfield:

$$\bar{D}_{\dot\alpha}\Phi = 0 \quad \Phi = \varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar\theta)\partial_\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar\theta - \frac{1}{4}(\theta\theta)(\bar\theta\bar\theta)\partial_\mu\partial^\mu\varphi$$

Vector superfield:

$$V(x,\theta,\bar\theta) = V^\dagger(x,\theta,\bar\theta)$$

$$\begin{aligned} V(x,\theta,\bar\theta) = {} & C + i\theta\chi - i\bar\theta\bar\chi + \frac{i}{2}(\theta\theta)(M+iN) - \frac{i}{2}(\bar\theta\bar\theta)(M-iN) + \theta\sigma^\mu\bar\theta V_\mu \\ & + (\theta\theta)\bar\theta\left(i\bar\lambda - \frac{1}{2}\bar\sigma^\mu\partial_\mu\chi\right) + (\bar\theta\bar\theta)\theta\left(-i\lambda - \frac{1}{2}\sigma^\mu\partial_\mu\bar\chi\right) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\left(D - \frac{1}{2}\partial_\mu\partial^\mu C\right) \end{aligned}$$

$$V \mapsto V + i(\Lambda - \Lambda^\dagger) \qquad \Phi \mapsto e^{-2iq\Lambda}\Phi$$

$$V_{WZ}(x,\theta,\bar\theta) = (\theta\sigma^\mu\bar\theta)V_\mu + i(\theta\theta)\bar\theta\bar\lambda - i(\bar\theta\bar\theta)\theta\lambda + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D$$

D- and F-terms:

$$(\theta\theta)(\bar\theta\bar\theta)D(x) \qquad (\theta\theta)F(x)$$

Field strength:

$$W_\alpha = -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V = -i\lambda_\alpha(y) + \theta_\alpha D(y) + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + (\theta\theta)\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta}(y)$$

Lagrangian:

$$\mathcal{L} = \sum_i \Phi_i^\dagger e^{2q_i V}\Phi_i\Big|_D + \Big(W(\Phi_i)\Big|_F + h.c.\Big) + \Big(f(\Phi_i)(W^\alpha W_\alpha)\Big|_F + h.c.\Big) + \xi V\Big|_D$$

Non-abelian vector superfields:

$$V_\mu = V_\mu^a T^a \quad D = D^a T^a \quad \lambda = \lambda^a T^a$$

$$\Phi \to e^{-2i\Lambda q} \qquad \Lambda = \Lambda^a T^a \qquad e^{2qV} \to e^{2qV'} = e^{-2i\Lambda^\dagger q} e^{2qV} e^{2i\Lambda q}$$

$$W_\alpha = -\frac{1}{8q}(\bar{D}\bar{D})\left(e^{-2qV}D_\alpha e^{2qV}\right) \qquad W_\alpha \to e^{2iq\Lambda^\dagger} W_\alpha e^{2iq\Lambda}$$

$$W_\alpha^a = -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) + (\theta\theta)\sigma^\mu_{\alpha\dot{\beta}} D_\mu \bar{\lambda}^{a\dot{\beta}}(y)$$

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + q f^{abc} V_\mu^b V_\mu^c \qquad D_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + q V_\mu^b \bar{\lambda}^c f^{abc}$$

$$\frac{1}{4}\Big(\mathrm{tr}\, W^\alpha W_\alpha\Big|_F + h.c.\Big) = \frac{1}{2}D^a D^a - i\lambda^a \sigma^\mu D_\mu \bar{\lambda}^a - \frac{1}{4}F^{a\mu\nu}F_{\mu\nu}^a$$

$$\mathcal{L}_D = \frac{1}{2}D^a D^a + q\varphi_m^\dagger D^a T_{mn}^a \varphi_n \qquad V_D(\varphi) = \frac{q^2}{2}\left(\varphi_m^\dagger T_{mn}^a \varphi_n\right)\left(\varphi_p^\dagger T_{pq}^a \varphi_q\right)$$

Supersymmetry breaking:

$$Q_\alpha|0\rangle \neq 0 \quad \bar{Q}_{\dot{\alpha}}|0\rangle \neq 0$$

$$\sum_{\alpha=1}^{2}\left(Q_\alpha Q_\alpha^\dagger + Q_\alpha^\dagger Q_\alpha\right) = 4E \Rightarrow \text{broken SUSY: } E_{vac} > 0\,,\ \text{unbroken SUSY: } E_{vac} = 0$$

F-term breaking:

$$\langle F\rangle \neq 0 \Rightarrow \langle\delta\varphi\rangle = \langle\delta F\rangle = 0 \quad \langle\delta\psi\rangle \neq 0$$

$$V_F = |\frac{\partial W}{\partial\varphi}|^2 = |F|^2 \Rightarrow V_F > 0 \text{ for} F - \text{term breaking}$$

O'Raifeartaigh model:

$$K = \Phi_i^\dagger \Phi_i \qquad W = g\Phi_1(\Phi_3^2 - m^2) + M\Phi_2\Phi_3$$

D-term breaking:

$$\langle D\rangle \neq 0 \Rightarrow \langle\delta\lambda\rangle = \varepsilon\langle D\rangle \neq 0$$

Supertrace:

$$\mathrm{Str} M^2 = \sum_j (-1)^{2j+1}(2j+1)m_j^2 = 0$$

MSSM:

vector	$SU(3)_C \times SU(2)_L \times U(1)_Y$	spin-1/2	spin-1
G	$(8,1,0)$	gluino $\widetilde{g}$	gluon g
W	$(1,3,0)$	wino $\widetilde{w}$	W-boson W^μ
B	$(1,1,0)$	bino $\widetilde{b}$	hypercharge boson B^μ
chiral		spin-0	spin-1/2
$Q_i = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix}$	$(3,2,1/6)$	squarks $\begin{pmatrix} \widetilde{u}_{Li} \\ \widetilde{d}_{Li} \end{pmatrix}$	quarks $\begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix}$
$L_i = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}$	$(1,2,-1/2)$	sleptons $\begin{pmatrix} \widetilde{\nu}_{Li} \\ \widetilde{e}_{Li} \end{pmatrix}$	leptons $\begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}$
$\bar{u}_{iR}$	$(\bar{3},1,-2/3)$	antisquark $\widetilde{u}^*_{iR}$	antiquark $\bar{u}_{iR}$
$\bar{d}_{iR}$	$(\bar{3},1,1/3)$	antisquark $\widetilde{d}^*_{iR}$	antiquark $\bar{d}_{iR}$
$\bar{e}_{iR}$	$(1,1,1)$	slepton $\widetilde{e}^*_{iR}$	lepton $\bar{e}_{iR}$
$H_1 = \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix}$	$(1,2,-1/2)$	Higgs $\begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix}$	Higgsino $\begin{pmatrix} \widetilde{H}_1^0 \\ \widetilde{H}_1^- \end{pmatrix}$
$H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}$	$(1,2,1/2)$	Higgs $\begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}$	Higgsino $\begin{pmatrix} \widetilde{H}_2^+ \\ \widetilde{H}_2^0 \end{pmatrix}$

$$W_{RP} = (Y_U)_{ij} Q_i H_2 \bar{u}_{Rj} - (Y_D)_{ij} Q_i H_1 \bar{d}_{Rj} - (Y_E)_{ij} L_i H_1 \bar{e}_{jR} + \mu H_1 H_2$$

$$W_{\not{RP}} = \frac{1}{2}\lambda_{ijk} L_i L_j \bar{e}_{kR} + \lambda'_{ijk} L_i Q_j \bar{d}_k + \kappa_i L_i H_2 + \frac{1}{2}\lambda''_{ijk} \bar{u}_{iR} \bar{d}_{jR} \bar{d}_{kR}$$

R-parity:

$$R = (-1)^{3(B-L)+2S} \qquad +1 \text{ for Standard Model particles} \qquad -1 \text{ for superpartners}$$

2 Superset-up

2.1 Basics

Metric signature Our metric signature is mostly minus:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \mathrm{diag}\,(1,-1,-1,-1)$$

Representation theory If these were proper notes there would be a discussion of representations of the Lorentz group and $SL(2,\mathbb{C})$ here as well as an explanation of what spinors are. However, there isn't.

Raising and lowering indices Spinor indices are raised and lowered using the epsilon tensor:

$$\varepsilon^{\alpha\beta} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon^{\dot{\alpha}\dot{\beta}}$$

$$\varepsilon_{\alpha\beta} = -\varepsilon^{\alpha\beta} \qquad \varepsilon_{\dot\alpha\dot\beta} = -\varepsilon^{\dot\alpha\dot\beta} \quad \varepsilon_{\alpha\beta}\varepsilon^{\beta\gamma} = \delta_\alpha^\gamma$$

When raising or lowering the order is important:

$$\psi^\alpha \equiv \varepsilon^{\alpha\beta}\psi_\beta \qquad \psi_\alpha \equiv \varepsilon_{\alpha\beta}\psi^\beta$$

Index-free contraction We define the contraction of two spinors as follows:

$$\psi\chi \equiv \psi^\alpha\chi_\alpha$$

$$\bar\psi\bar\chi \equiv \bar\psi_{\dot\alpha}\bar\chi^{\dot\alpha}$$

Note that **left**-handed spinors have the index on the **left** raised, and **right**-handed spinors have the index on the **right** raised. The index free notation commutes:

$$\psi\chi = \chi\psi \qquad \bar\psi\bar\chi = \bar\chi\bar\psi$$

Proof: This is because $\chi\psi = \chi^\alpha\psi_\alpha = -\psi_\alpha\chi^\alpha = -\varepsilon_{\alpha\beta}\psi^\beta\chi^\alpha = +\psi^\beta\varepsilon_{\beta\alpha}\chi^\alpha = \psi^\beta\chi_\beta = \psi\chi$, and similarly for the barred spinors.

Sigma matrices In 4-component notation,

$$\sigma^\mu = (\mathbb{I}, \sigma^1, \sigma^2, \sigma^3)$$

with index structure

$$\sigma^\mu_{\alpha\dot\alpha}$$

Recall that the Pauli matrices are hermitian and traceless, and $\sigma^i\sigma^j = \delta^{ij} + i\varepsilon^{ijk}\sigma^k$. We can also define

$$(\bar\sigma^\mu)^{\dot\alpha\alpha} \equiv \varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\beta\dot\beta}$$

which has the numerical form

$$\bar\sigma^\mu = (\mathbb{I}, -\sigma^1, -\sigma^2, -\sigma^3)$$

Proof: To demonstrate this it is convenient to use matrix notation and note that fact that $\varepsilon^{\alpha\beta} = \varepsilon^{\dot\alpha\dot\beta} = i\sigma^2$. Then from the index structure we have $\bar\sigma^\mu = -(\varepsilon\sigma^\mu\varepsilon)^T = (\sigma^2\sigma^\mu\sigma^2)^T$, from which the result easily follows as σ^1, σ^3 and the identity are equal to their transpose while $\sigma^2 = -(\sigma^2)^T$.

Sigma matrix identities We can prove some identities involving products of these sigma matrix objects:

$$\mathrm{tr}\,(\sigma^\mu\bar\sigma^\nu) = 2\eta^{\mu\nu}$$

Proof: This is most easily seen by noting that the product of any two of the matrices involved is a Pauli matrix and so traceless if the two matrices are distinct; if on the other hand $\mu = \nu$ then we get $\pm\mathbb{I}$, with trace ± 2, with the plus sign corresponding to $\mu = \nu = 0$ and the minus sign corresponding to $\mu = i$, recalling that $\sigma^i = -\bar\sigma^i$.

Another result is

$$\sigma^\mu_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\beta\beta} = 2\delta_\alpha^\beta\delta_{\dot\alpha}^{\dot\beta}$$

Proof: One can argue that the right-hand side must be proportional to the two Kronecker deltas as they are the only available tensors with the right index structure, and then evaluate for specific components to get the constant of proportionality.

The previous result with $\dot\alpha$ and $\dot\beta$ contracted gives

$$(\sigma^\mu\bar\sigma_\mu)_\alpha^{\ \beta} = 4\delta_\alpha^\beta$$

Left- and right-handed representations The matrices

$$(\sigma^{\mu\nu})_\alpha^{\ \beta} = \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right) \quad (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta} = \frac{i}{4}\left(\bar\sigma^\mu\sigma^\nu - \bar\sigma^\nu\sigma^\mu\right)$$

furnish us with left- and right-handed representations of the Lorentz group on two-component spinors.

Hermitian conjugation Hermitian conjugation exchanges left- and right-handed spinors:

$$(\psi_\alpha)^\dagger = \bar\psi_{\dot\alpha} \qquad (\psi^\alpha)^\dagger = \bar\psi^{\dot\alpha}$$

When taking the hermitian conjugate of a product **do not include minus signs from interchanges**

$$(\psi_\alpha\chi_\beta)^\dagger = \chi^\dagger_\beta\psi^\dagger_\alpha$$

This means that

$$(\psi\chi)^\dagger = (\psi^\alpha\chi_\alpha)^\dagger = \bar\chi_{\dot\alpha}\bar\psi^{\dot\alpha} = \bar\chi\bar\psi$$

Recall that σ^μ is hermitian, so

$$(\psi\sigma^\mu\bar\chi)^\dagger = \chi\sigma^\mu\bar\psi$$

The rule for indices is to swap order than trade dots for no dots and vice versa, i.e.

$$(\sigma^\mu_{\alpha\dot\beta})^\dagger = \sigma^\mu_{\beta\dot\alpha}$$

2.2 Spinor identities

A very important simplification We have

$$\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}(\theta\theta)$$

Proof: By the definition of the contraction of two spinors, $\theta\theta = \theta^\alpha\theta_\alpha = \varepsilon_{\alpha\beta}\theta^\alpha\theta^\beta = -\theta^1\theta^2 + \theta^1\theta^2 = -2\theta^1\theta^2 = +2\theta^2\theta^1$. The result follows by recalling that $\varepsilon^{12} = +1$ and $\varepsilon^{21} = -1$.

Similarly,

$$\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = +\frac{1}{2}\varepsilon^{\dot\alpha\dot\beta}(\bar\theta\bar\theta)$$

Proof: Identical to the previous, by using $\bar\theta\bar\theta = \bar\theta_{\dot\alpha}\bar\theta^{\dot\alpha} = \varepsilon_{\dot\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\theta^{\dot\alpha} = -\bar\theta^{\dot 2}\bar\theta^{\dot 1} + \bar\theta^{\dot 1}\bar\theta^{\dot 2} = +2\bar\theta^{\dot 1}\bar\theta^{\dot 2} = -2\bar\theta^{\dot 2}\bar\theta^{\dot 1}$, and recalling $\varepsilon^{\dot 1\dot 2} = +1$, $\varepsilon^{\dot 2\dot 1} = -1$.

Similarly we have

$$\theta_\alpha\theta_\beta = +\frac{1}{2}\varepsilon_{\alpha\beta}(\theta\theta) \qquad \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\varepsilon_{\dot\alpha\dot\beta}(\bar\theta\bar\theta)$$

Various rearrangement identities The previous result is very useful when we have some expression in which a spinor θ occurs twice:

$$(\theta\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\psi\chi)$$

Proof: Simply by writing left-hand side as

$$-\theta^\alpha\theta^\beta\psi_\alpha\chi_\beta = +\frac{1}{2}\varepsilon^{\alpha\beta}(\theta\theta)\psi_\alpha\chi_\beta = -\frac{1}{2}(\theta\theta)\psi^\alpha\chi_\alpha = -\frac{1}{2}(\theta\theta)(\psi\chi)$$

Note that one can apply this with for example $\psi = \sigma^\mu\bar\psi$, as this is a left-handed spinor (from index structure), so

$$(\theta\sigma^\mu\bar\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\chi\sigma^\mu\bar\psi)$$

Fierz identity The same methods apply to show

$$(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta)$$

Proof: Write the left-hand side as

$$\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\theta^\beta\sigma^\nu_{\beta\dot\beta}\bar\theta^{\dot\beta} = +\frac{1}{4}\varepsilon^{\alpha\beta}\varepsilon^{\dot\alpha\dot\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}(\theta\theta)(\bar\theta\bar\theta) = \frac{1}{4}\sigma^\mu_{\alpha\dot\alpha}(\bar\sigma^\nu)^{\dot\alpha\alpha}(\theta\theta)(\bar\theta\bar\theta) = \frac{1}{4}\mathrm{tr}\,(\sigma^\mu\sigma^\nu)(\theta\theta)(\bar\theta\bar\theta)$$

from which the result follows as $\mathrm{tr}\,(\sigma^\mu\sigma^\nu) = 2\eta^{\mu\nu}$.

Fierz identity A similar result is

$$(\theta\psi)(\bar\chi\bar\eta) = -\frac{1}{2}(\theta\sigma^\mu\bar\eta)(\bar\chi\bar\sigma_\mu\psi)$$

Proof: Write the right-hand side as

$$-\frac{1}{2}\theta^\alpha\bar\eta^{\dot\alpha}\bar\chi_{\dot\beta}\psi_\beta\sigma^\mu_{\alpha\dot\alpha}(\bar\sigma_\mu)^{\dot\beta\beta} = -\theta^\alpha\bar\eta^{\dot\alpha}\bar\chi_{\dot\beta}\psi_\beta\delta^\beta_\alpha\delta^{\dot\beta}_{\dot\alpha} = -\theta^\alpha\psi_\alpha\bar\eta^{\dot\alpha}\bar\chi_{\dot\alpha} = +(\theta\psi)(\bar\chi\bar\eta)$$

Bilinear forms One has some freedom to rewrite bilinear forms involving the Pauli matrices:

$$\psi\sigma^\mu\bar\chi = -\bar\chi\bar\sigma^\mu\psi$$

Proof: Write the left-hand side as

$$\psi^\alpha\sigma^\mu_{\alpha\dot\alpha}\bar\chi^{\dot\alpha} = -\bar\chi_{\dot\beta}\varepsilon^{\dot\alpha\dot\beta}\varepsilon^{\alpha\beta}\sigma^\mu_{\alpha\dot\alpha}\psi_\beta = -\bar\chi_{\dot\beta}(\bar\sigma^\mu)^{\dot\beta\beta}\psi_\beta = -\bar\chi\bar\sigma^\mu\psi$$

More manipulation using ε tensors gives

$$\psi\sigma^\mu\bar{\sigma}^\nu\chi = \chi\sigma^\nu\bar{\sigma}^\mu\psi$$

3 Supermultiplets

$\mathcal{N}=1$ **SUSY algebra** The SUSY algebra extends the familiar Poincare algebra

$$[P^\mu, P^\nu] = 0$$

$$[M^{\mu\nu}, P^\sigma] = i(P^\mu\eta^{\nu\sigma} - P^\nu\eta^{\mu\sigma})$$

$$[M^{\mu\nu}, M^{\rho\sigma}] = i(M^{\mu\sigma}\eta^{\nu\rho} + M^{\nu\rho}\eta^{\mu\sigma} - M^{\mu\rho}\eta^{\nu\sigma} - M^{\nu\sigma}\eta^{\mu\rho})$$

by introducing *fermionic* symmetry generators Q_α and $\bar{Q}_{\dot{\alpha}}$.

We can motivate the SUSY algebra as follows. Consider first the transformation of Q_α as a spinor under a Lorentz transformation:

$$Q_\alpha \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right)_\alpha^{\ \beta} Q_\beta \approx Q_\alpha - \frac{i}{2}\omega_{\mu\nu}(\sigma^{\mu\nu})_\alpha^{\ \beta}Q_\beta$$

It also transforms as an operator under $U = \exp(-\frac{i}{2}\omega_{\mu\nu}M^{\mu\nu})$ as $Q \to U^\dagger Q U$, so to first order

$$Q_\alpha \to Q_\alpha - \frac{i}{2}\omega_{\mu\nu}[Q_\alpha, M^{\mu\nu}]$$

hence we derive

$$[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta}Q_\beta$$

Similarly, we have

$$\bar{Q}^{\dot{\alpha}} \to \exp\left(-\frac{i}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right)^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}} \approx \bar{Q}^{\dot{\alpha}} - \frac{i}{2}\omega_{\mu\nu}(\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{Q}^{\dot{\beta}}$$

and this similarly transforms as $\bar{Q} \to U^\dagger \bar{Q} U$, so

$$[\bar{Q}^{\dot{\alpha}}, M^{\mu\nu}] = (\bar{\sigma}^{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}}\bar{Q}^{\dot{\beta}}$$

Next consider

$$[Q_\alpha, P^\mu] = c\sigma^\mu_{\alpha\dot{\alpha}}\bar{Q}^{\dot{\alpha}}$$

where c is a (complex) constant and the right-hand side follows from the index structure and the requirement of linearity. Similarly, we should have

$$[\bar{Q}^{\dot{\alpha}}, P^\mu] = c^*(\bar{\sigma}^\mu)^{\dot{\alpha}\beta}Q_\beta$$

Now use the Jacobi identity:

$$\begin{aligned}0 &= [P^\mu, [P^\nu, Q_\alpha]] + [P^\nu, [Q_\alpha, P^\mu]] + [Q_\alpha, [P^\mu, P^\nu]] \\ &= -c\sigma^\nu_{\alpha\dot\alpha}[P^\mu, \bar Q^{\dot\alpha}] + c\sigma^\mu_{\alpha\dot\alpha}[P^\nu, \bar Q^{\dot\alpha}] \\ &= |c|^2 \left(\sigma^\nu \bar\sigma^\mu - \sigma^\mu \bar\sigma^\nu\right)_\alpha^{\ \beta} Q_\beta \neq 0\end{aligned}$$

This means that we must have $c = 0$, so

$$[Q_\alpha, P_\mu] = [\bar Q^{\dot\alpha}, P_\mu] = 0$$

Next, consider

$$\{Q_\alpha, Q^\beta\} = k(\sigma^{\mu\nu})_\alpha^{\ \beta} M_{\mu\nu}$$

where the right-hand side follows again from index structure and linearity. However the left-hand side commutes with P^μ and the right-hand side does not, unless $k = 0$. Hence

$$\{Q_\alpha, Q^\beta\} = \{\bar Q_{\dot\alpha}, \bar Q^{\dot\beta}\} = 0$$

Finally, index structure and convention takes

$$\{Q_\alpha, \bar Q_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu$$

We also have that Q_α commutes with any generators of internal symmetries, with the exception of the R-symmetry transformation

$$Q_\alpha \to e^{-iR\gamma} Q_\alpha e^{iR\gamma} = e^{i\gamma} Q_\alpha$$

which means

$$[Q_\alpha, R] = Q_\alpha \quad [\bar Q_{\dot\alpha}, R] = -\bar Q_{\dot\alpha}$$

Casimir operators The Casimir operators for the super-Poincare algebra are

$$C_1 = P_\mu P^\mu \qquad \tilde C_2 = C_{\mu\nu} C^{\mu\nu}$$

where

$$C_{\mu\nu} = B_\mu P_\nu - B_\nu P_\mu$$

with

$$B_\mu = W_\mu - \frac{1}{4} \bar Q_{\dot\alpha} (\bar\sigma_\mu)^{\dot\alpha\alpha} Q_\alpha$$

with the Pauli-Ljubanski vector

$$W_\mu = \frac{1}{2} \varepsilon_{\mu\nu\rho\sigma} P^\nu M^{\rho\sigma}$$

We take $\varepsilon_{0123} = +1 = -\varepsilon^{0123}$.

$\mathcal{N}=1$ **massless supermultiplets** We can take a standard momentum vector $p^\mu = (E,0,0,E)$, for which $C_1 = \tilde{C}_2 = 0$. We can characterise a massless state by its momentum p^μ and its helicity λ, where $W^\mu|p^\mu,\lambda\rangle = \lambda p^\mu |p^\mu,\lambda\rangle$. Now, for the supersymmetry generators we have

$$\{Q_\alpha, \bar{Q}_{\dot\alpha}\} = 2\sigma^\mu_{\alpha\dot\alpha} P_\mu = 2E(\sigma^0 - \sigma^3)_{\alpha\dot\alpha} = 4E \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot\alpha}$$

This implies that $\{Q_1, \bar{Q}_{\dot 1}\} = 0$. Now,take a state $|p^\mu,\lambda\rangle$ in the multiplet to be such that $Q_\alpha|p^\mu,\lambda\rangle = 0$ (if not we can just consider $|p^\mu,\lambda'\rangle = Q_\alpha|p^\mu,\lambda\rangle$ instead, and by the anticommutation relations $Q_\alpha Q_\alpha|p^\mu,\lambda\rangle = 0$). We can form new states from this one by applying $\bar{Q}_{\dot 1}$ or $\bar{Q}_{\dot 2}$, but by the anticommutation relations

$$0 = \langle p^\mu,\lambda|\{Q_1,\bar{Q}_{\dot 1}\}|p^\mu,\lambda\rangle = \langle p^\mu,\lambda|Q_1\bar{Q}_{\dot 1}|p^\mu,\lambda\rangle$$

so $\bar{Q}_{\dot 1}$ creates a state of zero norm; this would apply for any state formed by acting with $\bar{Q}_{\dot 1}$ so we can therefore take $\bar{Q}_{\dot 1} \equiv 0$ in this supermultiplet. Thus the only state other than $|p^\mu,\lambda\rangle$ is found by applying $\bar{Q}_{\dot 2}$. Now, we have

$$[W_\mu, \bar{Q}^{\dot\alpha}] = \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu[M^{\rho\sigma},\bar{Q}^{\dot\alpha}] = -\frac{1}{2}\varepsilon_{\mu\nu\rho\sigma}P^\nu(\bar\sigma^{\rho\sigma})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}$$

and in particular

$$[W_0, \bar{Q}^{\dot\alpha}] = -\frac{1}{2}\varepsilon_{03\rho\sigma}E(\bar\sigma^{\rho\sigma})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta} = -p_0 E(\bar\sigma^{12})^{\dot\alpha}{}_{\dot\beta}\bar{Q}^{\dot\beta}$$

using antisymmetry and the fact that $p_0 = E$. Now,

$$\bar\sigma^{12} = \frac{i}{4}\left(\bar\sigma^1\sigma^2 - \bar\sigma^2\sigma^1\right) = \frac{i}{4}\left(-\sigma^1\sigma^2 + \sigma^2\sigma^1\right) = \frac{1}{2}\sigma^3$$

as $\sigma^1\sigma^2 = -\sigma^2\sigma^1 = i\sigma^3$. Thus,

$$[W_0, \bar{Q}^{\dot\alpha}] = -\frac{1}{2}p_0(\sigma^3\bar{Q})^{\dot\alpha}$$

or explicitly

$$[W_0, \bar{Q}^{\dot 1}] = -\frac{1}{2}\bar{Q}^{\dot 1} \quad [W_0, \bar{Q}^{\dot 2}] = +\frac{1}{2}\bar{Q}^{\dot 2}$$

the former implying that

$$[W_0, \bar{Q}_{\dot 2}] = -\frac{1}{2}\bar{Q}_{\dot 2}$$

and so we have

$$W_0\bar{Q}_{\dot 2}|p^\mu,\lambda\rangle = \left([W_0,\bar{Q}_{\dot 2}] + \bar{Q}_{\dot 2}W_0\right)|p^\mu,\lambda\rangle = \left(\lambda - \frac{1}{2}\right)\bar{Q}_{\dot 2}|p^\mu,\lambda\rangle$$

Hence we see that $\bar{Q}_{\dot 2}$ reduces helicity of a state by 1/2. We define

$$|p^\mu, \lambda - 1/2\rangle \equiv \frac{1}{\sqrt{4E}}\bar{Q}_{\dot 2}|p^\mu,\lambda\rangle$$

We can generate no further new states. Note that $\frac{1}{\sqrt{4E}}Q_2$ and $\frac{1}{\sqrt{4E}}\bar{Q}_{\dot{2}}$ form a pair of creation and annihilation operators, satisfying

$$\left\{\frac{1}{\sqrt{4E}}Q_2, \frac{1}{\sqrt{4E}}\bar{Q}_{\dot{2}}\right\} = 1$$

Finally we should include CPT conjugates of negative helicity, and so conclude that our multiplet consists of the states

$$|p^\mu, \lambda\rangle \quad |p^\mu, \lambda - 1/2\rangle$$

along with the CPT conjugates

$$|p^\mu, -\lambda\rangle \quad |p^\mu, -(\lambda - 1/2)\rangle$$

Note that to show the Casimir $\tilde{C}_2$ is zero we calculate as follows:

$$\tilde{C}_2 = 2(B_\mu P_\nu B^\mu P^\nu - B_\mu P_\nu B^\nu P^\mu) = -2(B_\mu P^\mu)^2 = -2E^2(B_0 + B_3)^2$$

but as $B_\mu = W_\mu - \frac{1}{4}\bar{Q}_{\dot{\alpha}}(\bar{\sigma}_\mu)^{\dot{\alpha}\alpha}Q_\alpha$ we have

$$B_0 = \frac{1}{2}\varepsilon_{03\rho\sigma}EM^{\rho\sigma} - \frac{1}{4}\left(\bar{Q}_{\dot{1}}Q_1 + \bar{Q}_{\dot{2}}Q_2\right) \quad B_3 = \frac{1}{2}\varepsilon_{30\rho\sigma}EM^{\rho\sigma} - \frac{1}{4}\left(\bar{Q}_{\dot{1}}Q_1 - \bar{Q}_{\dot{2}}Q_2\right)$$

using $\bar{\sigma}_3 = -\bar{\sigma}^3 = \sigma^3$. So $B_0 + B_3 = -\frac{1}{2}\bar{Q}_{\dot{1}}Q_1$ and is therefore zero for massless states.

Examples of massless supermultiplets We can take $\lambda = 1/2$, giving us a chiral multiplet with two $|p, 0\rangle$ states and the states $|p, \pm 1/2\rangle$. The latter correspond to quarks, leptons, Higgsinos, and the former correspond to squarks, sleptons and Higgses.

We can take $\lambda = 1$, giving a vector multiplet with the states $|p, \pm 1/2\rangle$ (photino, gluino, Zino, Wino) and $|p, \pm 1\rangle$ (photon, gluon, Z-boson, W-boson). Note that we don't construct for instance quark-quarkino pairs as a spin-1 particle only leads to a renormalisable QFT if it is a gauge boson.

We can also take $\lambda = 2$, giving a gravitino-graviton pair.

$\mathcal{N} = 1$ **massive supermultiplets** For a massive particle we can go to the centre of mass frame, $p^\mu = (m, 0, 0, 0)$. Now the Casimirs become $C_1 = m^2$ and $\tilde{C}_2 = 2m^4 Y^i Y_i$ where the superspin is $Y_i = J_i + \frac{1}{4m}\bar{Q}_{\dot{\alpha}}\bar{\sigma}_i^{\dot{\alpha}\alpha}Q_\alpha$ (the plus sign is a minus in the notes which confuses me but never mind, I've taken $J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk}$ and $\varepsilon_{0ijk} \equiv \varepsilon_{ijk}$, perhaps we can absorb a minus sign into the Pauli matrix and its down or up index or something) and satisfies $[Y_i, Y_j] = i\varepsilon_{ijk}Y_k$. We can therefore label states by their mass m and the number y where $y(y+1)$ is the eigenvalue of Y_iY_i. The supersymmetry generators now obey

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}}P_\mu = 2m\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\alpha}}$$

Now, let $|\Omega\rangle$ be the vacuum state, which is annihilated by Q_1 and Q_2. For this state the ordinary spin agrees with the superspin, $Y_i|\Omega\rangle = J_i|\Omega\rangle$. Hence for a given m, y we have

$$|\Omega\rangle = |m, j = y; p^\mu, j_3\rangle$$

From $[Q_\alpha, M^{\mu\nu}] = (\sigma^{\mu\nu})_\alpha^{\ \beta} Q_\beta$ and $[\bar{Q}^{\dot\alpha}, M^{\mu\nu}] = (\bar\sigma^{\mu\nu})^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{\dot\beta}$ we can derive

$$[\bar{Q}^{\dot\alpha}, J_3] = [\bar{Q}^{\dot\alpha}, M^{12}] = (\bar\sigma^{12})^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{\dot\beta} = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}^{\dot\alpha}_{\ \dot\beta}\bar{Q}^{\dot\beta}$$

remembering that $J_i = \frac{1}{2}\varepsilon_{ijk}M^{jk}$ and using the definition of $\bar\sigma^{\mu\nu}$ again. Hence we have that

$$[\bar{Q}^{\dot 1}, J_3] = \frac{1}{2}\bar{Q}^{\dot 1} \qquad [\bar{Q}^{\dot 2}, J_3] = -\frac{1}{2}\bar{Q}^{\dot 2}$$

or

$$[J_3, \bar{Q}_{\dot 1}] = \frac{1}{2}\bar{Q}_{\dot 1} \qquad [J_3, \bar{Q}_{\dot 2}] = -\frac{1}{2}\bar{Q}_{\dot 2}$$

Thus we find we can use $\bar{Q}_{\dot 1}$ to raise the value of j_3:

$$J_3\bar{Q}_{\dot 1}|j_3\rangle = \left([J_3, \bar{Q}_{\dot 1}] + \bar{Q}_{\dot 1}J_3\right)|j_3\rangle = (j_3 + 1/2)\bar{Q}_{\dot 1}|j_3\rangle$$

and similarly $\bar{Q}_{\dot 2}$ lowers j_3 by 1/2. So we have

$$|j_3 + 1/2\rangle \equiv \frac{\bar{Q}_{\dot 1}}{\sqrt{2m}}|j_3\rangle \qquad |j_3 - 1/2\rangle \equiv \frac{\bar{Q}_{\dot 2}}{\sqrt{2m}}|j_3\rangle$$

and $\frac{1}{\sqrt{2m}}Q_{1,2}$ have the opposite effect. The main point is that $\bar{Q}_{\dot\alpha}$ acting on $|\Omega\rangle$ behaves like the combination of spins j and 1/2, i.e. $j \otimes 1/2 = (j - 1/2) \oplus (j + 1/2)$. The only way this works is if we have a decomposition of the form

$$\frac{\bar{Q}_{\dot 1}}{\sqrt{2m}}|\Omega\rangle = k_1|m, j = y + 1/2; p^\mu, j_3 + 1/2\rangle + k_2|m, j = y - 1/2; p^\mu, j_3 + 1/2\rangle$$

$$\frac{\bar{Q}_{\dot 2}}{\sqrt{2m}}|\Omega\rangle = k_3|m, j = y + 1/2; p^\mu, j_3 - 1/2\rangle + k_4|m, j = y - 1/2; p^\mu, j_3 - 1/2\rangle$$

The only other states are of the form

$$|\Omega'\rangle = \frac{1}{2m}\bar{Q}_{\dot 2}\bar{Q}_{\dot 1}|\Omega\rangle$$

Note that $Q_1|\Omega\rangle = 0$ but $Q_1|\Omega'\rangle = -\bar{Q}_{\dot 2}|\Omega\rangle \neq 0$ so that $|\Omega\rangle \neq |\Omega'\rangle$ and $|\Omega'\rangle$ therefore constitutes a different set of states of spin $j = y$.

The states in the massive supermultiplet are then $|\Omega\rangle$ and $|\Omega'\rangle$ of the form

$$|m, j = y; p^\mu, j_3\rangle$$

giving a total of $2(2y + 1)$ states, states

$$|m, j = y + 1/2; p^\mu, j_3\rangle$$

giving a further $2(y+1/2)+1 = 2y+2$ states, and also

$$|m, j = y - 1/2; p^\mu, j_3\rangle$$

giving another $2(y-1/2)+1 = 2y$ states.

A slight exception is provided by the case $y = 0$. There we have

$$|\Omega\rangle = |m, j = 0; p^\mu, j_3 = 0\rangle$$

$$\frac{\bar{Q}_{\dot{1},\dot{2}}}{\sqrt{2m}}|\Omega\rangle = |m, j = 1/2; p^\mu, j_3 = \pm 1/2\rangle$$

$$|\Omega'\rangle = \frac{1}{2m}\bar{Q}_{\dot{2}}\bar{Q}_{\dot{1}}|\Omega\rangle = |m, j = 0; p^\mu, j_3 = 0\rangle$$

The states $|\Omega\rangle$ and $|\Omega'\rangle$ differ in their handedness and are exchanged under parity. There are two eigenstates of parity

$$|\pm\rangle = \frac{1}{\sqrt{2}}\left(|\Omega\rangle \pm |\Omega'\rangle\right)$$

corresponding to a scalar and a pseudoscalar particle.

Let's also outline the $y = 1/2$ case. Here we start with the two states $|m, j = 1/2; p^\mu, j_3 = \pm 1/2\rangle$. Acting with $\bar{Q}_{\dot{1}}$ produces the states $|m, j = 1; p^\mu, j_3 = 1, 0\rangle$ while acting with $\bar{Q}_{\dot{2}}$ produces the states $|m, j = 0; p^\mu, j_3 = 0\rangle$ and $|m, j = 1; p^\mu, j_3 = -1\rangle$. Acting with both gives us then another $j = 1/2$, $j_3 = \pm 1/2$ pair.

Extended SUSY We extend supersymmetry by including $\mathcal{N}$ copies of the supersymmetry generators, labelled by an index A, with new anticommutation relations

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2\sigma^\mu_{\alpha\dot{\beta}}P_\mu\delta^A_B$$

$$\{Q^A_\alpha, Q^B_\beta\} = \varepsilon_{\alpha\beta}Z^{AB}$$

$$\{\bar{Q}_{\dot{\alpha}A}, \bar{Q}_{\dot{\beta}B}\} = \varepsilon_{\dot{\alpha}\dot{\beta}}(Z^\dagger)_{AB}$$

where Z^{AB} commutes with everything and is antisymmetric $Z^{AB} = -Z^{BA}$. We are also using a "perverse but essential" convention where $Z_{AB} = -Z^{AB}$.

Note that if $Z^{AB} = 0$ then there is an internal U($\mathcal{N}$) symmetry $Q^A_\alpha \to U^A_B Q^B_\alpha$, known as R-symmetry. If some $Z^{AB} \neq 0$ then the above anticommutation relations break this symmetry to some subgroup of U($\mathcal{N}$).

$\mathcal{N} > 1$ **massless supermultiplets** Again we have $p^\mu = (E, 0, 0, E)$ and now

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 4E\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}}\delta^A_B$$

We can again find that $\bar{Q}_{\dot{1}A} = 0$, and from the anticommutators we then must have all $Z^{AB} = 0$. We now have that each of the $\mathcal{N}$ operators $\bar{Q}_{\dot{2}A}$ lower the helicity by 1/2. To construct an extended SUSY massless supermultiplet we therefore start with a state of maximal helicity λ_{max} and apply all possible combinations of these operators.

Explicitly, we start with the single state $|p^\mu, \lambda_{max}\rangle$. Applying $\bar{Q}_{\dot{2}A}$ gives us $\mathcal{N}$ states with helicity $\lambda_{max} - 1/2$. Applying two operators $\bar{Q}_{\dot{2}A}\bar{Q}_{\dot{2}B}$ gives $\mathcal{N}(\mathcal{N}-1)/2$ states with helicity $\lambda_{max} - 1$. We continue in this way until we reach the single state with helicity $\lambda_{max} - \mathcal{N}/2$ formed by applying all operators. Note that the total number of states with helicity $\lambda_{max} - k$ is $\binom{\mathcal{N}}{k}$ so the total number of states is $2^{\mathcal{N}}$.

Note that $\lambda_{max} - \lambda_{min} = \frac{1}{2}\mathcal{N}$ in all cases. For renormalisable theories we should have $|\lambda| \leq 1$, which implies $\mathcal{N} \leq 4$. However we find that $\mathcal{N} > 1$ is non-chiral, which does not work with the Standard Model, which contains chiral particles.

Examples of $\mathcal{N} > 1$ massless supermultiplets Consider the $\mathcal{N} = 2$ vector multiplet, which has $\lambda_{max} = 1$. Acting with a single lowering operator gives states with $\lambda = 1/2$, and acting with both we get a state with $\lambda = 0$ (and we should also include the CPT conjugates of negative helicity). We can decompose this multiplet into an $\mathcal{N} = 1$ chiral multiplet, consisting of the $\lambda = 0$ and one $\lambda = 1/2$ state, and an $\mathcal{N} = 1$ vector multiplet, consisting of the other $\lambda = 1/2$ state and the $\lambda = 1$ state (plus CPT conjugates in both these cases).

An $\mathcal{N} = 2$ hypermultiplet has $\lambda_{max} = 0$, and so consists of one state with $\lambda = 1/2$, two with $\lambda = 0$ and one with $\lambda = -1/2$. This multiplet is CPT self-conjugate and decomposes into a sum of a chiral and antichiral $\mathcal{N} = 1$ multiplet.

An $\mathcal{N} = 4$ vector multiplet has $\lambda_{max} = 1$, consisting of one state with $\lambda = 1$, four with $\lambda = 1/2$, six with $\lambda = 0$, four with $\lambda = -1/2$ and one with $\lambda = -1$. This decomposes into one $\mathcal{N} = 2$ vector multiplet and two $\mathcal{N} = 2$ hypermultiplets, or else one $\mathcal{N} = 1$ vector multiplet and three $\mathcal{N} = 1$ chiral multiplets.

$\mathcal{N} > 1$ massive supermultiplets For massive particles we go to the centre of mass frame $p^\mu = (m, 0, 0, 0)$. We have

$$\{Q^A_\alpha, \bar{Q}_{\dot{\beta}B}\} = 2m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}_{\alpha\dot{\beta}} \delta^A_B$$

Unlike the massless case, Z^{AB} may now be non-zero. We consider the two cases $Z^{AB} \neq 0$ and $Z^{AB} = 0$ separately, starting with the latter.

If $Z^{AB} = 0$ then we have $2\mathcal{N}$ raising and lowering operators given by

$$a^A_\alpha = \frac{1}{\sqrt{2m}} Q^A_\alpha \qquad a^{A\dagger}_{\dot{\alpha}} = \frac{1}{\sqrt{2m}} \bar{Q}^A_{\dot{\alpha}}$$

These allow us to create $2^{2\mathcal{N}}$ states. For example, consider $\mathcal{N} = 2$ with $y = 0$. We have the ground state $|\Omega\rangle$ and then four states of the form $a^{A\dagger}_{\dot{\alpha}}|\Omega\rangle$, which have spin $j = y = 1/2$, and $j_3 = \pm 1/2$ depending on whether they were created by $a^{A\dagger}_{\dot{1}}$ or $a^{A\dagger}_{\dot{2}}$. We then have six states formed by acting with two creation operators. The possibilities are $a^{1\dagger}_{\dot{1}} a^{2\dagger}_{\dot{1}}$, giving one state with $j_3 = 1$, $a^{1\dagger}_{\dot{2}} a^{2\dagger}_{\dot{2}}$, giving one state with $j_3 = -1$, and four states of the form $a^{A\dagger}_{\dot{1}} a^{B\dagger}_{\dot{2}}$, each of which has $j_3 = 0$. These six states split up into three states with spin $j=0$ and three with spin $j=1$. After this we then have four spin 1/2 states formed by acting with three creation operators, and one state with spin 0 formed by acting with all creation operators.

Note in general if we start with a state of superspin y then we end up with $(2y+1)2^{2\mathcal{N}}$ states, as the vacuum state $|\Omega\rangle$ is $(2y+1)$ dimensional.

If some $Z^{AB} \neq 0$, then we proceed by defining the scalar quantity

$$\mathcal{H} = (\bar{\sigma}^0)^{\dot{\beta}\alpha}\{Q^A_\alpha - \Gamma^A_\alpha, \bar{Q}_{\dot{\beta}A} - \bar{\Gamma}_{\dot{\beta}A}\}$$

where

$$\Gamma^A_\alpha = \varepsilon_{\alpha\beta} U^{AB} \bar{Q}_{\dot{\alpha}B} (\bar{\sigma}^0)^{\dot{\alpha}\beta}$$

for U^{AB} any unitary $\mathcal{N} \times \mathcal{N}$ matrix. Note that $\mathcal{H} \geq 0$ as it is a sum of quantities of the form $X^\dagger X$.

To evaluate $\mathcal{H}$, we first use that

$$(\bar{\sigma}^0)^{\dot{\beta}\alpha}\{Q^A_\alpha, \bar{Q}_{\dot{\beta}A}\} = 2m\delta^A_A (\bar{\sigma}^0)^{\dot{\beta}\alpha}\sigma^0_{\alpha\dot{\beta}} = 4m\mathcal{N}$$

and also

$$\{\Gamma^A_\alpha, \bar{Q}_{\dot{\beta}A}\} = \varepsilon_{\alpha\beta} U^{AB} (\bar{\sigma}^0)^{\dot{\alpha}\beta}\{\bar{Q}_{\dot{\alpha}B}, \bar{Q}_{\dot{\beta}A}\} = \varepsilon_{\alpha\beta} U^{AB} (\bar{\sigma}^0)^{\dot{\alpha}\beta} \varepsilon_{\dot{\alpha}\dot{\beta}} Z^\dagger_{BA} = -\sigma^0_{\alpha\dot{\beta}} U^{AB} Z^\dagger_{BA}$$

As we're using the ridiculous convention that $Z_{AB} = -Z^{AB}$, we thus get

$$(\bar{\sigma}^0)^{\dot{\beta}\alpha}\{Q^A_\alpha, \bar{\Gamma}_{\dot{\beta}A}\} + (\bar{\sigma}^0)^{\dot{\beta}\alpha}\{\Gamma^A_\alpha, \bar{Q}_{\dot{\beta}A}\} = 2\text{tr}\left(ZU^\dagger + UZ^\dagger\right)$$

where we've added the hermitian conjugate term. Finally we write

$$\Gamma_{\dot{\alpha}A} = \varepsilon_{\dot{\beta}\dot{\alpha}} Q^B_\alpha (\bar{\sigma}^0)^{\dot{\beta}\alpha} (U^\dagger)_{BA}$$

with $U_{BA} = -U^{BA}$ also, so that

$$\begin{aligned}
\{\Gamma^A_\alpha, \bar{\Gamma}_{\dot{\beta}A}\} &= \varepsilon_{\alpha\beta}(\bar{\sigma}^0)^{\dot{\alpha}\beta}\varepsilon_{\dot{\beta}\dot{\gamma}}(\bar{\sigma}^0)^{\dot{\gamma}\gamma} U^{AB} U^\dagger_{CA}\{\bar{Q}_{\dot{\alpha}B}, Q^C_\gamma\} \\
&= 2m\varepsilon_{\alpha\beta}\varepsilon_{\dot{\beta}\dot{\gamma}}(\sigma^0)_{\gamma\dot{\alpha}}(\bar{\sigma}^0)^{\dot{\alpha}\beta}(\bar{\sigma}^0)^{\dot{\gamma}\gamma} U^{AB} U^\dagger_{BA} \\
&= -2m\varepsilon_{\alpha\gamma}\varepsilon_{\dot{\beta}\dot{\gamma}}(\bar{\sigma}^0)^{\dot{\gamma}\gamma} U^{AB} (U^\dagger)^{BA} \\
&= +2m\mathcal{N}(\sigma^0)_{\alpha\beta}
\end{aligned}$$

So we get

$$\mathcal{H} = 8m\mathcal{N} - 2\text{tr}\left(ZU^\dagger + UZ^\dagger\right) \geq 0$$

Now, according to the polar decomposition theorem for matrices we can write $Z = HV$ for H hermitian and V unitary. Let's take $V = U$ then $ZU^\dagger = H$ and we have

$$8m\mathcal{N} - 4\text{tr}\, H \geq 0 \Rightarrow m \geq \frac{1}{2\mathcal{N}}\text{tr}\, H$$

and as $H = ZU^\dagger$, $H^2 = HH^\dagger = ZZ^\dagger$ we can write the so-called BPS bound:

$$m \geq \frac{1}{2\mathcal{N}} \mathrm{tr}\, \sqrt{ZZ^\dagger}$$

States saturating this bound are called BPS states; they correspond to $\mathcal{H} = 0$ and thus to vanishing $Q^A_\alpha - \Gamma^A_\alpha$, leading to shorter multiplets as some generators vanish.

For instance, for $\mathcal{N} = 2$ we can write Z^{AB} in the form

$$Z^{AB} = \begin{pmatrix} 0 & q_1 \\ -q_1 & 0 \end{pmatrix} \Rightarrow m \geq \frac{1}{2} q_1$$

For $\mathcal{N} > 2$ and even we can express Z^{AB} in block diagonal form, with each block of the form

$$\begin{pmatrix} 0 & q_i \\ -q_i & 0 \end{pmatrix}$$

and then the BPS condition holds block by block, $2m \geq q_i$. We can define $\mathcal{H}$ for each block. If k of the q_i are equal to $2m$ then there are $2\mathcal{N} - 2k$ creation operators and so $2^{2(\mathcal{N}-k)}$ states. The cases $k = 0, 0 < k < \mathcal{N}/2$ and $k = \mathcal{N}/2$ are termed long, short and ultra-short multiplets respectively.

4 Superfields

Superspace A superfield may be thought of as living in superspace. For $\mathcal{N} = 1$ superspace this is defined to be the coset formed by quotienting the super-Poincare group, parametrised by $\{\omega^{\mu\nu}, a^\mu, \theta^\alpha, \bar{\theta}_{\dot{\alpha}}\}$, by the Lorentz group, parametrised by $\{\omega^{\mu\nu}\}$. Here θ^α and $\bar{\theta}_{\dot{\alpha}}$ are spinors of Grassmann variables.

We can write an element of superspace as

$$G(x, \theta, \bar{\theta}) = e^{i\left(-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q}\right)}$$

Under a supersymmetry transformation we have

$$\begin{aligned} G(x, \theta, \bar{\theta}) \to G(0, \varepsilon, \bar{\varepsilon}) G(x, \theta, \bar{\theta}) &= e^{i(\varepsilon Q + \bar{\varepsilon}\bar{Q})} e^{i\left(-x^\mu P_\mu + \theta Q + \bar{\theta}\bar{Q}\right)} \\ &= e^{i\left(-x^\mu P_\mu + (\theta+\varepsilon)Q + (\bar{\theta}+\bar{\varepsilon})\bar{Q}\right) + \frac{i^2}{2}\left[\varepsilon Q + \bar{\varepsilon}\bar{Q}, \theta Q + \bar{\theta}\bar{Q}\right]} \end{aligned}$$

using the CBH formula. One can work out the commutators to find that the superspace parameters $x, \theta, \bar{\theta}$ transform as

$$x^\mu \to x^\mu - i\varepsilon\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\varepsilon} \qquad \theta \to \theta + \varepsilon \qquad \bar{\theta} + \bar{\varepsilon}$$

General scalar superfield The general scalar superfield has the form

$$\begin{aligned} S(x^\mu, \theta_\alpha, \bar{\theta}^{\dot{\alpha}}) = \varphi(x^\mu) &+ \theta\psi(x^\mu) + \bar{\theta}\bar{\chi}(x^\mu) \\ &+ (\theta\theta)M(x^\mu) + (\bar{\theta}\bar{\theta})N(x^\mu) + (\theta\sigma^\nu\bar{\theta})V_\nu(x^\mu) \\ &+ (\theta\theta)\bar{\theta}\bar{\lambda}(x^\mu) + (\bar{\theta}\bar{\theta})\theta\rho(x^\mu) + (\theta\theta)(\bar{\theta}\bar{\theta})D(x^\mu) \end{aligned}$$

Transformation properties of general scalar superfield The general scalar superfield transforms as an operator as

$$S \mapsto e^{-i(\varepsilon Q+\bar{\varepsilon}\bar{Q})} S e^{+i(\varepsilon Q+\bar{\varepsilon}\bar{Q})}$$

and as a Hilbert space vector by

$$S \mapsto e^{+i(\varepsilon \mathcal{Q}+\bar{\varepsilon}\bar{\mathcal{Q}})} S(x,\theta,\bar{\theta}) = S(x^\mu - i\varepsilon\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\varepsilon}, \theta+\varepsilon, \bar{\theta}+\bar{\varepsilon})$$

A Taylor expansion implies that the transformation properties are implemented by the operators

$$\mathcal{Q}_\alpha = -i\frac{\partial}{\partial\theta^\alpha} - \sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\frac{\partial}{\partial x^\mu}$$

$$\bar{\mathcal{Q}}_{\dot{\alpha}} = i\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + \theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\frac{\partial}{\partial x^\mu}$$

and we have

$$\delta S = i(\varepsilon\mathcal{Q} + \bar{\varepsilon}\bar{\mathcal{Q}})S$$

One can then work out the transformation properties of the various fields making up S. In doing so we need to make use of the identities which follow from $\theta^\alpha\theta^\beta = -\frac{1}{2}\varepsilon^{\alpha\beta}(\theta\theta)$ and $\bar{\theta}^{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} = +\frac{1}{2}\varepsilon^{\dot{\alpha}\dot{\beta}}(\bar{\theta}\bar{\theta})$ in order to end up with an expression with the same structure as the original superfield.

Covariant derivative One can define a covariant derivative which commutes with $\varepsilon\mathcal{Q} + \bar{\varepsilon}\bar{\mathcal{Q}}$:

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu$$

$$\bar{D}_{\dot{\alpha}} = -\bar{\partial}_{\dot{\alpha}} - i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

These anticommute with $\mathcal{Q}_\alpha$ and $\bar{\mathcal{Q}}_{\dot{\alpha}}$ and themselves, apart from

$$\{D_\alpha, \bar{D}_{\dot{\alpha}}\} = -2i\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu$$

Chiral superfield A chiral superfield Φ obeys

$$\bar{D}_{\dot{\alpha}}\Phi = 0$$

General form of chiral superfield It is convenient here (and sometimes elsewhere) to define

$$y^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$$

One has that $\bar{D}_{\dot{\alpha}}\theta^\alpha = \bar{D}_{\dot{\alpha}}y^\mu = 0$, while $\bar{D}_{\dot{\alpha}}\bar{\theta}^{\dot{\beta}} \neq 0$, so a scalar superfield is chiral if it is a function of just y and θ:

$$\Phi(y,\theta) = \varphi(y) + \sqrt{2}\theta\psi(y) + (\theta\theta)F(y)$$

We can expand this and use the results $(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta})$ and $(\theta\sigma^\mu\bar{\theta})(\theta\partial_\mu\psi) = -\frac{1}{2}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar{\theta}$ to get the general chiral superfield in the form

$$\begin{aligned}\Phi(x,\theta,\bar{\theta}) = \varphi(x) + \sqrt{2}\theta\psi(x) + (\theta\theta)F(x)& \\ + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi(x) - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi(x)\sigma^\mu\bar{\theta}& \\ - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi(x)&\end{aligned}$$

Here $\varphi(x)$ is a scalar field, $\psi(x)$ a spin-1/2 field and $F(x)$ an auxiliary field.

F- and D-terms For a general scalar superfield, the D-term

$$(\theta\theta)(\bar{\theta}\bar{\theta})D(x)$$

transforms as $D \to D+$ total derivative under a supersymmetry transformation. For a chiral superfield, the F-term

$$(\theta\theta)F(x)$$

transforms as $F \to F+$ total derivative under a supersymmetry transformation. Thus we can use these terms to construct supersymmetry invariant Lagrangians.

Calculation of D-terms of $\Phi^\dagger\Phi$ We have

$$\Phi = \varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi$$

$$\Phi^\dagger = \varphi^* + \sqrt{2}\bar{\theta}\bar{\psi} + (\bar{\theta}\bar{\theta})F^* - i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi^* + \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\theta\sigma^\mu\partial_\mu\bar{\psi} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi^*$$

The terms involving two θs and two $\bar{\theta}$s in $\Phi^\dagger\Phi$ are

$$\left(-\frac{1}{2}\varphi^*\partial_\mu\partial^\mu\varphi + F^*F\right)(\theta\theta)(\bar{\theta}\bar{\theta}) + \partial_\mu\varphi^*\partial_\nu\varphi(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) + \left(i(\bar{\theta}\bar{\theta})\theta\sigma^\mu\partial_\mu\bar{\psi}\theta\psi + h.c.\right)$$

Using the identities

$$(\theta\sigma^\mu\bar{\theta})(\theta\sigma^\nu\bar{\theta}) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar{\theta}\bar{\theta}) \quad (\theta\psi)(\theta\chi) = -\frac{1}{2}(\theta\theta)(\psi\chi) \quad \psi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\psi$$

and integrating by parts we get

$$\Phi^\dagger\Phi\Big|_D = \partial_\mu\varphi^*\partial^\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^*F$$

Calculation of F-terms of Φ^2 and Φ^3 We have

$$\Phi = \varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi$$

so we just need to consider

$$\varphi + \sqrt{2}\psi\theta + \theta\theta F$$

Squaring this gives

$$\varphi^2 + 2(\psi\theta)(\psi\theta) + 2\varphi(\sqrt{2}\psi\theta + \theta\theta F)$$

hence

$$\Phi^2\Big|_F = 2F\varphi - \psi\psi$$

We then need to work out

$$(\varphi + \sqrt{2}\psi\theta + \theta\theta F)(\varphi^2 + 2\sqrt{2}\varphi\psi\theta + \theta\theta(2F\varphi - \psi\psi))$$

Taking just the terms with two θs:

$$4\varphi\psi\theta\psi\theta + (\theta\theta)\left(3F\varphi^2 - \varphi(\psi\psi)\right)$$

hence

$$\Phi^3\Big|_F = 3(\varphi^2 F - \varphi(\psi\psi))$$

Lagrangians for chiral superfields A general Lagrangian for chiral superfields Φ_i is of the form

$$\mathcal{L} = K(\Phi_i, \Phi_j^\dagger)\Big|_D + \left(W(\Phi_i)\Big|_F + h.c.\right)$$

where K is known as the Kahler potential and W is known as the superpotential. We can Taylor expand the latter about $\Phi_i = \varphi_i$:

$$W(\Phi_i) = W(\varphi_i) + (\Phi_i - \varphi_i)\frac{\partial W}{\partial \varphi_i} + \frac{1}{2}(\Phi_i - \varphi_i)(\Phi_j - \varphi_j)\frac{\partial^2 W}{\partial \varphi_i \varphi_j} + \dots$$

where

$$\frac{\partial W}{\partial \varphi_i} \equiv \frac{\partial W}{\partial \Phi_i}\Big|_{\Phi_i = \varphi_i}$$

Extracting the F-terms via

$$\Phi_i - \varphi_i = \sqrt{2}\theta\psi_i + (\theta\theta)F_i + \dots$$

gives

$$\mathcal{L} = K(\Phi_i, \Phi_j^\dagger)\Big|_D + \left(F_i\frac{\partial W}{\partial \varphi_i} + h.c.\right) - \frac{1}{2}\left(\frac{\partial^2 W}{\partial \varphi_i \varphi_j}\psi_i\psi_j + h.c.\right)$$

For the Kahler term it is usual to take

$$K(\Phi_i, \Phi_j^\dagger) = \Phi_i^\dagger \Phi_i$$

for which one has

$$\mathcal{L} = \partial_\mu \varphi_i^* \partial_\mu \varphi_i - i\bar{\psi}_i \bar{\sigma}^\mu \partial_\mu \psi_i - \frac{1}{2}\left(\frac{\partial^2 W}{\partial \varphi_i \varphi_j}\psi_i\psi_j + h.c.\right) + \mathcal{L}_F$$

with

$$\mathcal{L}_F = F_i F_i^* + F_i \frac{\partial W}{\partial \varphi_i} + F_i^* \frac{\partial W^*}{\partial \varphi_i^*}$$

One can solve for the auxiliary field equation of motion

$$F_i = -\frac{\partial W^*}{\partial \varphi_i^*}$$

and hence

$$\mathcal{L}_F = -\left|\frac{\partial W}{\partial \varphi_i}\right|^2 \equiv -V_F$$

We thus obtain V_F, the scalar potential.

One can constrain the form of the superpotential on dimensional grounds. We must have $[\mathcal{L}] = 4$, and as φ and ψ are normal scalar and spin-1/2 fields they have dimensions $[\varphi] = 1$, $[\psi] = 3/2$ implying that $[\Phi] = 1$ and $[\theta] = [\bar{\theta}] = -1/2$. Now we want $[W|_F] = 4$, and as we have $W = + \dots (\theta\theta) W|_F + \dots$ we should have $[W] = 3$. If we are to avoid couplings of negative mass dimension it follows the allowed form of W is

$$W = \alpha + \lambda_i \Phi_i + \frac{1}{2} m_{ij} \Phi_i \Phi_j + \frac{1}{3} g_{ijk} \Phi_i \Phi_j \Phi_k$$

Similarly one constrains $[K] = 2$.

Wess-Zumino model The Wess-Zumino model involves one chiral superfield Φ, and has superpotential

$$W = \frac{1}{2} m \Phi^2 + \frac{1}{3} g \Phi^3$$

The F-terms of this superpotential are

$$m(\varphi F - \frac{1}{2}(\psi\psi)) + g(\varphi^2 F - \varphi(\psi\psi))$$

Vector superfield A vector superfield V satisfies

$$V(x, \theta, \bar{\theta}) = V^\dagger(x, \theta, \bar{\theta})$$

General form of vector superfield It is convenient to take the following general form of a vector superfield:

$$
\begin{aligned}
V(x,\theta,\bar{\theta}) = C(x) + i\theta\chi(x) - i\bar{\theta}\bar{\chi}(x) \\
&+ \frac{i}{2}(\theta\theta)(M(x)+iN(x)) - \frac{i}{2}(\bar{\theta}\bar{\theta})(M(x)-iN(x)) + \theta\sigma^\mu\bar{\theta}V_\mu(x) \\
&+ (\theta\theta)\bar{\theta}\left(i\bar{\lambda}(x) - \frac{1}{2}\bar{\sigma}^\mu\partial_\mu\chi(x)\right) + (\bar{\theta}\bar{\theta})\theta\left(-i\lambda(x) - \frac{1}{2}\sigma^\mu\partial_\mu\bar{\chi}(x)\right) \\
&+ \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})\left(D(x) - \frac{1}{2}\partial_\mu\partial^\mu C(x)\right)
\end{aligned}
$$

Gauge transformations A generalised gauge transformation of a vector field is of the form

$$V \mapsto V + i(\Lambda - \Lambda^\dagger)$$

for Λ a chiral superfield. Under this $V_\mu \mapsto V_\mu - \partial_\mu(\varphi + \varphi^\dagger)$ which is the usual transformation of a vector field.

We can choose φ, ψ and F to set C, M, N and χ to zero. This give us a vector superfield in Wess-Zumino gauge:

$$V_{WZ}(x,\theta,\bar{\theta}) = (\theta\sigma^\mu\bar{\theta})V_\mu(x) + i(\theta\theta)\bar{\theta}\bar{\lambda}(x) - i(\bar{\theta}\bar{\theta})\theta\lambda(x) + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D(x)$$

The component fields are now $V_\mu(x)$, a gauge boson, λ and $\bar{\lambda}$ representing a fermion gaugino, and $D(x)$ an auxiliary field. Note that $V_{WZ}^2 = \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})V^\mu V_\mu$ and all higher powers are zero.

Supersymmetry transformations take us out of Wess-Zumino gauge; however we can always augment a supersymmetry transformation with an additional gauge transformation to return to Wess-Zumino gauge.

Couplings of vector superfields to chiral superfields The supersymmetric generalisation of the familiar U(1) transformations of a complex scalar field coupled to a vector field is to have

$$\Phi \mapsto e^{-2iq\Lambda}\Phi$$

for Φ a chiral superfield. As under the same transformation $V \mapsto V + i(\Lambda - \Lambda^\dagger)$ a gauge invariant coupling we can use in Lagrangians is

$$\Phi^\dagger e^{2qV}\Phi$$

Supersymmetric field strength The supersymmetric analogue of the field strength $F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ is given by

$$W_\alpha = -\frac{1}{4}(\bar{D}\bar{D})D_\alpha V$$

which is chiral and gauge invariant. It is convenient to evaluate this using $y^\mu = x^\mu + i\theta\sigma^\mu\bar\theta$. Recall that

$$D_\alpha = \partial_\alpha + i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu \qquad \bar D_{\dot\alpha} = -\bar\partial_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial_\mu$$

and one can show that

$$\partial_\alpha\theta^\beta = \delta^\beta_\alpha \quad \partial_\alpha\theta_\beta = \varepsilon_{\beta\alpha} \quad \partial_\alpha(\theta\theta) = 2\theta_\alpha \quad \partial_\alpha(\theta\lambda) = \lambda_\alpha$$

Now, one has

$$D_\alpha y^\mu = 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha} \quad \bar D_{\dot\alpha} y^\mu = 0$$

hence on a function of y, $\bar D_{\dot\alpha} \equiv 0$, $D_\alpha \equiv 2i\sigma^\mu_{\alpha\dot\alpha}\bar\theta^{\dot\alpha}\partial_\mu$, where we now mean a derivative with respect to y^μ. Hence rewrite V in terms of y:

$$\begin{aligned} V &= (\theta\sigma^\mu\bar\theta)V_\mu(y^\mu - i\theta\sigma^\mu\bar\theta) + i(\theta\theta)\bar\theta\bar\lambda(y) - i(\bar\theta\bar\theta)\theta\lambda(y) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)D(y) \\ &= (\theta\sigma^\mu\bar\theta)V_\mu(y) + i(\theta\theta)\bar\theta\bar\lambda(y) - i(\bar\theta\bar\theta)\theta\lambda(y) + \frac{1}{2}(\theta\theta)(\bar\theta\bar\theta)\left(D(y) - i\partial_\mu V^\mu\right) \end{aligned}$$

having Taylor expanded and used $(\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta)$. We can now use this expression to calculate

$$(\bar D\bar D)D_\alpha V \equiv (\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha})(\partial_\alpha + 2i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu)V$$

where ∂_μ is derivative with respect to y. Because of the $(\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha})$ derivatives only terms involving two $\bar\theta$s need be kept from working out $D_\alpha V$. We need the facts that

$$\partial_\alpha\theta\lambda = \lambda_\alpha \quad \partial_\alpha\theta\theta = 2\theta_\alpha$$

then we have

$$(\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha})\left(-i\lambda_\alpha(\bar\theta\bar\theta) + \theta_\alpha(\bar\theta\bar\theta)(D - i\partial^\mu V_\mu) + 2i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\sigma^\nu\bar\theta)\partial_\mu V_\nu - 2\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}(\theta\theta)\bar\theta\partial_\mu\bar\lambda\right)$$

Now,

$$2i\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\theta^\beta\sigma^\nu_{\beta\dot\gamma}\bar\theta^{\dot\gamma} = -i\varepsilon^{\dot\beta\dot\gamma}(\bar\theta\bar\theta)\sigma^\mu_{\alpha\dot\beta}\sigma^\nu_{\beta\dot\gamma}\varepsilon^{\beta\gamma}\theta_\gamma = i\sigma^\mu_{\alpha\dot\beta}(\bar\sigma^\nu)^{\dot\beta\gamma}\theta_\gamma(\bar\theta\bar\theta)$$

and

$$-2\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\bar\theta_{\dot\alpha}\partial_\mu\bar\lambda^{\dot\alpha} = 2\sigma^\mu_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu\bar\theta^{\dot\alpha}\bar\lambda_{\dot\alpha} = \sigma^\mu_{\alpha\dot\beta}\varepsilon^{\dot\beta\dot\alpha}\partial_\mu\bar\lambda_{\dot\alpha}(\bar\theta\bar\theta) = \sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta}(\bar\theta\bar\theta)$$

Hence we have

$$(\bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}\bar\theta\bar\theta)\left(-i\lambda_\alpha + \theta_\alpha(D - i\partial^\mu V_\mu) + i(\sigma^\mu\bar\sigma^\nu\theta)_\alpha\partial_\mu V_\nu + (\sigma^\mu\partial_\mu\bar\lambda)_\alpha\theta\theta\right)$$

Now,

$$\bar{\partial}_{\dot\alpha}\bar{\partial}^{\dot\alpha}\bar\theta\bar\theta = \varepsilon^{\dot\alpha\dot\beta}\bar{\partial}_{\dot\alpha}\bar{\partial}_{\dot\beta}\bar\theta\bar\theta = -2\varepsilon^{\dot\alpha\dot\beta}\bar{\partial}_{\dot\alpha}\bar\theta_{\dot\beta} = -2\delta^{\dot\alpha}_{\dot\alpha} = -4$$

and as we have

$$\sigma^{\mu\nu} = \frac{i}{4}\left(\sigma^\mu\bar\sigma^\nu - \sigma^\nu\bar\sigma^\mu\right) = \frac{i}{4}(2\sigma^\mu\bar\sigma^\nu + 2\eta^{\mu\nu}) \Rightarrow \sigma^\mu\bar\sigma^\nu = -2i\sigma^{\mu\nu} + \eta^{\mu\nu}$$

the V_μ terms combine as

$$-\theta_\alpha i\partial^\mu V_\mu + i(-2i\sigma^{\mu\nu} + \eta^{\mu\nu}\theta)_\alpha\partial_\mu V_\nu = (\sigma^{\mu\nu}\theta)_\alpha(\partial_\mu V_\nu - \partial_\nu V_\mu)$$

and so we find

$$W_\alpha(y,\theta) = -i\lambda_\alpha(y) + \theta_\alpha D(y) + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}(y) + (\theta\theta)\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta}(y)$$

This expression is sufficient for working out the F-terms of $W^\alpha W_\alpha$ as we can replace y with x and calculate away. We only need to consider

$$\left(-i\lambda^\alpha + \theta^\alpha D + (\sigma^{\mu\nu}\theta)^\alpha F_{\mu\nu} + \varepsilon^{\alpha\beta}\theta\theta\sigma^\mu_{\beta\dot\beta}\partial_\mu\bar\lambda^{\dot\beta}\right)\left(-i\lambda_\alpha + \theta_\alpha D + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu} + \theta\theta\sigma^\mu_{\alpha\dot\beta}\partial_\mu\bar\lambda^{\dot\beta}\right)\Big|_F$$

which gives

$$-2i(\theta\theta)\lambda\sigma^\mu\partial_\mu\bar\lambda + (\theta\theta)D^2 + 2F_{\mu\nu}\theta\sigma^{\mu\nu}\theta D + F_{\mu\nu}F_{\rho\sigma}(\sigma^{\mu\nu}\theta)^\alpha(\sigma^{\rho\sigma}\theta)_\alpha$$

Now,

$$\theta^\alpha\sigma^{\mu\nu}_\alpha{}^\beta\theta_\beta = -\theta^\alpha\theta^\beta\sigma^{\mu\nu}_{\alpha\beta} = \frac{1}{2}\theta\theta\varepsilon^{\alpha\beta}\sigma^{\mu\nu}_{\alpha\beta} = -\frac{1}{2}\theta\theta\,\mathrm{tr}\,\sigma^{\mu\nu} = 0$$

The final result needs an identity involving traces of $\sigma^{\mu\nu}$ which I won't bother giving here. The end-product is

$$W^\alpha W_\alpha\Big|_F = D^2 - 2i\lambda\sigma^\mu\partial_\mu\bar\lambda - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}F_{\mu\nu}\widetilde{F}^{\mu\nu}$$

with

$$\widetilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

Lagrangians for vector and chiral superfields For a theory consisting of several chiral superfields Φ_i coupled to a vector superfield V, we take the Lagrangian

$$\mathcal{L} = \sum_i \Phi_i^\dagger e^{2q_iV}\Phi_i\Big|_D + \left(W(\Phi_i)\Big|_F + h.c.\right) + \left(f(\Phi_i)(W^\alpha W_\alpha)\Big|_F + h.c.\right) + \xi V\Big|_D$$

Here q_i denotes the U(1) charge of each chiral superfield, $W(\Phi_i)$ is a superpotential which must be U(1) invariant, $f(\Phi)$ is called the gauge kinetic function, and the final term involving the constant ξ is known as the Fayet-Iliopolous term. For a renormalisable theory we should take $f(\Phi_i) = \tau = $ constant. For the case of supersymmetric QED, we take $f(\Phi_i) = 1/4$.

In Wess-Zumino gauge we have

$$V_{WZ} = (\theta\sigma^\mu\bar{\theta})V_\mu + i(\theta\theta)\bar{\theta}\bar{\lambda} - i(\bar{\theta}\bar{\theta})\theta\lambda + \frac{1}{2}(\theta\theta)(\bar{\theta}\bar{\theta})D$$

so

$$e^{2qV} = 1 + 2q(\theta\sigma^\mu\bar{\theta})V_\mu + 2iq(\theta\theta)\bar{\theta}\bar{\lambda} - 2iq(\bar{\theta}\bar{\theta})\theta\lambda + (\theta\theta)(\bar{\theta}\bar{\theta})(qD + q^2 V_\mu V^\mu)$$

and using

$$\Phi = \varphi + \sqrt{2}\theta\psi + (\theta\theta)F + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi - \frac{i}{\sqrt{2}}(\theta\theta)\partial_\mu\psi\sigma^\mu\bar{\theta} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi$$

$$\Phi^\dagger = \varphi^* + \sqrt{2}\bar{\theta}\bar{\psi} + (\bar{\theta}\bar{\theta})F^* - i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi^* + \frac{i}{\sqrt{2}}(\bar{\theta}\bar{\theta})\theta\sigma^\mu\partial_\mu\bar{\psi} - \frac{1}{4}(\theta\theta)(\bar{\theta}\bar{\theta})\partial_\mu\partial^\mu\varphi^*$$

we can work out that

$$\begin{aligned} \Phi^\dagger e^{2qV}\Phi\Big|_D &= \Phi^\dagger\Phi\Big|_D \\ &\quad + (\varphi^* + \sqrt{2}\bar{\theta}\bar{\psi} - i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi^*) \\ &\quad\quad \times (2q(\theta\sigma^\mu\bar{\theta})V_\mu + 2iq(\theta\theta)\bar{\theta}\bar{\lambda} - 2iq(\bar{\theta}\bar{\theta})\theta\lambda + (\theta\theta)(\bar{\theta}\bar{\theta})(qD + q^2 V_\mu V^\mu)) \\ &\quad\quad \times (\varphi + \sqrt{2}\theta\psi + i(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi)\Big|_D \end{aligned}$$

The new terms we need to consider are

$$(\theta\theta)(\bar{\theta}\bar{\theta})\varphi^*(qD + q^2 V_\mu V^\mu)\varphi$$

which is fine, and

$$2iq\varphi^*(\theta\sigma^\mu\bar{\theta})V_\mu(\theta\sigma^\mu\bar{\theta})\partial_\mu\varphi + h.c. = iq\varphi^* V^\mu\partial_\mu\varphi(\theta\theta)(\bar{\theta}\bar{\theta}) + h.c.$$

as well as

$$-2\sqrt{2}iq\varphi^*(\bar{\theta}\bar{\theta})(\theta\lambda)(\theta\psi) + h.c = (\theta\theta)(\bar{\theta}\bar{\theta})\sqrt{2}iq\varphi^*(\lambda\psi) + h.c.$$

and finally

$$4q(\bar{\theta}\bar{\psi})(\theta\sigma^\mu\bar{\theta})V_\mu(\theta\psi) = -q(\bar{\theta}\bar{\theta})(\theta\theta)\bar{\psi}\bar{\sigma}^\mu V_\mu\psi$$

using the usual tricks.

Hence we get

$$\begin{aligned} \Phi^\dagger e^{2qV}\Phi\Big|_D &= \partial_\mu\varphi^\dagger\partial^\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu\partial_\mu\psi + F^\dagger F \\ &\quad - q\bar{\psi}\bar{\sigma}^\mu V_\mu\psi + iq\varphi^\dagger V^\mu\partial_\mu\varphi - iq\partial^\mu\varphi^\dagger V_\mu\varphi \\ &\quad + \sqrt{2}iq\left(\varphi^\dagger(\lambda\psi) - (\bar{\psi}\bar{\lambda})\varphi\right) + q\varphi^\dagger\left(D + q^2 V_\mu V^\mu\right)\varphi \end{aligned}$$

or

$$\Phi^\dagger e^{2qV}\Phi\Big|_D = (D_\mu\varphi)^\dagger D^\mu\varphi - i\bar{\psi}\bar{\sigma}^\mu D_\mu\psi + F^\dagger F + q\varphi^\dagger D\varphi + \sqrt{2}iq\left(\varphi^\dagger(\lambda\psi) - (\bar{\psi}\bar{\lambda})\varphi\right)$$

using the covariant derivative

$$D_\mu = \partial_\mu - iqV_\mu$$

Now, we had

$$W^\alpha W_\alpha\Big|_F = D^2 - 2i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{2}F_{\mu\nu}F^{\mu\nu} - \frac{i}{2}F_{\mu\nu}\widetilde{F}^{\mu\nu}$$

so

$$\frac{1}{4}W^\alpha W_\alpha\Big|_F + h.c = \frac{1}{2}D^2 - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

allowing us to write the total Lagrangian explicitly.

$$\begin{aligned}\mathcal{L} &= \partial_\mu\varphi_i^*\partial_\mu\varphi_i - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\varphi_i\varphi_j}\psi_i\psi_j + h.c.\right) + \mathcal{L}_F \\ &+ (D_\mu\varphi_i)^\dagger D^\mu\varphi_i - i\bar{\psi}_i\bar{\sigma}^\mu D_\mu\psi_i + q\varphi_i^\dagger D\varphi_i + \sqrt{2}iq\left(\varphi_i^\dagger(\lambda\psi_i) - (\bar{\psi}_i\bar{\lambda})\varphi_i\right) \\ &+ \frac{1}{2}D^2 - i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\xi D\end{aligned}$$

with

$$\mathcal{L}_F = F_i^\dagger F_i + F_i\frac{\partial W}{\partial\varphi_i} + F_i^\dagger\frac{\partial W^\dagger}{\partial\varphi_i^\dagger}$$

One pick out the terms involving the auxiliary field D:

$$\mathcal{L}_D = \left(q\varphi_i^\dagger\varphi_i + \frac{1}{2}\xi\right)D + \frac{1}{2}D^2 \Rightarrow D = -q\varphi_i^\dagger\varphi_i - \frac{1}{2}\xi$$

This gives us a part of the scalar potential

$$\mathcal{L}_D = -\frac{1}{2}(q\varphi_i^\dagger\varphi_i + \frac{1}{2}\xi)^2 \equiv -V_D(\varphi)$$

We can rewrite the Lagrangian as

$$\begin{aligned}\mathcal{L} &= \partial_\mu\varphi_i^*\partial_\mu\varphi_i - i\bar{\psi}_i\bar{\sigma}^\mu\partial_\mu\psi_i - \frac{1}{2}\left(\frac{\partial^2 W}{\partial\varphi_i\varphi_j}\psi_i\psi_j + h.c.\right) \\ &+ (D_\mu\varphi_i)^\dagger D^\mu\varphi_i - i\bar{\psi}_i\bar{\sigma}^\mu D_\mu\psi_i + \sqrt{2}iq\left(\varphi_i^\dagger(\lambda\psi_i) - (\bar{\psi}_i\bar{\lambda})\varphi_i\right) \\ &- i\lambda\sigma^\mu\partial_\mu\bar{\lambda} - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - V(\varphi_i)\end{aligned}$$

with the scalar potential given by

$$V(\varphi_i) = \sum_i\left|\frac{\partial W}{\partial\varphi_i}\right|^2 + \frac{1}{8}(\xi + 2q\varphi_i^\dagger\varphi_i)^2$$

Non-abelian vector superfields In the non-abelian case the vector superfield is now valued in some representation of a Lie algebra, so we have $V = V^a T^a$ with T^a denoting the Lie algebra generators in the particular representation used, satisfying $[T^a, T^b] = i f^{abc} T^c$. This means we have

$$V_\mu = V_\mu^a T^a \quad D = D^a T^a \quad \lambda = \lambda^a T^a$$

We also consider our chiral superfields Φ_i as transforming in the same representation. In particular we have

$$\Phi \to e^{-2i\Lambda q} \qquad \Lambda = \Lambda^a T^a$$

and want $\Phi^\dagger e^{2qV}\Phi$ to be invariant as before. This is possible if we define the transformation law for V by

$$e^{2qV} \to e^{2qV'} = e^{-2i\Lambda^\dagger q} e^{2qV} e^{2i\Lambda q}$$

which by the CBH formula $e^A e^B = e^{A+B+[A,B]/2+\dots}$ leads to

$$V' = V + i(\Lambda - \Lambda^\dagger) + iq[V, \Lambda + \Lambda^\dagger] + \dots$$

We can still use this to put V in Wess-Zumino gauge.

The definition of the field strength is modified to:

$$W_\alpha = -\frac{1}{8q}(\bar{D}\bar{D})\left(e^{-2qV} D_\alpha e^{2qV}\right)$$

which transforms as

$$W_\alpha \to e^{2iq\Lambda^\dagger} W_\alpha e^{2iq\Lambda}$$

and so we use

$$\mathrm{tr}\, W^\alpha W_\alpha \Big|_F$$

in our Lagrangians.

In Wess-Zumino gauge it can be shown that

$$W_\alpha^a = -i\lambda_\alpha^a(y) + \theta_\alpha D^a(y) + (\sigma^{\mu\nu}\theta)_\alpha F_{\mu\nu}^a(y) + (\theta\theta)\sigma^\mu_{\alpha\dot{\beta}} D_\mu \bar{\lambda}^{a\dot{\beta}}(y)$$

with

$$F_{\mu\nu}^a = \partial_\mu V_\nu^a - \partial_\nu V_\mu^a + q f^{abc} V_\mu^b V_\mu^c$$

and

$$D_\mu \bar{\lambda}^a = \partial_\mu \bar{\lambda}^a + q V_\mu^b \bar{\lambda}^c f^{abc}$$

which is just the usual non-abelian generalisation of our previous expression. Similarly, one gets

$$\frac{1}{4}\left(\mathrm{tr}\, W^\alpha W_\alpha \Big|_F + h.c.\right) = \frac{1}{2} D^a D^a - i\lambda^a \sigma^\mu D_\mu \bar{\lambda}^a - \frac{1}{4} F^{a\mu\nu} F_{\mu\nu}^a$$

Our previous expression for the Lagrangian of chiral superfields coupled to a vector superfield can be easily carried over to the non-abelian case, noting that there is now no Fayet-Iliopolous term, i.e. $\xi = 0$, and that each chiral superfield now carries an internal representation index. For instance, the part of the Lagrangian involving the auxiliary field $D = D^a T^a$ is now

$$\mathcal{L}_D = \frac{1}{2} D^a D^a + q\varphi^\dagger_m D^a T^a_{mn} \varphi_n$$

where m, n denote the representation index. We thus get that $D^a = -q\varphi^\dagger_m T^a_{mn}\varphi_n$, so that

$$V_D(\varphi) = \frac{q^2}{2}\left(\varphi^\dagger_m T^a_{mn}\varphi_n\right)\left(\varphi^\dagger_p T^a_{pq}\varphi_q\right)$$

Renormalisation For $\mathcal{N} = 1$ supersymmetry the Kahler potential K, superpotential W, gauge kinetic function $f(\Phi)$ and Fayet-Iliopolous constant ξ completely specify the structure of the theory. It turns out that K gets quantum corrections at all orders in perturbation theory, $f(\Phi)$ only gets corrections at one-loop, and W and ξ are not renormalised at all.

5 Superbreaking

Conditions for supersymmetry breaking Supersymmetry is broken if the vacuum state is not annihilated by the generators, $Q_\alpha|0\rangle \neq 0$. Consider

$$\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 2\sigma^\mu_{\alpha\dot{\alpha}} P_\mu$$

and contract with $(\bar{\sigma}^\nu)^{\dot{\beta}\alpha}$ to get

$$(\bar{\sigma}^\nu)^{\dot{\beta}\alpha}\{Q_\alpha, \bar{Q}_{\dot{\alpha}}\} = 4P^\nu$$

(as the trace of $\bar{\sigma}^\nu\sigma^\mu$ is $2\eta^{\mu\nu}$). Specialising to $\nu = 0$, we have that

$$\sum_{\alpha=1}^{2}\left(Q_\alpha Q^\dagger_\alpha + Q^\dagger_\alpha Q_\alpha\right) = 4E$$

Taking the vacuum expectation value of this we see that broken supersymmetry means $E_{vac} > 0$, while unbroken supersymmetry means $E_{vac} = 0$.

F-term breaking Consider a chiral superfield, for which we have the supersymmetry transformations

$$\delta\varphi = \sqrt{2}\varepsilon\psi \quad \delta\psi = \sqrt{2}\varepsilon F + i\sqrt{2}\sigma^\mu\varepsilon\partial_\mu\varphi \quad \delta F = i\sqrt{2}\bar{\varepsilon}\sigma^\mu\partial_\mu\psi$$

Supersymmetry will be broken if one of these variations has a non-zero vacuum expectation value. Lorentz invariance however requires that $\langle\psi\rangle = \langle\partial_\mu\varphi\rangle = 0$. Hence the only way to achieve supersymmetry breaking via a chiral superfield is to have

$$\langle F\rangle \neq 0 \Rightarrow \langle\delta\varphi\rangle = \langle\delta F\rangle = 0 \quad \langle\delta\psi\rangle \neq 0$$

The spinor field ψ then becomes a Goldstone fermion (or Goldstino).

As $V_F = |\frac{\partial W}{\partial \varphi}|^2 = |F|^2$ then we must have $V_F > 0$ for F-term breaking.

O'Raifeartaigh model An example of F-term breaking is provided by the O'Raifeartaigh model. This model consists of three chiral superfields, with Kahler potential

$$K = \Phi_i^\dagger \Phi_i$$

and superpotential

$$W = g\Phi_1(\Phi_3^2 - m^2) + M\Phi_2\Phi_3 \quad M >> m$$

Recall that

$$F_i = -\frac{\partial W^*}{\partial \varphi_i^*} \equiv -\frac{\partial W^*}{\partial \Phi_i^*}\bigg|_{\Phi_i = \varphi_i}$$

so that

$$F_1 = -g(\varphi_3^{*2} - m^2) \quad F_2 = -M\varphi_3^* \quad F_3 = -2g\varphi_1^*\varphi_2^* - M\varphi_2^*$$

We observe that $\langle F_1 \rangle = 0 \Rightarrow \langle F_2 \rangle \neq 0$ and $\langle F_2 \rangle = 0 \Rightarrow \langle F_1 \rangle \neq 0$, so it is unavoidable that we cannot have all $\langle F_i \rangle = 0$ simultaneously, and thus have F-term breaking.

The scalar potential is

$$V_F = g^2|\varphi_3^2 - m^2|^2 + M^2|\varphi_3|^2 + |2g\varphi_1 + M|^2|\varphi_2|^2$$

The minimum of this potential is achieved for

$$\langle \varphi_3 \rangle = \langle \varphi_2 \rangle = 0$$

and $\langle \varphi_1 \rangle$ arbitrary; then we get $V_F = g^2 m^4 > 0$. To calculate the scalar masses let $\varphi_3 = \frac{1}{\sqrt{2}}(a + ib)$, then

$$\begin{aligned} V_F &= \frac{1}{4}g^2|a^2 - b^2 - 2m^2 + 2iab|^2 + \frac{1}{2}M^2(a^2 + b^2) + |\varphi_2|^2 M^2 + \text{cubic terms} \\ &= -\frac{1}{4}g^2 4m^2(a^2 - b^2) + \frac{1}{2}M^2(a^2 + b^2) + |\varphi_2|^2 M^2 + \text{cubic terms} \end{aligned}$$

from which we see that

$$m_1^2 = 0 \quad m_2^2 = M^2 \quad m_a^2 = M^2 - 2g^2m^2 \quad m_b^2 = M^2 + 2g^2m^2$$

The fermion masses appear from the term

$$-\frac{1}{2}\frac{\partial^2 W}{\partial \varphi_i \varphi_j}\psi_i\psi_j = -\frac{1}{2}\begin{pmatrix}\psi_1 & \psi_2 & \psi_3\end{pmatrix}\begin{pmatrix} 0 & 0 & 2g\varphi_3 \\ 0 & 0 & M \\ 2g\varphi_3 & M & 2g\varphi_1 \end{pmatrix}\begin{pmatrix}\psi_1 \\ \psi_2 \\ \psi_3\end{pmatrix}$$

from which we extract the mass matrix

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & M \\ 0 & M & 0 \end{pmatrix}$$

implying there are two fermions of mass M and one massless fermion, ψ_1. (The Lagrangian for a two-component Majorana spinor is $-i\bar{\psi}\bar{\sigma}^\mu\partial_\mu - \frac{1}{2}m(\psi\psi + \bar{\psi}\bar{\psi})$.)

Supertrace in F-term breaking The supertrace is defined by

$$\mathrm{Str}M^2 = \sum_j (-1)^{2j+1}(2j+1)m_j^2$$

For a chiral superfield this reduces to

$$-\sum_{\text{scalars}} m^2 + 2\sum_{\text{fermions}} m^2$$

Now, the scalar mass terms arise from

$$V_F = \sum_j \frac{\partial W}{\partial \varphi_j}\frac{\partial W^*}{\partial \varphi_j^*}$$

Let's split the scalar fields into their real and imaginary parts:

$$\varphi_j = \frac{1}{\sqrt{2}}(a_j + ib_j)$$

The mass matrix is then given schematically by

$$M^2 \sim \begin{pmatrix} \frac{\partial^2 V}{\partial a_j \partial a_k} & \frac{\partial^2 V}{\partial a_j \partial b_k} \\ \frac{\partial^2 V}{\partial b_j \partial a_k} & \frac{\partial^2 V}{\partial b_j \partial b_k} \end{pmatrix}$$

where to be precise we should consider setting all fields $a_j = b_j = 0$ after taking the derivatives so only the quadratic terms are relevant. Now, we just want the trace,

$$\sum_j \frac{\partial^2 V}{\partial a_j^2} + \sum_j \frac{\partial^2 V}{\partial b_j^2}$$

Now,

$$\frac{\partial}{\partial a_j} = \frac{\partial \varphi_j}{\partial a_j}\frac{\partial}{\partial \varphi_j} + \frac{\partial \varphi_j^*}{\partial a_j}\frac{\partial}{\partial \varphi_j^*} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial \varphi_j} + \frac{\partial}{\partial \varphi_j^*}\right)$$

(no sum on j here) and similarly

$$\frac{\partial}{\partial b_j} = \frac{i}{\sqrt{2}}\left(\frac{\partial}{\partial \varphi_j} - \frac{\partial}{\partial \varphi_j^*}\right)$$

so

$$\frac{\partial^2}{\partial a_j^2} = \frac{1}{2}\left(\frac{\partial^2}{\partial \varphi_j^2} + \frac{\partial^2}{\partial \varphi_j^{*2}} + 2\frac{\partial^2}{\partial \varphi_j \varphi_j^*}\right)$$

$$\frac{\partial^2}{\partial b_j^2} = -\frac{1}{2}\left(\frac{\partial^2}{\partial \varphi_j^2} + \frac{\partial^2}{\partial \varphi_j^{*2}} - 2\frac{\partial^2}{\partial \varphi_j \varphi_j^*}\right)$$

hence we get for the trace of the scalar mass matrix

$$2\sum_j \frac{\partial^2 V}{\partial \varphi_j \varphi_j^*} = 2\sum_{i,j} \frac{\partial^2}{\partial \varphi_j \varphi_j^*}\frac{\partial W}{\partial \varphi_i}\frac{\partial W^*}{\partial \varphi_i^*} = 2\sum_{i,j} \frac{\partial^2 W}{\partial \varphi_i \varphi_j}\frac{\partial^2 W^*}{\partial \varphi_i^* \varphi_j^*}$$

The fermion mass matrix is

$$M = \frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j}$$

where again we should set the fields to zero afterwards to only consider the relevant terms. To get the sum of mass squares we need the trace of $MM^\dagger$ (because we can diagonalise M to have eigenvalues $m_i e^{i\phi_i}$; then $MM^\dagger$ has m_i^2 on the diagonal; $\mathrm{tr}\,(UMU^\dagger UM^\dagger U^\dagger) = \mathrm{tr}MM^\dagger$), but this is

$$\frac{\partial^2 W}{\partial \varphi_i \partial \varphi_j}\frac{\partial^2 W^*}{\partial \varphi_j^* \partial \varphi_i^*}$$

which gives just half the trace of the scalar mass squares. We therefore see that the supertrace vanishes for F-term supersymmetry breaking.

D-term breaking For a vector superfield V consisting of the fields λ, V_μ, D then Lorentz invariance only allows us have

$$\langle D\rangle \neq 0 \Rightarrow \langle \delta\lambda\rangle = \varepsilon\langle D\rangle \neq 0$$

so that λ becomes a Goldstino.

For an abelian vector superfield we have

$$D = -q\varphi_i^\dagger \varphi_i - \frac{1}{2}\xi$$

and

$$V_D = \frac{1}{2}(q\varphi_i^\dagger \varphi_i + \frac{1}{2}\xi)^2 = \frac{1}{2}D^2$$

If q and ξ have opposite signs then we can have $\langle D\rangle = 0$ with $\langle \varphi_i\rangle \neq 0$, and this minimises V_D leaving supersymmetry unbroken. If however they have the same sign then we can take $\langle \varphi_i\rangle = 0$, and $\langle D\rangle \neq 0$, so that $V_D > 0$ in the vacuum and supersymmetry is broken. Note that the Lagrangian then contains a term

$$\frac{1}{2}q\xi\varphi_i^\dagger \varphi_i$$

so each scalar field φ_i acquires a mass $m^2 = \frac{1}{2}q\xi$ while the fermions ψ_i remain massless.

F- and D-term breaking in non-abelian model For a model with a non-abelian vector superfield then there is no Fayet-Iliopolous term. For simplicity consider coupling a single chiral superfield Φ with representation index m to a vector superfield. We have

$$V = F_m^\dagger F_m + \frac{1}{2} D^a D^a$$

with

$$F_m = \frac{\partial W^*}{\partial F_m^\dagger} \quad D^a = \varphi_m^\dagger (T^a)_{mn} \varphi_n$$

Now, by definition the vacuum corresponds to a minimum of the potential, i.e.

$$0 = \left\langle \frac{\partial V}{\partial \varphi_n} \right\rangle = \frac{\partial^2 W}{\partial \varphi_m \partial \varphi_n} F_n + \varphi_m^\dagger (T^a)_{mn} D^a$$

where we have left the vacuum expectation brackets implicit on the right-hand side. Now, the superpotential W is gauge invariance by construction, so that

$$0 = \langle \delta^a W \rangle = \frac{\partial W}{\partial \varphi_m} \delta^a \varphi_m = F_m^\dagger (T^a)_{mn} \varphi_n$$

If our generators are hermitian this is equivalent to

$$\varphi_m^\dagger (T^a)_{mn} F_n = 0$$

We can combine this into the matrix condition

$$\begin{pmatrix} \frac{\partial^2 W}{\partial \varphi_m \partial \varphi_n} & \varphi_m^\dagger (T^a)_{mn} \\ -\varphi_m^\dagger (T^a)_{mn} & 0 \end{pmatrix} \begin{pmatrix} F_n \\ D^a \end{pmatrix} = 0$$

We can relate this matrix to the fermion mass matrix. The relevant terms in the Lagrangian are

$$-\frac{1}{2} \left(\frac{\partial^2 W}{\partial \varphi_m \varphi_n} \psi_m \psi_n + h.c. \right) + \sqrt{2} i q \varphi_m^\dagger (\lambda^a (T^a)_{mn} \psi_n) + h.c.$$

or

$$-\frac{1}{2} \begin{pmatrix} \psi_m & \lambda_a \end{pmatrix} \begin{pmatrix} \frac{\partial^2 W}{\partial \varphi_m \varphi_n} & \sqrt{2} i q \varphi_p^\dagger (T^a)_{pm} \\ -\sqrt{2} i q \varphi_p^\dagger (T^a)_{pn} & 0 \end{pmatrix} \begin{pmatrix} \psi_n \\ \lambda_a \end{pmatrix}$$

In verifying this note we get an additional minus sign from interchanging the spinors λ^a and ψ_n; recall again that a Lagrangian for a two-component Majorana spinor is $-i\bar{\psi}\bar{\sigma}^\mu \partial_\mu - \frac{1}{2} m(\psi\psi + \bar{\psi}\bar{\psi})$ justifying claiming that

$$\begin{pmatrix} \frac{\partial^2 W}{\partial \varphi_m \varphi_n} & \sqrt{2} i q \varphi_p^\dagger (T^a)_{pm} \\ -\sqrt{2} i q \varphi_p^\dagger (T^a)_{pn} & 0 \end{pmatrix}$$

represents the fermion mass matrix. As then

$$\begin{pmatrix} \frac{\partial^2 W}{\partial\varphi_m \varphi_n} & \sqrt{2} i q \varphi_p^\dagger (T^a)_{pm} \\ -\sqrt{2} i q \varphi_p^\dagger (T^a)_{pn} & 0 \end{pmatrix} \begin{pmatrix} F_n \\ \frac{1}{\sqrt{2}} D^a \end{pmatrix} = 0$$

we can conclude the fermion mass matrix has a zero eigenvalue, corresponding to the existence of a massless Goldstino.

6 Supermodel

The Standard Model Let's first review the basic successes and shortcomings of the current Standard Model of particle physics. The Standard Model is a gauge theory with gauge group $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ broken to $\mathrm{SU}(3)_C \times \mathrm{U}(1)_{em}$ by the Higgs mechanism, through which the particles acquire mass. The Standard Model describes electromagnetic, weak and strong interactions and is impressively in accord with experiment. The Higgs particle remains the only undiscovered Standard Model particle.

Apart from the obvious failure to include quantum gravity, there are a number of problems with the Standard Model. The hierarchy problem is the question of why the electroweak scale ($\sim 10^2$ GeV) is so much less than the Planck scale ($\sim 10^{19}$ GeV), and how do we ensure that the Higgs mass does not receive massive quantum corrections? We will discuss this further below. The cosmological constant problem asks why the cosmological constant (vacuum energy of the universe) is so small, $\Lambda/(M_{pl}^4) \sim 10^{-120}$. This would seem to require much fine tuning of the contributions of the Standard Model particles to the vacuum energy. The Standard Model also still involves ~ 20 free parameters which must be set by measurement, and does not describe dark matter.

MSSM field content Supersymmetry is one way of extending the Standard Model. The simplest possibility is the minimal supersymmetric Standard Model (MSSM). We can describe its field content in terms of $\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ quantum numbers. We have

- Vector superfields

field	$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	spin-1/2	spin-1
G	$(8, 1, 0)$	gluino $\widetilde{g}$	gluon g
W	$(1, 3, 0)$	wino $\widetilde{w}$	W-boson W^μ
B	$(1, 1, 0)$	bino $\widetilde{b}$	hypercharge boson B^μ

- Chiral superfields

field	$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$	spin-0	spin-1/2
$Q_i = \begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix}$	$(3, 2, 1/6)$	squarks $\begin{pmatrix} \widetilde{u}_{Li} \\ \widetilde{d}_{Li} \end{pmatrix}$	quarks $\begin{pmatrix} u_{Li} \\ d_{Li} \end{pmatrix}$
$L_i = \begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}$	$(1, 2, -1/2)$	sleptons $\begin{pmatrix} \widetilde{\nu}_{Li} \\ \widetilde{e}_{Li} \end{pmatrix}$	leptons $\begin{pmatrix} \nu_{Li} \\ e_{Li} \end{pmatrix}$
$\bar{u}_{iR}$	$(\bar{3}, 1, -2/3)$	antisquark $\widetilde{u}^*_{iR}$	antiquark $\bar{u}_{iR}$
$\bar{d}_{iR}$	$(\bar{3}, 1, 1/3)$	antisquark $\widetilde{d}^*_{iR}$	antiquark $\bar{d}_{iR}$
$\bar{e}_{iR}$	$(1, 1, 1)$	slepton $\widetilde{e}^*_{iR}$	lepton $\bar{e}_{iR}$

The indices L and R denote whether the fermions are left-handed or right-handed, and $i = 1, 2, 3$ labels the generation.

- Higgs doublets

field	$SU(3)_C \times SU(2)_L \times U(1)_Y$	spin-0	spin-1/2
$H_1 = \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix}$	$(1, 2, -1/2)$	Higgs $\begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix}$	Higgsino $\begin{pmatrix} \widetilde{H}_1^0 \\ \widetilde{H}_1^- \end{pmatrix}$
$H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}$	$(1, 2, 1/2)$	Higgs $\begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}$	Higgsino $\begin{pmatrix} \widetilde{H}_2^+ \\ \widetilde{H}_2^0 \end{pmatrix}$

We need two Higgses firstly so that we can give mass to both up- and down-type quarks, as we cannot use $H_1^\dagger$ in the superpotential as it is meant to be holomorphic, and secondly so as to cancel an anomaly arising from a triangle Feynman diagram with hypercharge bosons as external particles (this diagram is proportional to $(\sum_{\text{lh fermions}} - \sum_{\text{rh fermions}})(Y/2)^3$).

MSSM superpotential The MSSM superpotential involves writing all field terms which are renormalisable and invariant under the gauge symmetries. The superpotential in fact splits into two parts, the first of which is

$$W_{RP} = (Y_U)_{ij} Q_i H_2 \bar{u}_{Rj} - (Y_D)_{ij} Q_i H_1 \bar{d}_{Rj} - (Y_E)_{ij} L_i H_1 \bar{e}_{jR} + \mu H_1 H_2$$

In writing this we have suppressed internal indices, so really

$$Q_i H_2 \bar{u}_{Rj} \equiv \varepsilon_{ab} Q_i^{xa} H_2^b \bar{u}_{Rxj}$$

with $x = 1, 2, 3$ an SU(3) index. The matrices Y_{ij} amount to matrices of Yukawa couplings, and μ is a mass term for the Higgses.

In fact we can write

$$(Y_U)_{ij} Q_i H_2 \bar{u}_{Rj} = (Y_U)_{ij} \left(-u_{Li} H_2^0 \bar{u}_{Rj} + d_{Li} H_2^+ \bar{u}_{Rj} \right)$$

and then apply the Higgs mechanism by writing $H_2^0 = \frac{1}{\sqrt{2}}(v_2 + h_2^0)$, thus obtaining a mass matrix

$$\frac{1}{\sqrt{2}} v_2 (Y_U)_{ij}$$

for up-type quarks and squarks. Similarly, we have

$$-(Y_D)_{ij} Q_i H_1 \bar{d}_{Rj} = -(Y_D)_{ij} \left(-u_{Li} H_1^- \bar{d}_{Rj} + d_{Li} H_1^0 \bar{d}_{Rj} \right)$$

leading to a mass matrix

$$\frac{1}{\sqrt{2}} v_1 (Y_D)_{ij}$$

for down-type quarks and squarks, and

$$-(Y_E)_{ij}L_iH_1\bar{e}_{jR} = -(Y_E)_{ij}\left(-\nu_{Li}H_1^-\bar{e}_{jR} + e_{Li}H_1^0\bar{e}_{jR}\right)$$

leading to a mass matrix

$$\frac{1}{\sqrt{2}}v_1(Y_E)_{ij}$$

for leptons and sleptons.

R-parity violating terms and proton decay Other possible terms for the superpotential are

$$W_{\not{R}\not{P}} = \frac{1}{2}\lambda_{ijk}L_iL_j\bar{e}_{kR} + \lambda'_{ijk}L_iQ_j\bar{d}_k + \kappa_iL_iH_2 + \frac{1}{2}\lambda''_{ijk}\bar{u}_{iR}\bar{d}_{jR}\bar{d}_{kR}$$

The first three of these violate lepton number, with $\Delta L = 1$, and the last violates baryon number, with $\Delta B = 1$. An unwanted consequence of this is that including these terms would lead to proton decay.

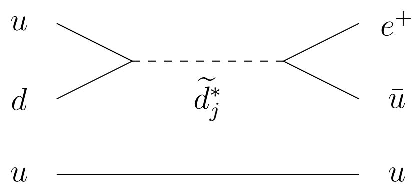

Figure 1: Proton decay

This can be seen in figure 1. Now, the term $\frac{1}{2}\lambda''_{ijk}\bar{u}_{iR}\bar{d}_{jR}\bar{d}_{kR}$ contributes

$$\frac{1}{2}\lambda''_{ijk}\bar{u}_{iR}\bar{d}_{jR}\bar{d}_{kR}\Big|_F + h.c.$$

to the Lagrangian; writing

$$\bar{u}_{iR} = \tilde{\bar{u}}_{iR} + \sqrt{2}\theta\bar{u}_{iR} + \ldots \qquad \bar{d}_{iR} = \tilde{\bar{d}}_{iR} + \sqrt{2}\theta\bar{d}_{iR} + \ldots$$

we get quark-quark-squark interaction terms of the form

$$\sim \lambda^*_{11k}\tilde{\bar{d}}^*_{kR}\bar{u}^\dagger_R\bar{d}^\dagger_R$$

Similarly from the F-terms of $L_iQ_j\bar{d}_k$ we get interaction terms

$$\sim \lambda'^*_{11k}\tilde{\bar{d}}^*_{kR}e^\dagger_{Li}u^\dagger_{Lj}$$

which mediate the $ud \to \tilde{\bar{d}}^* \to e^+\bar{u}$ interaction; the amplitude for proton decay is then proportional to $\lambda^*_{11k}\lambda'^*_{11k}$. The probability will then be proportional to $|\lambda_{11k}\lambda'_{11k}|^2$; and also to $m_{\tilde{d}}^{-}4$ (the propagator contains an inverse mass squared); and so an estimate for the proton decay rate on dimensional grounds ([mass] = [time]$^{-1}$) is

$$\Gamma \sim m_p^5 m_{\tilde{d}}^{-4} |\lambda_{11k}\lambda'_{11k}|^2$$

Experiment suggests the proton lifetime is $\tau \sim 10^{40} s$ so

$$10^{40} s \sim m_p^{-5} m_{\tilde{d}}^{4} |\lambda_{11k}\lambda'_{11k}|^{-2}$$

Using the facts that $1s \sim 10^{24}$ GeV^{-1}, $m_p \sim 1$ GeV and supposing $m_{\tilde{d}} \sim 1$ TeV we have

$$10^{64} \sim 10^{36-30} |\lambda_{11k}\lambda'_{11k}|^{-2} \Rightarrow |\lambda_{11k}\lambda'_{11k}|^2 \sim 10^{-60}$$

so one or other of the couplings must be absolutely tiny. Conversely if we had assumed the couplings were of $O(1)$, we would obtain $\tau \sim 10^{-18}$ s.

To rule out proton decay it is convenient to impose a new symmetry on the MSSM Lagrangian which forbids the superpotential $W_{\not{R}\not{P}}$. This symmetry is R-parity and it is defined by

$$R = (-1)^{3(B-L)+2S}$$

where B and L are the baryon and lepton numbers and S is the spin (note that superpartners inherit the baryon and lepton numbers of the original Standard Model particles). All standard model particles have $R = +1$, and their superpartners have $R = -1$. Imposing R-parity conservation has the effect of ruling out all interaction terms stemming from $W_{\not{R}\not{P}}$ (this can be seen by expanding the chiral superfields and observing which terms have two Standard Model fields interacting with a single superpartner field, or a single Standard Model field interacting with a single superpartner field). It also means that a single supersymmetric particle cannot decay into Standard Model particles alone - there must be an odd number of supersymmetric particle present in the decay. An interesting effect of this is that it implies the lightest supersymmetric particle (LSP) must be stable (as there is nothing else supersymmetric for it to decay into). If the LSP is neutral then we obtain a good candidate for dark matter in the form a WIMP neutralino (mass eigenstate of neutral supersymmetric particles such as higgsino, photino). In the context of the LHC, R-parity conservation implies that even numbers of supersymmetric particles would be produced in every proton-proton collision. One way to search for these is by looking for missing transverse momentum.

MSSM gauge couplings The (chiral) matter superfields we have described above couple to vector gauge superfields. This coupling is provided by the Kahler potential

$$K = \sum_k \Phi_k^\dagger e^{\sum_{i=1} 2 g_i T^a_{R(i)} V^a} \Phi_k$$

where we sum over all chiral superfields Φ_k. Each chiral superfield is in a particular representation $R(i)$ of the three gauge groups of the model, which we label by $i = 1, 2, 3$ for U(1), SU(2) and SU(3). The coupling for group i is g_i, and $T^a_{R(i)}$ denotes the a^{th} generator of the group i in the representation $R(i)$

which Φ_k transforms in.

MSSM Higgs potential We know from our general theory of non-abelian fields that the D-term potential for a scalar field φ coupled to a vector superfield with coupling constant g and in a representation with hermitian generators T^a is

$$V_D = \frac{1}{2} g^2 D^a D^a \qquad D^a = -\varphi^* T^a \varphi$$

In the MSSM we have scalar Higgs fields

$$H_1 = \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix} \quad H_2 = \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix}$$

transforming trivially under SU(3) but in the fundamental representation of SU(2) and with hypercharges $-1/2$ and $+1/2$ respectively under $\mathrm{U}(1)_Y$. We let the latter have coupling constant g' and the former have coupling constant g.

The contribution from the $\mathrm{U}(1)_Y$ generators to V_D is then just

$$\frac{1}{8} g'^2 \left(H_2^\dagger H_2 - H_1^\dagger H_1 \right)^2 = \frac{1}{8} g'^2 \left(|H_2^0|^2 + |H_2^+|^2 - |H_1^0|^2 - |H_1^-|^2 \right)^2$$

The generators of SU(2) can be taken to be $\sigma^a/2$ with σ^a the Pauli sigma matrices. Letting D_2^a denote the corresponding D fields we have

$$\begin{aligned} 2D_2^1 &= \begin{pmatrix} H_1^{0*} & H_1^{-*} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix} + \begin{pmatrix} H_2^{+*} & H_2^{0*} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix} \\ &= H_1^{0*} H_1^- + H_1^{-*} H_1^0 + H_2^{+*} H_2^0 + H_2^{0*} H_2^+ \end{aligned}$$

$$\begin{aligned} 2D_2^2 &= \begin{pmatrix} H_1^{0*} & H_1^{-*} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix} + \begin{pmatrix} H_2^{+*} & H_2^{0*} \end{pmatrix} \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix} \\ &= i \left(-H_1^{0*} H_1^- + H_1^{-*} H_1^0 - H_2^{+*} H_2^0 + H_2^{0*} H_2^+ \right) \end{aligned}$$

$$\begin{aligned} 2D_2^3 &= \begin{pmatrix} H_1^{0*} & H_1^{-*} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} H_1^0 \\ H_1^- \end{pmatrix} + \begin{pmatrix} H_2^{+*} & H_2^{0*} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} H_2^+ \\ H_2^0 \end{pmatrix} \\ &= |H_1^0|^2 - |H_1^-|^2 + |H_2^+|^2 - |H_2^0|^2 \end{aligned}$$

The contribution to the potential is then

$$
\begin{aligned}
\frac{1}{8}g^2\Big(& \left[H_1^{0*}H_1^- + H_1^{-*}H_1^0 + H_2^{+*}H_2^0 + H_2^{0*}H_2^+\right]^2 - \left[-H_1^{0*}H_1^- + H_1^{-*}H_1^0 - H_2^{+*}H_2^0 + H_2^{0*}H_2^+\right]^2 \\
& + \left[|H_1^0|^2 - |H_1^-|^2 + |H_2^+|^2 - |H_2^0|^2\right]^2 \Big) \\
= \frac{1}{8}g^2\Big(& 4\left[|H_1^0|^2|H_1^-|^2 + H_1^{0*}H_1^-H_2^{0*}H_2^+ + H_1^{-*}H_1^0H_2^{+*}H_2^0 + |H_2^+|^2|H_2^0|^2\right] \\
& + |H_1^0|^4 + |H_1^-|^4 + |H_2^+|^4 + |H_2^0|^4 - 2|H_1^0|^2|H_1^-|^2 + 2|H_1^0|^2|H_2^+|^2 \\
& - 2|H_1^0|^2|H_2^0|^2 - 2|H_1^-|^2|H_2^+|^2 + 2|H_1^-|^2|H_2^0|^2 - 2|H_2^0|^2|H_2^+|^2\Big) \\
= \frac{1}{8}g^2\Big(& 4\left[|H_2^+|^2|H_1^0|^2 + |H_2^0|^2|H_1^-|^2 + H_2^+H_1^{0*}H_2^{0*}H_1^- + H_2^{+*}H_1^0H_2^0H_1^{-*}\right] \\
& + |H_1^0|^4 + |H_1^-|^4 + |H_2^+|^4 + |H_2^0|^4 + 2|H_1^0|^2|H_1^-|^2 + 2H_2^0|^2|H_2^+|^2 \\
& - 2|H_2^+|^2|H_1^0|^2 - 2|H_1^-|^2|H_2^0|^2 - 2|H_1^0|^2|H_2^0|^2 - 2|H_1^-|^2|H_2^+|^2\Big)
\end{aligned}
$$

which is just

$$
\frac{1}{2}g^2\left|H_2^+H_1^{0*} + H_2^0H_1^{-*}\right|^2 + \frac{1}{8}g^2\left(|H_2^0|^2 + |H_2^+|^2 - |H_1^0|^2 - |H_1^-|^2\right)^2
$$

We can also consider the F-term potential, which can be written

$$
V_F = \left|\frac{\partial W}{\partial H_1}\right|^2 + \left|\frac{\partial W}{\partial H_2}\right|^2 = |\mu|^2\left(|H_2|^2 + |H_1|^2\right)
$$

so that the full Higgs potential from the F- and D-terms is

$$
\frac{g^2}{2}\left|H_2^+H_1^{0*} + H_2^0H_1^{-*}\right|^2 + \frac{g^2+g'^2}{8}\left(|H_2^0|^2 + |H_2^+|^2 - |H_1^0|^2 - |H_1^-|^2\right)^2 + |\mu|^2\left(|H_1^0|^2 + |H_1^-|^2 + |H_2^0|^2 + |H_2^+|^2\right)
$$

Supersymmetry breaking in the real world Owing to the conspicuous lack of superpartners with the same mass as Standard Model particles supersymmetry if present in nature must be a broken symmetry. The question then arises of finding the mechanism by which supersymmetry is broken. The methods of F- and D-term breaking are unsuitable as breaking supersymmetry directly in this fashion always leads to a vanishing supertrace:

$$
\mathrm{Str}M^2 = \sum_j (-1)^{2j+1}(2j+1)m_j^2 = 0
$$

As we need supersymmetric particles to be heavier than their Standard Model partners this cannot hold. Thus we should have a "hidden sector" of fields which do not directly indirect with the Standard Model; supersymmetry is broken out of direct reach and this is mediated to the observable fields by some process. Examples include considering the gauge group $E_8 \times E_8$ from heterotic string theory: we break supersymmetry in the first E_8 factor, with the Standard Model fields contained in the second E_8, with the two interacting via supergravity.

In general one obtains additional "soft breaking terms" as part of our Lagrangian of observable fields.

For the MSSM the general R-parity conserving soft supersymmetry breaking Lagrangian is

$$\begin{aligned}\mathcal{L} = &(Y_U)_{ij}\widetilde{Q}_{Li}H_2\widetilde{u}^*_{Rj} - (Y_D)_{ij}\widetilde{Q}_{Li}H_1\widetilde{d}^*_{Rj} - (Y_E)_{ij}\widetilde{L}_{Li}H_1\widetilde{e}^*_{Rj}\\ &+ \widetilde{Q}^*_{Li}m^2_{\widetilde{Q}ij}\widetilde{Q}_{Lj} + \widetilde{L}^*_{Li}m^2_{\widetilde{L}ij}\widetilde{L}_{Lj} + \widetilde{e}^*_{Ri}m^2_{\widetilde{e}ij}\widetilde{e}^*_{Rj} + \widetilde{u}^*_{Ri}m^2_{\widetilde{u}ij}\widetilde{u}^*_{Rj} + \widetilde{d}^*_{Ri}m^2_{\widetilde{d}ij}\widetilde{d}^*_{Rj}\\ &+ \mu B H_1 H_2 + h.c. + m_1^2|H_1|^2 + m_2|H_2|^2 + \frac{M_3}{2}\widetilde{g}^a\widetilde{g}^a + \frac{M_2}{2}\widetilde{W}^b\widetilde{W}^b + \frac{M_1}{2}\widetilde{B}\widetilde{B}\end{aligned}$$

The complete model then has over 100 parameters.

Hierarchy problem The hierarchy problem comes in two parts. The first part asks why the electroweak symmetry breaking scale $M_{EW} \sim 10^2$ GeV is so much less than the Plank mass $M_{Pl} \sim 10^{19}$ GeV defining the scale of quantum gravity. The second part asks if this hierarchy is stable under quantum corrections.

The second part of the problem results from the fact that the electroweak scale is set by the Higgs mass term in the Standard Model Lagrangian (by gauge symmetry the Higgs is the only particle in the Standard Model which can have a mass term). Corrections to the Higgs mass arise from diagrams with fermions and boson loops, and in the presence of a momentum cut-off Λ have the form

$$\delta m_H^2 = \frac{\Lambda^2}{16\pi^2}\left(a\lambda - b\lambda_f^2\right)$$

Here λ is the coupling constant for the quartic Higgs self-interaction and λ_f is a coupling constant for a Higgs Yukawa coupling to fermions. The problem is that for large Λ this is much bigger than M_{EW}. One can fine-tune the theory to avoid this instability, but nobody likes fine-tuning. Supersymmetry however provides exactly the right relationship between λ and λ_f^2 to cancel this mass correction. This is one of the primary motivations behind considering supersymmetry as a physical symmetry. Note however that we must break supersymmetry in such a way as to preserve it as a solution to the hierarchy problem. (See discussion in Quevedo's lecture notes for more details.)

7 Supergravity

Elements of supergravity One can extend supersymmetry to a local symmetry to obtain supergravity. There, we introduce a spin-3/2 field ψ_μ^α known as the Rarita-Schwinger field, representing a gravitino, which couples to the conserved supercurrent J^μ_α. Construction of invariant Lagrangians is a little more involved. For the F-terms scalar potential one gets

$$V_F = e^{K/M_{Pl}^2}\left((K_{i\bar{j}})^{-1}D_iW(D_{\bar{j}}W)^* - \frac{3|W|^2}{M_{Pl}^2}\right)$$

Here W is a superpotential, K a Kahler potential, and

$$K_{i\bar{j}} = \frac{\partial^2 K}{\partial\Phi_i\Phi_j^*}$$

with Φ_i denoting the chiral superfields of the theory. Note the subtle fact that the index structure of the inverse Kahler metric is

$$(K_{i\bar{j}})^{-1} = K^{\bar{i}j}$$

We also have the derivatives

$$D_i W = \frac{\partial W}{\partial \Phi_i} + \frac{W}{M_{Pl}^2} \frac{\partial W}{\partial \Phi_i}$$

which are in fact essentially the auxiliary fields F_i are

$$F_i \propto D_i W$$

For supersymmetry breaking in supergravity models one has $\langle F_i \rangle \neq 0$. Note that it's possible to have $V > 0$ or $V < 0$ even after breaking supersymmetry. This is important for the cosmological constant problem, as we could have $\langle V_F \rangle \approx 0$ after breaking in accord with observation of the cosmological constant (whereas global supersymmetry breaking leads to a cosmological constant of order $(100\text{GeV})^4$).

There is also a phenomenon known as the super Higgs effect (note this does not refer to the normal Higgs effect in a supersymmetric theory). In this effect the goldstino resulting from the breaking gets eaten by the gravitino, which thereby obtains a mass. Thus a massive gravitino is a hallmark of broken supergravity. The gravitino mass is $m_{\tilde{G}} = \langle e^{K/2} |W| \rangle$.

8 Supersources

The basic notes for this course, asides from the ones I took down in Prof Ben Allanach's lectures, were

- F. Quevedo, "Cambridge Lectures on Supersymmetry and Extra Dimensions," `http://arxiv.org/abs/1011.1491`

There were various inconsistencies between Quevedo's notes and those from the actual classes I attended, due either to changing conventions or mistakes. Generally I've followed the conventions from my class notes (and inevitably taken the accidental liberty of introducing my own mistakes). Useful resources for comparison and illumination included:

- D. Bailin, A. Love, *Supersymmetric Gauge Field Theory and String Theory*, IOP
- J. Wess, J. Bagger, *Supersymmetry and Supergravity*, PUP
- M. Srednicki, *Quantum Field Theory*, CUP
- J.D. Lykken, "Introduction to Supersymmetry," `http://arxiv.org/abs/hep-th/9612114`
- S. Dawson, "The MSSM and Why It Works," `http://arxiv.org/abs/hep-ph/9712464`

I tended to refer to the first two of the above for information on superspace and superfields. Srednicki's book summarises about half the course in two typically short and carefully constructed chapters. Lykken's notes are quite good and he lists all his conventions, basic identities and commutation relations at the end (the world would be a far better place if all texts on the subject did something similar). Dawson's notes, as the name suggests, were helpful for the MSSM.