PY1002: Special Relativity

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These notes cover the Junior Freshman course given by Dr. Barklie in Michaelmas Term 2006.

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1 Galilean Transformations

1.1 Reference Frames

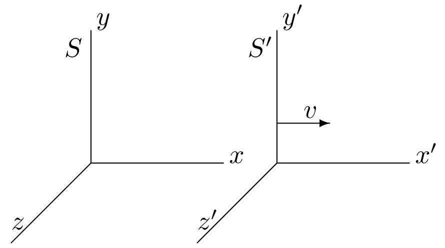
A key concept in special relativity is that of a *reference frame.* A reference frame is basically a set of axes (a co-ordinate system) which a particular observer uses to record events. A reference frame is either inertial (accelerating) or non-inertial (not accelerating). In special relativity, an *event* consists of a particular set of space-time co-ordinates of the form (x, y, z, t).

1.2 Galilean Invariance

The *principle of Galilean Invariance* states that the basic laws of physics are identical in all inertial reference frames. Essentially, this means that there is no way to tell by experiment if the reference frame you are in is moving or not.

1.3 Galilean Transformations

Consider an inertial reference frame S and a second inertial frame S' moving with velocity v away from it along the x-axis.



Suppose the origins of the two frames coincide at time $t = t' = 0$. Then we have the following *Galilean transformations* for events in the two frames:

Galilean transformations

$$x = x' + vt'$$
$$y = y'$$
$$z = z'$$
$$t = t'$$

and the velocity transformations:

Galilean velocity transformations

$$v_x = v_{x'} + v$$
$$v_y = v_{y'}$$
$$v_z = v_{z'}$$

However these transformations are only correct when the velocities involved are far less than the speed of light, c.

2 Michelson-Morley Experiment

2.1 The Ether

Nineteenth century physicists believed in the existence of a stationary, all-permeating medium called the *ether* which allowed light to travel through space.

2.2 Brief Outline of The Experiment

The movement of the Earth through the ether was believed to cause an ether wind of velocity v. It followed that light travelling in the direction of the ether wind would have a velocity of $v + c$, and when travelling in the opposite direction would have a velocity of $v - c$.

The *Michelson-Morley experiment* used a device called a Michelson interferometer to split a beam of light into two beams, that would each travel an identical distance l but with different velocities with respect to the ether, as a result of the Earth's movement. When recombined, the two split beams would have been travelling for slightly different times, and so would be out of phase with each other, producing an interference effect.

Rotating the apparatus would produce a second time difference leading to a different interference pattern - essentially the idea was to measure the fringe shift caused by this change. Using the known velocity of the Earth through the ether, this fringe shift could be calculated theoretically. However, when the experiment was performed the measured fringe shift was far less than the expected one, and within experimental error corresponded to a complete absence of a fringe shift.

The failure of the Michelson-Morley experiment implied that the ether did not in fact exist, though physicists at the time were not quick to accept this. It also highlighted the need for a new understanding of space and time.

For a more thorough analysis of this experiment, see just about any physics book that discusses special relativity.

3 Lorentz Transformations

3.1 The Special Theory Of Relativity

In 1905, Albert Einstein published a paper entitled *On The Electrodynamics of Moving Bodies* in which he developed a theory whereby transformations could be found that left Maxwell's equations of electromagnetism invariant when going from a stationary frame to a moving one. This theory, which has come to be known as the *Special Theory of Relativity*, also accounted for the results of the Michelson-Morley experiment and did away with the need for the ether.

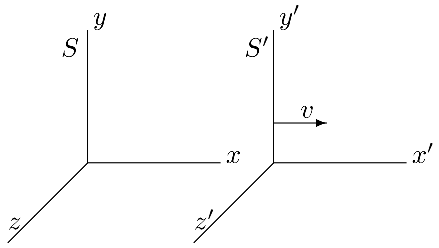
The two postulates of Einstein's Special Theory of Relativity are:

1. The basic laws of physics are identical in all inertial frames.
2. The speed of light in space will always be measured to have the same value, c, independent of the motion of the light source.

Einstein's paper concerned itself first with the kinematic effects of these two postulates, then with the electrodynamic effects. In what follows, we shall deal almost wholly with the former.

3.2 Lorentz Transformations

The essential transformations that Einstein found, and which actually had been previously discovered by others, are known as the *Lorentz Transformations.* Consider two reference frames, as before:



Suppose again that the origins coincide at $t = t' = 0$, and that at this time there is a burst of light at the origins. As light travels with speed c in both frames, at time t in frame S, the wavefront of the light forms a sphere of radius $r = ct$ centred on the origin of S, and in S' the wavefront forms a similar sphere of radius $r' = ct'$.
Now,

$$r = \sqrt{x^2 + y^2 + z^2} = ct$$
$$\Rightarrow x^2 + y^2 + z^2 = c^2t^2 \tag{3.1}$$

and similarly

$$x'^2 + y'^2 + z'^2 = c^2t'^2 \tag{3.2}$$

We wish to find a transformation between (3.1) and (3.2). Starting with the Galilean transformation $x' = x - vt, y' = y, z' = z, t' = t$, we find that:

$$x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2$$

which does not agree with (3.1). We now try $t' = t + fx$ for some constant f, obtaining:

$$x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2(t^2 + 2tfx + f^2x^2)$$

and we note that the $-2xvt$ term on the left cancels with the $2tfx$ term on the right if $f = -\frac{v}{c^2}$.
Hence for $t' = t - \frac{vx}{c^2}$, we get:

$$x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2\left(t^2 - \frac{2tvx}{c^2} + \frac{v^2x^2}{c^4}\right)$$

$$\Rightarrow x^2 - 2xvt + v^2t^2 + y^2 + z^2 = c^2t^2 - 2tvx + \frac{v^2x^2}{c^2}$$

$$\Rightarrow x^2(1-\frac{v^2}{c^2}) + y^2 + z^2 = c^2t^2(1-\frac{v^2}{c^2})$$

and we can remove the $(1-\frac{v^2}{c^2})$ terms by altering our transformations so that:

$$x' = \frac{x - vt}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

$$t' = \frac{t - \frac{vx}{c^2}}{\left(1-\frac{v^2}{c^2}\right)^{\frac{1}{2}}}$$

Letting $\beta = \frac{v}{c}$ and $\gamma = \frac{1}{(1-\beta^2)^{\frac{1}{2}}}$, we can now write the Lorentz transformations as:

Lorentz transformations

$$x' = \gamma(x - vt)$$

$$t' = \gamma\Big(t - \frac{vx}{c^2}\Big)$$

$$y' = y,\ z' = z$$

And equivalently:

Lorentz transformations

$$x = \gamma(x' + vt')$$

$$t = \gamma\Big(t' + \frac{vx'}{c^2}\Big)$$

$$y = y',\ z = z'$$

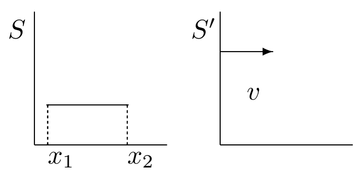
Note that for $v << c$, $\gamma \approx 1$, and these reduce to the Galilean transformations. Also note that $\gamma \geq 1$.

4 Relativistic Effects

4.1 Length Contraction

Length contraction

Consider two frames S, S' with S' moving away from S with velocity v along the x-axis, as before. Let there be a rod of length l_0 in frame S. This rod is at rest with respect to S, but moving relative to S'.



Now, $l_0 = x_2 - x_1$ in S. In S' the length l of the rod is the distance between points x'_1 and x'_2, recorded at time $t'_1 = t'_2 = t'$.

Using the Lorentz transformations,

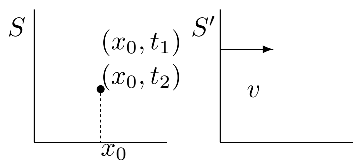
$$x_1 = \gamma(x'_1 + vt')$$

$$x_2 = \gamma(x_1' + vt')$$
$$\Rightarrow x_2 - x_1 = \gamma(x_2' - x_1')$$
$$l_0 = \gamma\, l$$

As $\gamma \geq 1$, this means we have *length contraction* when there is relative motion between the length being measured and the observer compared to when there is no relative motion.

4.2 Time Dilation

Consider two frames as before, and an event occuring at x_0 at times t_1, t_2. The time interval between these two events is known as the *proper time interval* τ_0 - it is the shortest possible time interval between two events.



In S' using the Lorentz transformations for time,

$$t_1' = \gamma\Big(t_1 - \frac{vx_0}{c^2}\Big)$$
$$t_2' = \gamma\Big(t_2 - \frac{vx_0}{c^2}\Big)$$
$$\Rightarrow t_2' - t_1' = \gamma(t_2 - t_1)$$
$$\tau = \gamma\tau_0$$

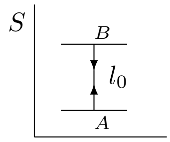
Time dilation

where τ is the time interval as measured in S'. Thus we have *time dilation*: time runs slower for moving clocks. This has been verified experimentally, by comparing the number of mu-mesons that decay when at rest in the laboratory with the number that decay while moving at close to the speed of light while travelling through the atmosphere. For the moving mu-mesons, time moves slower meaning that less decay than when they are at rest. (Another experiment involves synchronising two atomic clocks and sending one on a plane journey. When the two are compared afterwords, the one that has been moving is found to have run slower and be behind the other.)

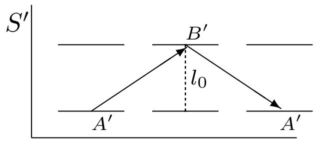
4.3 Light Pulse Clock Method

Time Dilation We can also derive the expressions for both time dilation and length contraction using the idea of a 'light pulse clock.' Consider a light source moving with velocity v relative to a frame S' along the x-axis. We let S be the frame in which the light source is at rest.

Now, in S suppose the light source emits a pulse of light that travels from a point A to a point B and returns to A in a time interval $\tau_0 = \frac{2l_0}{c}$:



The situation in S' is different:



Here the light pulse clock itself travels a distance of $v\tau$ where τ is the time interval in S' while the light pulse travels from A' to B' and back. We can use Pythagoras to show that in S' the light pulse travels a total distance of $2\left[l_0^2 + (\frac{v\tau}{2})^2\right]^{\frac{1}{2}}$

$$\Rightarrow \tau = \frac{2\left[l_0^2 + (\frac{v\tau}{2})^2\right]^{\frac{1}{2}}}{c}$$

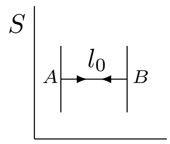
$$\Rightarrow \tau^2 = \frac{4\left[l_0^2 + (\frac{v^2\tau^2}{4})\right]}{c^2}$$

$$\Rightarrow \tau^2(\frac{c^2}{4} - \frac{v^2}{4}) = l_0^2$$

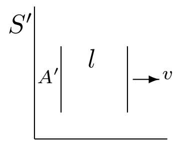
$$\Rightarrow \tau = \frac{2l_0}{(c^2 - v^2)^{\frac{1}{2}}} = \frac{\frac{2l_0}{c}}{(1 - \frac{v^2}{c^2})^{\frac{1}{2}}}$$

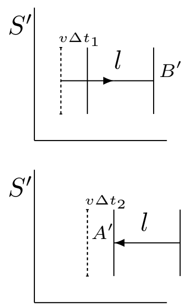
$$\Rightarrow \tau = \gamma\tau_0$$

Length Contraction Consider the same light pulse clock and frames as before, but with the light pulse clock rotated through 90 degrees. In S, we have:



and the time for the light pulse to travel from A to B and back is again $\tau_0 = \frac{2l_0}{c}$. In S' we can consider the journey in the following stages:





The fixed distance between A' and B' is l. As the light pulse travels from A' to B', the point B' moves a distance $v\Delta t_1$ to the right, and as it returns, A' moves a distance $v\Delta t_2$. Hence, we have the folowing two expressions for the distance travelled by the light pulse:

$$l + v\Delta t_1 = c\Delta t_1$$

$$l - v\Delta t_2 = c\Delta t_2$$

$$\Rightarrow \tau = \Delta t_1 + \Delta t_2 = \frac{l}{c-v} + \frac{l}{c+v} = \frac{2lc}{c^2 - v^2}$$

$$\Rightarrow \tau = \frac{\frac{2l}{c}}{(1 - \frac{v^2}{c^2})} = \gamma^2 \frac{2l}{c}$$

and as $\tau = \gamma \tau_0$ we have

$$\gamma^2 \frac{2l}{c} = \gamma \frac{2l_0}{c}$$

$$\Rightarrow \gamma l = l_0$$

4.4 Simultaneity

In Special Relativity, simultaneity is relative. That is, two events that are simultaneous in one frame may not be simultaneous in another.

Consider the Lorentz Transformation for time, $t = \gamma(t' + \frac{vx'}{c^2})$ and two events in S', (x'_1, t'_1) and (x'_2, t'_2). Then in S,

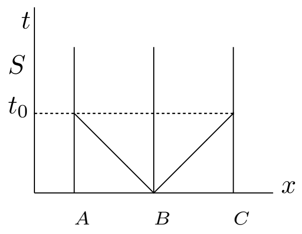
$$t_2 - t_1 = \gamma(t'_2 - t'_1) + \frac{\gamma v}{c^2}(x'_2 - x'_1)$$

So if $t'_2 = t'_1$ *and* $x'_2 = x'_1$ then $t_2 = t_1$ and the events are simultaneous in both frames. If $x'_2 \neq x'_1$ then the events are simultaneous in S' but not S.

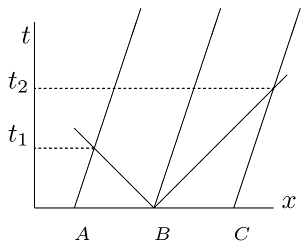
4.5 Minkowski Diagrams

Space-time graphs are a useful aid in special relativity. In their most basic form, we can represent the one-dimensional motion of a body on a graph of t or ct against x. The path of a body is known as its world line.

As an example, we can consider the problem of simultaneity again, and construct a space-time diagram for a frame S in which the points A, B and C are stationary. The diagram shows a beam of light emitted from the point B at time $t = 0$ and arriving at A and C at time t_0.



If now we let the points A, B and C be moving relative to S we find that the same beam of light does not arrive simultaneously at A and C.



and note that the arrival of the light at A and C is a simultaneous event in some other frame S'. The time axis of this frame would be perpendicular to the line joining the points of intersection of the light beam with A and C.

Very often ct is chosen instead of t for the time axis. Then both axes express distances, and then in all frames the world line of a beam of light starting at the origin bisects the ct and x axes. (Note that these axes are not always perpendicular to each other.)

Space-time diagrams are also called Minkowski diagrams after their inventor, Hermann Minkowski.

4.6 Composition Of Velocities

Consider two frames S, S' as before, moving relative to each other with velocity v. Suppose the components of the velocity of a body in frame S are (u_x, u_y, u_z), and those of the body as measured in frame S' are (u'_x, u'_y, u'_z).

Now, using the Lorentz transformations:

$$dx = \gamma(dx' + vdt')$$

$$\Rightarrow dx = \gamma(\frac{dx'}{dt'} + v)dt' = \gamma(u'_x + v)dt' \tag{4.1}$$

and

$$dt = \gamma\Big(dt' + \frac{vdx'}{c^2}\Big) \tag{4.2}$$

Dividing (4.1) by (4.2), we then have:

$$\frac{dx}{dt} = \frac{\gamma(u'_x + v)dt'}{\gamma\Big(dt' + \frac{vdx'}{c^2}\Big)}$$

$$\Rightarrow u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

Composition of velocities

And starting with $dy = dy'$, we can say that $dy = \frac{dy'}{dt'}dt' = u'_y dt'$, and dividing by 4.2 as before, we get the transformation:

$$u_y = \frac{\frac{u'_y}{\gamma}}{1 + \frac{vu'_x}{c^2}}$$

Composition of velocities

Similarly we have:

$$u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u'_y = \frac{\frac{u_y}{\gamma}}{1 - \frac{vu_x}{c^2}}$$

These are the formulae for the *composition of velocities* in special relativity. Note that it is impossible to measure a velocity to be faster than c using these expressions. A body travelling at $0.75c$ with respect to one frame will have its velocity observed to be $\frac{24c}{25}$ by an observer in a frame moving towards it at $0.75c$, rather than the velocity of $1.5c$ predicted by the Galilean velocity transformation.

4.7 Angle Transformation ('Headlight Effect')

Consider a light source moving with velocity v in frame S. In frame S', which is moving with velocity v away from S, then light source is at rest.

Consider a photon travelling at angle θ' as measured in S'. Clearly the x-component of the velocity of the photon is $u'_x = c\,\cos\theta'$. In S, the x-component is $u_x = c\,\cos\theta$.

From the velocity transformations:

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}}$$

$$\Rightarrow c\,\cos\theta = \frac{c\,\cos\theta' + v}{1 + \frac{vc\,\cos\theta'}{c^2}}$$

$$\Rightarrow \cos\theta = \frac{\cos\theta' + \frac{v}{c}}{1 + \frac{v\cos\theta'}{c}}$$

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

Angle transformation

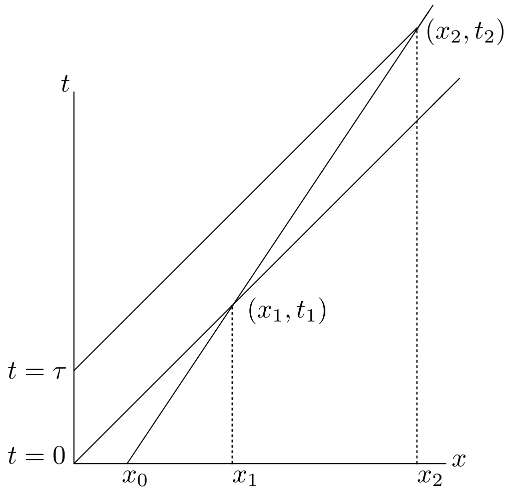
the *angle transformation.*

A consequence of this result is the *headlight effect*: light from a moving source is observed to not be emitted in all directions. Rather, it appears to be emitted in a beam (like from a car headlight).

4.8 Relativistic Doppler Effect

Consider a source of light at rest in a frame S at $x = 0$. It emits a crest of light at time $t = 0$ and a second crest at time $t = \tau =$ period.

Consider now an observer in a frame S' moving with velocity v away from the source of light along the x-axis, and at $x = x_0$ at time $t = t' = 0$. We can construct a space-time diagram to illustrate the situation:



Now,

$$x_1 = ct_1 = x_0 + vt_1$$
$$x_2 = c(t_2 - \tau) = x_0 + vt_2$$

and so

$$x_0 = ct_1 - vt_1 = c(t_2 - \tau) - vt_2$$
$$\Rightarrow t_1(c - v) = t_2(c - v) - c\tau$$
$$\Rightarrow t_2 - t_1 = \frac{c\tau}{c - v}$$

and

$$x_2 - x_1 = v(t_2 - t_1)$$
$$\Rightarrow x_2 - x_1 = \frac{vc\tau}{c - v}$$

In the frame S', using the Lorentz transformation for time $t' = \gamma(t - \frac{vx}{c^2})$, we have that

$$\tau' = t_2' - t_1' = \gamma\left[t_2 - t_1 - \frac{v}{c^2}(x_2 - x_1)\right]$$

$$\Rightarrow \tau' = \gamma\left[\frac{c\tau}{c - v} - \frac{v}{c^2}\frac{vc\tau}{c - v}\right]$$

$$\Rightarrow \tau' = \frac{\gamma c\tau}{c - v}\left[1 - \frac{v^2}{c^2}\right]$$

$$\Rightarrow \tau' = \frac{\gamma\tau}{1 - \frac{v}{c}}\left[1 - \frac{v^2}{c^2}\right]$$

$$\Rightarrow \tau' = \frac{\gamma\tau}{1 - \beta}\left[1 - \beta^2\right] = \gamma\tau(1 + \beta)$$

$$\Rightarrow \tau' = \frac{\tau(1 + \beta)}{(1 - \beta^2)^{\frac{1}{2}}} = \frac{\tau(1 + \beta)}{(1 - \beta)^{\frac{1}{2}}(1 + \beta)^{\frac{1}{2}}}$$

$$\Rightarrow \tau' = \left(\frac{1 + \beta}{1 - \beta}\right)^{\frac{1}{2}}\tau$$

Relativistic Doppler Effect Change in Period the expression for the relativistic Doppler effect change in period. As $T = \frac{1}{f}$ we also have the expression for the change in frequency:

$$f' = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} f$$

Change in Frequency

In the case that S' is moving towards the source, switch all the signs:

$$\tau' = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} \tau$$

$$f' = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} f$$

Relativistic Doppler Effect (Observer Moving Towards)

5 Relativistic Dynamics

5.1 Relativistic Energy And Momentum

The Special Theory of Relativity necessitated that revisions be made to classical dynamics. Traditionally, we have Newton's Second Law, $F = \frac{dP}{dt} = ma$ if mass m is constant. A consequence of this is that a constant force would produce a constant acceleration, leading to the velocity v of the body increasing indefinitely to infinite speeds. However, in reality nothing can travel at speeds greater than c. This suggests that as the body's speed increases, its mass does not remain constant.

Indeed, it is found that:

$$m(v) = \frac{m_0}{\left(1 - \frac{v^2}{c^2}\right)^{\frac{1}{2}}} = m_0\gamma$$

Relativistic Mass

where $m_0 = m(v = 0)$ is the body's *rest mass.*

Now, we can expand γ in a Taylor expansion to get:

$$m = m_0(1 + \frac{1}{2}\frac{v^2}{c^2} + \frac{3}{8}\frac{v^4}{c^4}...)$$

$$\Rightarrow \Delta m = m - m_0 = \frac{1}{2}\frac{v^2}{c^2}m_0 + \frac{3}{8}\frac{v^4}{c^4}m_0...$$

$$\Rightarrow \Delta m = \frac{\frac{1}{2}m_0v^2}{c^2}(1 + \frac{3}{4}\frac{v^2}{c^2}...)$$

$$\Rightarrow \Delta mc^2 = \frac{1}{2}m_0v^2(1 + \frac{3}{4}\frac{v^2}{c^2}...)$$

and this is the kinetic energy of the body.

So:

$$\begin{aligned} K.E. &= \Delta mc^2 = mc^2 - m_0c^2 \\ &= m_0c^2(\gamma - 1) \end{aligned}$$

Relativistic Kinetic Energy

This implies that $\Delta m = \frac{\Delta E}{c^2}$, leading to:

$$E = mc^2$$

Relativistic Energy

where $m = m_0\gamma$.

This is the relativistic expression for energy. It introduces a new concept - that of mass-energy equivalence. Mass can be converted into energy, and vice-versa.

The expression for momentum is just:

$$P = mv = m_0 \gamma v$$

Relativistic Momentum

Using the above two equations, we note that

$$E^2 - P^2c^2 = m_0^2\gamma^2c^4 - m_0^2\gamma^2v^2c^2$$
$$= m_0^2\gamma^2c^4(1 - \frac{v^2}{c^2})$$

But as $\gamma = \frac{1}{(1-\frac{v^2}{c^2})^{\frac{1}{2}}}$, we then have the following useful result:

$$E^2 - P^2c^2 = m_0^2c^4 = \text{constant in all frames}$$

An Energy-Momentum Invariant

For a collection of particles, E = the sum of energies, P = vector sum of momentum, and $E^2 - P^2c^2 = E_0^2$ where E_0 = energy in a frame where the total momentum is zero.

For a photon, $E = hf$, $v = c$ for all observers and the rest mass is zero, so:

$$E^2 - p^2c^2 = 0$$
$$\Rightarrow P = \frac{E}{c} = \frac{hf}{c}$$

Momentum of a Photon

the expression for the momentum of a photon.

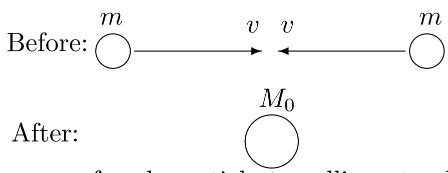
Combined with the laws of conservation of energy (more properly mass-energy) and momentum, the above expressions are used to solve problems of relativistic dynamics.

5.2 Problems Involving Relativistic Dynamics

In this section, we will look at some examples of problems involving relativistic dynamics.

5.2.1 Inelastic collision

Two identical particles of rest mass m_0 travelling in opposite directions, each with velocity v collide head on. The particles stick together after the collision forming a new particle of rest mass M_0, which is at rest (thus conserving momentum).



Let m be the relativistic mass of each particle travelling at velocity v. Then from the conservation of energy we have:

$$2mc^2 = M_0c^2$$
$$\Rightarrow 2m = M_0$$

Now, $m = \gamma m_0 \Rightarrow 2m > 2m_0 \Rightarrow M_0 > 2m_0$. As $mc^2 = m_0c^2+$ K.E., we have that:

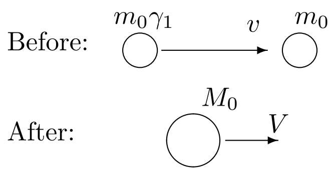
$$2(m_0c^2 + \text{K.E.}) = M_0c^2$$
$$M_0c^2 - 2m_0c^2 = 2\text{K.E.}$$

This example illustrates that though energy and momentum are always conserved, rest mass need not be.

5.2.2 Positron-Electron Collision and Annihilation

A positron is a particle with the same rest mass as an electron but the opposite charge. Here, a positron moving with speed v collides inelastically with an electron at rest, and forms a so-called positronium atom which recoils freely. The positronium atom subsequently annihilates during the course of its motion, forming two γ-ray photons. We wish to work out the speed V of the positronium atom, and the maximum energy a photon so produced may have.

We consider first the collision of the positron with the electron.



From conservation of momentum:

$$m_0\gamma_1 v = M_0\gamma_2 V \tag{5.1}$$

And from conservation of energy,

$$m_0\gamma_1 c^2 + m_0 c^2 = M_0\gamma_2 c^2$$

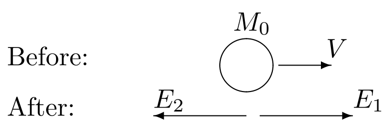
$$\Rightarrow m_0\gamma_1 + m_0 = M_0\gamma_2 \tag{5.2}$$

We then substitute (5.2) in (5.1) and obtain:

$$V = \frac{\gamma_1 v}{\gamma_1 + 1} \tag{5.3}$$

If we are given the positron's energy/kinetic energy rather than its speed, we can work out γ_1 (and hence v) from our expressions for energy.

We now consider the annihilation of the positron into two γ-ray photons. It can be seen (using conservation of momentum and $P = \frac{E}{c}$) that the case where one of the photons has the maximum possible energy is that in which the photons move in opposite directions along the x-axis:



We again use conservation of energy:

$$M_0\gamma_2 c^2 = E_1 + E_2 \tag{5.4}$$

and conservation of momentum:

$$M_0\gamma_2 V = \frac{E_1}{c} - \frac{E_2}{c} \tag{5.5}$$

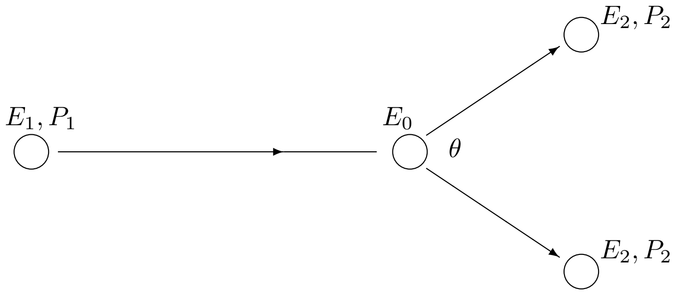
using $P = \frac{E}{c}$ for the momentum of the photons.

We have an expression for $M_0\gamma_2$ in (5.2), and so solving (5.4) and (5.5) for E_1 and E_2 is straightforward.

A version of this problem has occurred frequently on Junior Freshman examination papers.

5.2.3 Collision and Scattering Angle

In this problem we consider the collision between two identical particles, one of which is initially at rest with rest energy E_0. The other is moving with total energy $E_1 = E_0 + K.E.$. After the collision, the particles recoil, with θ being the separation angle between them. We consider this to be the special case of a symmetrical collision and assume both particles have total energy E_2 after the collision.



If the moving particle was travelling along the x-axis before the collision, then by the symmetrical nature of the collision the angle between the velocity of each particle after the collision and the x-axis is $\frac{\theta}{2}$.

From conservation of energy:

$$E_1 + E_0 = 2E_2 \tag{5.6}$$

and from conservation of momentum:

$$\begin{aligned} P_1 &= 2P_2 \cos\frac{\theta}{2} \\ \Rightarrow \frac{P_1}{P_2} &= 2\cos\frac{\theta}{2} \end{aligned} \tag{5.7}$$

and using $E^2 - P^2c^2$,

$$E_1^2 - P_1^2c^2 = E_2^2 - P_2^2c^2 = m_0c^2 = E_0^2 \tag{5.8}$$

Substituting $E_1 = E_0 + K.E.$ into (5.8) we get:

$$P_1^2c^2 = \Big(E_0 + K.E.\Big)^2 - E_0^2 = K.E.\Big(2E_0 + K.E.\Big) \tag{5.9}$$

And from (5.6) we have $E_2 = \frac{E_1}{2} + \frac{E_0}{2}$, which combines with (5.8) to give:

$$P_2^2c^2 = \Big(E_0 + \frac{K.E.}{2}\Big)^2 - E_0^2 = K.E.\Big(E_0 + \frac{K.E.}{4}\Big) \tag{5.10}$$

Dividing (5.9) by (5.10) gives:

$$\frac{P_1^2}{P_2^2} = \frac{2E_0 + K.E.}{E_0 + \frac{K.E.}{4}} \tag{5.11}$$

Using (5.7):

$$4\cos^2\frac{\theta}{2} = \frac{2E_0 + K.E.}{E_0 + \frac{K.E.}{4}} = \frac{8E_0 + 4K.E.}{4E_0 + K.E.}$$

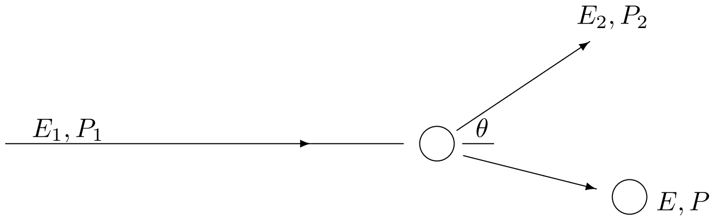
$$\Rightarrow \cos^2\frac{\theta}{2} = \frac{2E_0 + K.E.}{4E_0 + K.E.}$$

Using the trigonometric identity $\cos^2\frac{\theta}{2} = \frac{1}{2}(1+\cos\theta) \Rightarrow \cos\theta = 2\cos^2\frac{\theta}{2} - 1$ we get the final result for the scattering angle:

$$\cos\theta = \frac{4E_0 + 2K.E.}{4E_0 + K.E.} - 1$$
$$\Rightarrow \cos\theta = \frac{K.E.}{4E_0 + K.E.}$$

5.2.4 Compton Effect

The Compton effect is the collision of a photon with a free electron, such that the electron recoils and the photon is scattered with less energy and hence a greater wavelength.



We use the relationship $P = \frac{E}{c}$ to write the momentum of the photon before and after the collision as:

$$P_1 = \frac{E_1}{c}\hat{n}_1$$

and

$$P_2 = \frac{E_2}{c}\hat{n}_2$$

where $\hat{n}_1$ and $\hat{n}_2$ are unit vectors along the direction of travel of the photon.

The conservation laws give:

$$E_1 + m_0c^2 = E_2 + E$$
$$\frac{E_1}{c}\hat{n}_1 = \frac{E_2}{c}\hat{n}_2 + P$$

We solve these for E and P and subsitute into $E^2 - P^2c^2 = m_0^2c^4$, giving:

$$\left[(E_1 - E_2) + m_0c^2\right]^2 - \left[E_1\hat{n}_1 - E_2\hat{n}_2\right]^2 = m_0^2c^4$$
$$\Rightarrow E_1^2 - 2E_1E_2 + E_2^2 + 2(E_1 - E_2)m_0c^2 + m_0^2c^4 - \left[E_1^2 - 2E_1E_2\cos\theta + E_2^2\right] = m_0^2c^4$$

Note the use of the vector dot product when squaring the second bracket: $\hat{n}_1 \cdot \hat{n}_2 = \cos\theta$.

Tidying up the terms,

$$2(E_1 - E_2)m_0c^2 - 2E_1E_2(1-\cos\theta) = 0$$
$$\Rightarrow \frac{1}{E_2} - \frac{1}{E_1} = \frac{1}{m_0c^2}(1-\cos\theta)$$

having divided across by E_1E_2.

We now use the relationships $E = hf$ and $c = f\lambda \Rightarrow E = h\frac{c}{\lambda}$ to get our final result for the Compton effect change of wavelength:

$$\lambda_2 - \lambda_1 = \frac{h}{m_0c}(1-\cos\theta) \tag{5.12}$$

A Appendix: List of Formulae

Galilean Transformations

$$x = x' + vt' \qquad v_x = v_{x'} + v$$

$$y = y' \qquad v_y = v_{y'}$$

$$z = z' \qquad v_z = v_{z'}$$

$$t = t'$$

Lorentz Transformations

$$x = \gamma(x' + vt') \qquad x' = \gamma(x - vt)$$

$$t = \gamma\Big(t' + \frac{vx'}{c^2}\Big) \qquad t' = \gamma\Big(t - \frac{vx}{c^2}\Big)$$

$$y = y',\ z = z'$$

where

$$\gamma = \frac{1}{\Big(1 - \frac{v^2}{c^2}\Big)^{\frac{1}{2}}}$$

or

$$\gamma = \frac{1}{(1 - \beta^2)^{\frac{1}{2}}}$$

$$\beta = \frac{v}{c}$$

Length Contraction

$$l_0 = \gamma\, l$$

Time Dilation

$$\tau = \gamma\tau_0$$

Composition of Velocities

$$u_x = \frac{u'_x + v}{1 + \frac{vu'_x}{c^2}} \qquad u'_x = \frac{u_x - v}{1 - \frac{vu_x}{c^2}}$$

$$u_y = \frac{\frac{u'_y}{\gamma}}{1 + \frac{vu'_x}{c^2}} \qquad u'_y = \frac{\frac{u_y}{\gamma}}{1 - \frac{vu_x}{c^2}}$$

Angle Transformation

$$\cos\theta = \frac{\cos\theta' + \beta}{1 + \beta\cos\theta'}$$

Relativistic Doppler Effect

$$\textit{Observer moving away:}\quad \tau' = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}}\tau, \quad f' = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}} f$$

$$\textit{Observer moving towards:}\quad \tau' = \left(\frac{1-\beta}{1+\beta}\right)^{\frac{1}{2}}\tau, \quad f' = \left(\frac{1+\beta}{1-\beta}\right)^{\frac{1}{2}} f$$

Mass, Energy, Kinetic Energy, Momentum

$$m(v) = m_0\gamma$$

$$E = mc^2$$

$$K.E. = m_0c^2(\gamma - 1)$$

$$P = m_0\gamma v$$

An Energy-Momentum Invariant

$$E^2 - P^2c^2 = m_0^2c^4$$

Momentum of a Photon

$$P = \frac{E}{c} = \frac{hf}{c}$$

Compton Effect Change Of Wavelength

$$\lambda_2 - \lambda_1 = \frac{h}{m_0c}(1 - \cos\theta)$$