Character table of S_5

These notes describe how to find the character table of S_5, based on the methods described in the 2008-09 Group Representations course by Dr Timothy Murphy. We work both directly using eigenvalues of group elements and using induction from S_4 - there are obviously other possibilities not covered here.

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1 General information

1.1 Representations and characters

A **group representation** of a group G in a vector space V over a field k (which we will take to be $\mathbb{C}$) is given by a homomorphism from G to $GL(V)$. In general we think of $GL(V)$ as being composed of $n \times n$ invertible matrices, and think of our representation as a matrix representation. The degree of the representation is the dimension of the vector space. A subspace U of V is said to be stable if $gu \in U$ for all $u \in U, g \in G$. A representation is said to be **simple** if the only stable subspaces are V and 0. The sum of the dimensions of the simple representations over $\mathbb{C}$ of a group is equal to the order of G. A representation is said to be **semisimple** if it is expressible as a sum of simple representations.

The **character** of a representation $\alpha : g \to GL(V)$ is defined to be $\chi_\alpha(g) = \operatorname{tr} \alpha(g)$. Most of the properties of a representation are determined by its character. Recall that the **conjugacy class** of $g \in G$ is $\{h \in G : h = xgx^{-1}, x \in G\}$. Characters are constant on conjugacy classes. We also have that the number of simple representations over $\mathbb{C}$ is equal to the number of conjugacy classes. The character of the identity element equals the degree of the representation (as the identity element is represented by the identity matrix in the n-dimensional space V). The character satisfies $\chi_{\alpha+\beta}(g) = \chi_\alpha(g) + \chi_\beta(g)$, $\chi_{\alpha\beta}(g) = \chi_\alpha(g)\chi_\beta(g)$, $\chi_{\alpha^*}(g) = \chi_\alpha(g^1)$ and if $k = \mathbb{C}$, $\chi_{\alpha^*}(g) = \chi_\alpha(g^{-1}) = \overline{\chi_\alpha(g)}$.

The **intertwining number** $I(\alpha, \beta)$ of two representations in vector spaces U and V is defined to be the degree of the space of maps from U to V which leave the action of G invariant. If α, β are simple representations over $\mathbb{C}$ then $I(\alpha, \beta) = 1$ if $\alpha = \beta$ and is zero otherwise. If α is a semisimple representation and σ is a simple representation then $I(\alpha, \sigma)$ gives the number of times σ occurs in α. Note that **Maschke's theorem** states that every representation of a finite group over $\mathbb{C}$ (or $\mathbb{R}$) is semisimple. A formula for the intertwining number in terms of characters is

$$I(\alpha, \beta) = \frac{1}{||G||} \sum_{g \in G} \chi_\alpha(g^{-1})\chi_\beta(g)$$

where $||G||$ is the order of the group G. If $k = \mathbb{C}$ then we can replace $\chi_\alpha(g^{-1})$ by $\overline{\chi_\alpha(g)}$. The same formula can be given in terms of the classes $[g]$ in G as

$$I(\alpha, \beta) = \frac{1}{||G||} \sum_{[g] \in G} ||[g]|| \, \overline{\chi_\alpha([g])}\chi_\beta([g])$$

where $||[g]||$ gives the number of elements in the class $[g]$. The **character table** of a group gives the values of the characters of the simple representations on the classes in the groups.

1.2 Permutation groups

The symmetric group S_n consists of all permutations of n elements. It has order $||S_n|| = n!$. An element of the group can be written in the cycle notation $(abc \dots n)$ which means a goes to b, b goes to c and so on, with n going to 1. A permutation can be written as the union of disjoint cycles, eg in S^4 $(ab)(cd)$ which swaps a and b and swaps c and d. Two elements of S_n are conjugate if and only if they have are expressible as the same number of disjoint cycles of the same cycle lengths. S_n is also generated by the transpositions (ab).

The natural representation θ of S_n in k^n by permutation of coordinates splits into two simple parts, $\theta = 1 + \sigma$.

2 Finding the character table

2.1 Classes

The order of S_5 is $5! = 120$. The classes are $1^5, 21^3, 2^21, 31^2, 32, 41, 5$ (the notation means 21^3 is a two-cycle (ab), 32 is a three-cycle and a two-cycle $(abc)(de)$ etc). There is one element in the class 1^5 (the identity), $5{\cdot}4/2$ in the second (numbers of ways of choosing two elements with order not important), $(5{\cdot}4/2){\cdot}(3{\cdot}2/2)/2$ in the third (numbers of ways of choosing two elements and then two other elements with order not important), $5 \cdot 4 \cdot 3/3$ in the fourth (ways of choosing three elements with order not important), $(5 \cdot 4 \cdot 3/3) \cdot (2/2)$ in the fourth, $5 \cdot 4 \cdot 3 \cdot 2/4$ in the fifth and $5!/5$ in the last. Hence the top rows of our character table will look like

# elts	1	10	15	20	20	30	24
class	1^5	21^3	2^21	31^2	32	41	5

2.2 One dimensional representations

We know S_5 has two one-dimensional representations, the trivial representation 1 and the parity representation ε, which takes values $+1$ on even representations (odd cycle length) and -1 on odd representations (even cycle length). So we can fill in:

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1

2.3 Natural representation

The natural representation θ of S^5 on $\{x_1, x_2, x_3, x_4, x_5\}$ splits into two simple parts, 1 and σ. This natural representation is a permutation representation, and so its character is given by the number of elements it leaves invariant (consider the diagonal of a permutation matrix). So:

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
θ	5	3	1	2	0	1	0
$\sigma = \theta - 1$	4	2	0	1	-1	0	-1

Note that to check σ is simple, we calculate

$$I(\sigma,\sigma) = \frac{1}{120}\Big(1 \cdot 4^2 + 10 \cdot 2^2 + 15 \cdot 0^2 + 20 \cdot (-1)^2 + 20 \cdot (-1)^2 + 30 \cdot 0^2 + 24 \cdot (-1)^2\Big) = \frac{1}{120}(120) = 1$$

We can immediately find another simple representation, the product of ε and σ. For if σ is a simple representation of S_n, $I(\varepsilon\sigma, \varepsilon\sigma) = I(\sigma, \varepsilon^*\varepsilon\sigma) = I(\sigma, \varepsilon^2\sigma) = I(\sigma,\sigma) = 1$, and so $\varepsilon\sigma$ is a simple representation. To check that $\varepsilon\sigma$ is distinct from σ we note that as $\chi_\sigma((ab)) \neq 0$ and $\chi_\varepsilon((ab)) = -1$ then clearly $\chi_{\varepsilon\sigma}((ab)) \neq \chi_\sigma((ab))$, so $\varepsilon\sigma \neq \sigma$. This holds for any simple representation α of S_n such that $\chi_\alpha((ab)) \neq 0$, and thus for all representations of odd degree. We can see this by considering the eigenvalues of (ab), which as $(ab)^2 = 1$ are ± 1. This means (ab) is represented in matrix form by a matrix with ± 1 on the diagonal; if σ has odd degree then there are odd number of ± 1 terms, and we cannot add these to get zero, hence $\chi_\sigma((ab)) \neq 0$. Hence,

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
σ	4	2	0	1	-1	0	-1
$\varepsilon\sigma$	4	-2	0	1	1	0	-1

2.4 Dimensional analysis mod 8

We now use a trick to find the dimensions of the remaining representations. We know that there are $7-4=3$ simple representations left; let them have dimensions a, b, c. Then we must have $1^2+1^2+4^2+4^2+a^2+b^2+c^2 = 120 \Rightarrow a^2+b^2+c^2 = 86$, or $a^2+b^2+c^2 = 6 \bmod 8$. Quickly checking the values of $n^2 \bmod 8$ for $n = 1, \ldots, 10$ shows that $n^2 \bmod 8 = 0$ if n is a multiple of 4, 1 if n is odd, and 4 if $n \bmod 4 = 2$. Seeing as the only way we can make 6 out of three of $0, 1, 4$ is $1+1+4$ we see that a and b are odd and as odd representations come in pairs we must have $a = b$. Putting $2a^2 + c^2 = 86$ and trying $a = 3, 5, 7$ we find that $a = 5$ and $c = 6$. So,

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
σ	4	2	0	1	-1	0	-1
$\varepsilon\sigma$	4	-2	0	1	1	0	-1
φ	5						
$\varepsilon\varphi$	5						
ψ	6						

2.5 Eigenvalues

To proceed let us look at the 5-dimensional representation φ. Consider a 5-cycle $(abcde)$. As $(abcde)^5 = 1$, the eigenvalues will be chosen from $1, \omega, \omega^2, \omega^3, \omega^4$ (where $\omega = e^{2\pi i/5}$). Now $(abcde)^r$ ($r = 2, 3, 4$) is also a 5-cycle, with eigenvalues consisting of the eigenvalues of $(abcde)$ raised to the power of r. As conjugate cycles have the same eigenvalues, we see that if λ is an eigenvalue of $(abcde)$ then so are the powers of λ. This gives two possible choices for the eigenvalues, $\{1, 1, 1, 1, 1\}$ or $\{1, \omega, \omega^2, \omega^3, \omega^4\}$ (as there must be a total of five, ignoring repetition). Now the former case gives $\chi_\varphi((abcde)) = 5$, while the latter gives $\chi_\varphi((abcde)) = 0$. But in the formula for $I(\varphi, \varphi)$ this character will contribute $24 \cdot \chi_\varphi((abcde))^2$, which must be less than or equal to 120. This rules out the possibility of 1 being an eigenvalue with multiplicity 5.

Similarly consider a 4-cycle $(abcd)$ with eigenvalues $\pm 1, \pm i$. As $(abcd)^3$ is also a 4-cycle, if i occurs as an eigenvalue then so does $i^3 = -i$. Hence $\pm i$ occur together (or not at all) and will cancel out when we take the trace, leaving an odd number of ± 1 pairs. The two possibilities for $\chi_\varphi((abcd))$ are $\pm 3, \pm 1$ - we again see that the former contributes too much to $I(\varphi, \varphi)$, hence $\chi_\varphi((abcd)) = \pm 1$. We choose the plus option - the negative possibility then corresponds to $\varepsilon\varphi$.

Finally we can do the same for the 3-cycles (abc), which has eigenvalues $1, \omega, \omega^2$ (where $\omega = e^{2\pi i/3}$). As $(abc)^2$ is also a 3-cycle if an eigenvalue occurs then so does its square, meaning here that ω and ω^2 occur together. If they don't occur, we have $\chi_\varphi((abc)) = 5$, which is too large, if they occur once $\chi_\varphi((abc)) = 1+1+\omega+\omega^2 = 2$ which is too large (taking into account the contributions from the 4- and 5-cycles). Hence they must occur twice and $\chi_\varphi((abc)) = 1+\omega+\omega^2+\omega+\omega^2 = -1$. So we have

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
σ	4	2	0	1	-1	0	-1
$\varepsilon\sigma$	4	-2	0	1	1	0	-1
φ	5	x	y	-1	z	1	0
$\varepsilon\varphi$	5	- x	y	-1	$-z$	-1	0
ψ	6						

2.6 Orthogonality

To find x, y, z we use the orthogonality of the simple characters:

$$I(\varphi, 1) = 0 \Rightarrow 5 + 10x + 15y - 20 + 20z + 30 = 0 \Rightarrow 10x + 15y + 20z = -15$$

$$I(\varepsilon\varphi, 1) = 0 \Rightarrow 5 - 10x + 15y - 20 - 20z - 30 = 0 \Rightarrow 10x - 15y + 20z = -45$$

This gives $30y = 30 \Rightarrow y = 1$. We also have

$$I(\varphi, \sigma) = 0 \Rightarrow 20 + 20x - 20 - 20z = 0 \Rightarrow x = z$$

and filling this back in we find $x = -1$, giving

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
σ	4	2	0	1	-1	0	-1
$\varepsilon\sigma$	4	-2	0	1	1	0	-1
φ	5	-1	1	-1	-1	1	0
$\varepsilon\varphi$	5	1	1	-1	1	-1	0
ψ	6						

2.7 Regular representation

The regular representation of a group G is the representation induced by the action of the group on itself. Its character equals the order of the group for $g = e$ and is zero otherwise. Every simple representation σ_i occurs $\dim \sigma_i$ times in the regular representation, giving us a straightforward way of finding the representation ψ:

$$1 \cdot 1 + 1 \cdot (-1) + 4 \cdot 2 + 4 \cdot (-2) + 5 \cdot (-1) + 5 \cdot (1) + 6\chi_\psi((ab)) = 0 \Rightarrow \chi_\psi((ab)) = 0$$

$$1 \cdot 1 + 1 \cdot 1 + 4 \cdot 0 + 4 \cdot 0 + 5 \cdot 1 + 5 \cdot 1 + 6\chi_\psi((ab(cd))) = 0 \Rightarrow \chi_\psi((ab)(cd)) = -2$$

$$1 \cdot 1 + 1 \cdot 1 + 4 \cdot 1 + 4 \cdot 1 + 5 \cdot (-1) + 5 \cdot (-1) + 6\chi_\psi((abc)) = 0 \Rightarrow \chi_\psi((abc)) = 0$$

$$1 \cdot 1 + 1 \cdot (-1) + 4 \cdot (-1) + 4 \cdot 1 + 5 \cdot (-1) + 5 \cdot 1 + 6\chi_\psi((abc)(de)) = 0 \Rightarrow \chi_\psi((abc)(de)) = 0$$

$$1 \cdot 1 + 1 \cdot (-1) + 4 \cdot 0 + 4 \cdot 0 + 5 \cdot 1 + 5 \cdot (-1) + 6\chi_\psi((abcd)) = 0 \Rightarrow \chi_\psi((abcd)) = 0$$

$$1 \cdot 1 + 1 \cdot 1 + 4 \cdot (-1) + 4 \cdot (-1) + 5 \cdot 0 + 5 \cdot 0 + 6\chi_\psi((abcde)) = 0 \Rightarrow \chi_\psi((abcde)) = 1$$

and so

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	$2^2 1$	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
σ	4	2	0	1	-1	0	-1
$\varepsilon\sigma$	4	-2	0	1	1	0	-1
φ	5	-1	1	-1	-1	1	0
$\varepsilon\varphi$	5	1	1	-1	1	-1	0
ψ	6	0	-2	0	0	0	1

3 Alternative method - induced representations

3.1 Induced representations

Given H a subgroup of G and α a representation of H in U, then we can form the induced representation α^G of G. In terms of characters, we have

$$\chi_{\alpha^G}([g]) = \frac{||G||}{||H||\,||[g]||} \sum_{[h]\subset[g]} ||[h]||\,\chi_\alpha([h])$$

where $[g]$ are the classes in G and $[h]$ are the classes in H. Note that every class in h lies in a unique class in G, however the classes of G may be split in H. This does not happen for $S_{n-1} \subset S_n$, however.

3.2 Inducing from S_4 up to S_5

Suppose now we know the character table of S_4:

# elts	1	6	3	8	6
class:	1^4	21^2	2^2	31	4
1	1	1	1	1	1
ε	1	-1	1	1	-1
α	2	0	2	-1	0
β	3	1	-1	0	-1
$\varepsilon\beta$	3	-1	-1	0	1

The induced representations are given by

$$\chi_{\alpha^G}([g]) = \frac{5\,||[h]||}{||[g]||}\,\chi_\alpha([h])$$

giving for instance

$$\chi_{\alpha^G}(e) = \frac{5\cdot 1}{1}\cdot 2 = 10 \quad \chi_{\alpha^G}((ab)) = \frac{5\cdot 6}{10}\cdot 0 = 10 \quad \chi_{\alpha^G}((ab)(cd)) = \frac{5\cdot 3}{15}\cdot 2 = 2$$

$$\chi_{\alpha^G}((abc)) = \frac{5\cdot 8}{20}\cdot(-1) = -2 \qquad \chi_{\alpha^G}((abcd)) = \frac{5\cdot 6}{30}\cdot 0 = 0$$

and $\chi_{\alpha^G}((abc)(de)) = \chi_{\alpha^G}((abcde)) = 0$ as these classes do not occur in S_4. Hence, assuming we have found the first four representations of S_5 as before, we have

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
σ	4	2	0	1	-1	0	-1
$\varepsilon\sigma$	4	-2	0	1	1	0	-1
α^{S_5}	10	0	2	-2	0	0	0
β^{S_5}	15	3	-1	0	0	-1	0

We can also apply the mod trick to discover that the remaining simple representations have dimensions 5,5 and 6. Now, we find that

$$I(\alpha^{S_5}, \alpha^{S_5}) = 2 \qquad I(\beta^{S_5}, \beta^{S_5}) = 3$$

so α^{S_5} is composed of two simple parts and β^{S_5} is composed of three simple parts. For α^{S_5} this means that the dimensions of its components are either 4 and 6, or 5 and 5. But we can check that $I(\alpha^{S_5}, \sigma) = I(\alpha^{S_5}, \varepsilon\sigma) = 0$, hence $\alpha^{S_5} = \varphi + \varepsilon\varphi$ (using the fact that representations of odd degrees come in pairs - we can also check that $\alpha^{S_5} \neq 2\varphi$ by computing $I(\frac{1}{2}\alpha^{S_5}, \frac{1}{2}\alpha^{S_5}) = \frac{1}{2}$).

Now we can compute $I(\sigma, \beta^{S_5}) = 1$ and $I(\sigma, \beta^{S_5}) = 0$, meaning that we must have

$$\beta^{S_5} = \sigma + \begin{cases} \varphi \\ \varepsilon\varphi \end{cases} + \psi \Rightarrow \varepsilon\beta^{S_5} = \varepsilon\sigma + \begin{cases} \varepsilon\varphi \\ \varphi \end{cases} + \varepsilon\psi$$

Combining these give

$$\beta^{S_5} + \varepsilon\beta^{S_5} = \sigma + \varepsilon\sigma + \varphi + \varepsilon\varphi + \psi + \varepsilon\psi$$

But as ψ is of even degree, and as we have all the other simple representations, we must have $\psi = \varepsilon\psi$; and we also know that $\alpha^{S_5} = \varphi + \varepsilon\varphi$. This means we can find ψ using $\psi = \frac{1}{2}(\beta^{S_5} + \varepsilon\beta^{S_5} - \sigma - \varepsilon\sigma - \alpha^{S_5})$. This gives

# elts	1	10	15	20	20	30	24
class:	1^5	21^3	2^21	31^2	32	41	5
1	1	1	1	1	1	1	1
ε	1	-1	1	1	-1	-1	1
σ	4	2	0	1	-1	0	-1
$\varepsilon\sigma$	4	-2	0	1	1	0	-1
φ	5						
$\varepsilon\varphi$	5						
ψ	6	0	-2	0	0	0	1
α^{S_5}	10	0	2	-2	0	0	0
β^{S_5}	15	3	-1	0	0	-1	0
$\varepsilon\beta^{S_5}$	15	-3	-1	0	0	1	0

We can then finish off the character table using $\varepsilon\varphi = \beta^{S_5} - \sigma - \psi$, and the result is as before. (Note we have the freedom to call this representation $\varepsilon\varphi$ or φ as we wish, and choose the former to be consistent with our earlier notation.)