Two soliton bundles

The purpose of these notes is to define principal bundles and connections in principal bundles, and use the definitions to construct the monopole and instanton bundles. We follow the book by Nakahara and notes from course 448: K-theory and Solitons given by Dr. Sergey Cherkis. Corrections/suggestions to cblair[at]maths.tcd.ie.

Chris Blair, May 2009

1 Fibre bundles

A (differentiable) **fibre bundle** consists of a **total space** E, **base space** M and **fibre** F (with E, M and F being differentiable manifolds), and a surjection $\pi : E \to M$ called the **projection** such that $\pi^{-1}(p) = F$ for all $p \in M$, such that given an open covering $\{U_i\}$ of M there is a diffeomorphism $\phi_i : U_i \times F \to \pi^{-1}(U_i)$ such that $\pi \circ \phi_i(p, f) = p$. The map ϕ_i is called a **local trivialisation**.

The intuitive idea is that we have a space E which in local neighbourhoods U_i looks like (i.e. is diffeomorphic to) a simple product of two spaces, $U_i \times F$; we think of the total space as being formed by attaching the fibre F to every point on M. The projection π then tells us where on M we are (but ignores any information about the fibre). The local trivialisations are the maps that make the fibre bundle "look like" the product $U_i \times F$.

Given two open sets U_i and U_j with local trivialisations ϕ_i, ϕ_j, then on $U_i \cap U_j$ we relate the two trivialisations as follows: suppose we are in the U_j trivialisation, so that our fibre bundle has the (local) form $U_j \times F$. We use ϕ_j to map from $U_j \times F$ to $\pi^{-1}(U_i \cap U_j)$ and then use ϕ_i^{-1} to map to $U_i \times F$. The composition

$$t_{ij}(p) = \phi_i^{-1} \circ \phi_j$$

is called a **transition function**, and if we have $u \in E$ such that $\pi(u) = p$ (i.e. u is a point in the fibre bundle with base space coordinate p) then the local expressions of u in the ϕ_i and ϕ_j trivialisations are

$$\phi_i^{-1}(u) = (p, f_i) \quad \phi_j^{-1}(u) = (p, f_j)$$

with

$$f_i = t_{ij}(p) f_j$$

Transition functions can be thought of as mappings from $U_i \cap U_j$ into a particular set of transformations of the fibre, called the **structure group** of the fibre bundle.

Properties of transition functions: We have that transition functions satisfy $t_{ii}(p) =$ identity, $t_{ij}(p) = t_{ji}^{-1}(p)$ and $t_{ij}(p) \circ t_{jk}(p) = t_{ik}(p)$. Given two local trivialisations $\{\phi_i\}$ and $\{\tilde{\phi}_i\}$ with transition functions $t_{ij} = \phi_i^{-1} \circ \phi_j$ and $\tilde{t}_{ij} = \tilde{\phi}_i^{-1} \circ \tilde{\phi}_j$ define the function relating the two trivialisations to be $g_i = \phi_i^{-1} \circ \tilde{\phi}_i$, then we can write

$$\tilde{t}_{ij} = \tilde{\phi}_i^{-1} \circ \phi_i \circ \phi_i^{-1} \circ \phi_j \circ \phi_j^{-1} \circ \tilde{\phi}_j$$

so the transition functions transform as

$$\tilde{t}_{ij} = g_i^{-1} t_{ij} g_j$$

Indeed, two sets of transitions functions for a fibre bundle are equivalent if there exists a set of maps g_i relating them in this way.

A **section** of a fibre bundle is a smooth map $s : M \to E$ such that $\pi \circ s = \mathrm{id}_M$.

2 Principal bundles

A **principal bundle** is a fibre bundle such that the fibre F equals the structure group G. We denote this by $P(M, G)$, and sometimes call this a principal G-bundle over M.

Transition functions act on F on the left. We also have a **right action** of G on F (i.e. on itself). Given a local trivialisation $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ with $\phi_i^{-1}(u) = (p, g_i)$ we define the right action on $\pi^{-1}(U_i)$ by

$$\phi_i^{-1}(uh) = (p, g_i h) \qquad h \in G$$

or

$$uh = \phi_i(p, g_i h)$$

All that is happening here is that we have some local trivialisation where we map a point u in the principal bundle to the pair (p, g_i) with $p \in M$ and $g_i \in G$, and then define a right action of the group G on itself via $g_i \mapsto g_i h$.

The right action commutes with the left action, and is transitive and free (in other words given two points u_1, u_2 in $\pi^{-1}(p)$ we can find a group element h such that $u_1 = u_2 h$, and we have $uh = u \Rightarrow h = e$).

> **Canonical local trivialisation:** There is a nice way to define a local trivialisation on $P(M, G)$ using a section $s_i(p)$ defined on some open set U_i. Given some section $s_i(p)$ then clearly there exists a unique $g_u \in G$ such that $u = s_i(p)g_u$ for $u \in \pi^{-1}(p)$. Then we define a trivialisation $\phi_i : U_i \times G \to \pi^{-1}(U_i)$ by specifying how it takes an element of $\pi^{-1}(p)$ to $U_i \times G$, namely
>
> $$\phi_i^{-1}(u) = (p, g_u) \qquad u \in \pi^{-1}(p)$$
>
> So the trivialisation chooses one point $s_i(p)$ in each fibre over U_i and maps other points in the fibre to the group element that transforms the first point $s_i(p)$ into the second by the right action. In this trivialisation $s_i(p) = \phi_i(p, e)$ and by definition $\phi_i(p, g) = \phi_i(p, e)g = s_i(p)g$. Two sections s_i and s_j on the intersection of U_i and U_j are related by $s_i(p) = s_j(p)t_{ji}(p)$.

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We need to recall some important facts about pushforwards and pullbacks. Let us start by recalling the notion of a **tangent vector** to a manifold M. A tangent vector v is a linear map from smooth functions on M to $\mathbb{R}$, such that v satisfies the Leibniz property, $v(fg) = v(f)g + fv(g)$. Given some curve $\gamma(t)$ through M then a tangent vector v to the curve is defined by $vf = \frac{d}{dt} f(\gamma(t))$ for an arbitrary smooth function f. The set of all tangent vectors at a point $x \in M$ is called the **tangent space** to M at x, $T_x M$, and the set of all tangent spaces to M is called the **tangent bundle**, TM.

A **differential one-form** is a mapping which takes tangent vectors to $\mathbb{R}$. The space of one-forms at $x \in M$ can be thought of as dual to the tangent space $T_x M$, and is notated $T_x^* M$. Given some function f then we define a one-form df called the differential of f, defined by

$$df(v) = vf$$

i.e. the action of df on a tangent vector v is given by the action of v on f.

Given some differentiable map $\phi : M \to N$ between manifolds (or from a manifold to itself) then we have the following important features: given some map $f : N \to \mathbb{R}$ then we can define the **pullback** of f under ϕ by

$$\phi^* f(x) = f(\phi(x))$$

which says that the action of the pullback of f on $x \in M$ is given by the action of f on $\phi(x) \in N$. The map ϕ also induces the **pushforward** ϕ_* which maps the tangent space $T_x M$ at $x \in M$ to the tangent space $T_{\phi(x)} N$ at $\phi(x) \in N$. It is defined by

$$\phi_* v(f) = v(\phi^* f)$$

for arbitrary smooth $f : N \to \mathbb{R}$, i.e. the action of $\phi_* v$ on a function f in N is given by the action of v on the pullback of that function, so by the action of v on $f(\phi(x))$. If v is the tangent vector of some path $\gamma(t)$, then

$$\phi_* v(f) = v(\phi^* f) = v(f \circ \phi) \Rightarrow \phi_* v(f) = \frac{d}{dt} f(\phi(\gamma(t)))$$

We can also pullback one-forms: if ω a one-form on N then $\phi^*\omega$ is a one-form on M defined by

$$\phi^*\omega|_x(v) = \omega|_{\phi(x)}\phi_* v$$

where $v \in T_xM$, i.e. the action of the pullback $\phi^*\omega$ on a tangent vector at x is given by the action of ω on the pushforward of that tangent vector at $\phi(x)$.

We can extend the idea of one-forms to produce multilinear functions acting on two and more tangent vectors. An r-form η is an alternating (i.e. skew-symmetric) real-valued form which maps r tangent vectors to $\mathbb{R}$. We denote by $\Omega^r(M)$ the space of all r forms on the manifold M, where we view functions $f : M \to \mathbb{R}$ as zero forms.

We also have the exterior derivative $d : \Omega^r(M) \to \Omega^{r+1}(M)$ which takes an r-form to an $(r+1)$-form. For example, $df = \sum_i \frac{\partial f}{\partial x^i} dx^i$ is the one-form obtained by applying d to a function f.

Careful referral to these definitions should guide you through the next section of these notes.

3 Connections

We now want to develop a method whereby we can distinguish (loosely speaking) motion in the fibre (vertical) direction from motion in the base space (horizontal) direction. We do this by studying the tangent space TP to the principal bundle $P(M, G)$.

Let us first recall some facts about Lie groups.

> **Properties of Lie groups and Lie algebras:** Let G be a Lie group (i.e. a differentiable manifold with a group structure). We define the left and right actions by $L_g h = gh$ and $R_g h = hg$ for $g, h \in G$. We have an induced map of tangent spaces, $L_{g*} : T_h(G) \to T_g h(G)$, and have the notion of left invariant vector fields X which satisfy $L_{g*}X|_h = X|_{gh}$. The Lie algebra $\mathfrak{g}$ of G is defined to be the tangent space of G at the identity, and consists of left-invariant vector fields (as left-invariant vector fields are specified by their value at the identity). We have the adjoint action $\mathrm{ad} : G \to G$ defined by $\mathrm{ad}_g h = ghg^{-1}$, which induces the adjoint map $Ad : T_h(G) \to T_{ghg^{-1}}(G)$. The latter maps $\mathfrak{g}$ onto itself.

Given a principal bundle $P(M, G)$ with $u \in P(M, G)$ and fibre G_p at $\pi(u) = p$, then the **vertical subspace** V_uP of the tangent space T_pP is defined to be the subspace of T_pP which is tangent to G_p at u. More precisely, we define V_uP to be $\ker \pi_*|_u$ where $\pi_* : T_uP \to T_{\pi(u)}M$ is the differential or push-forward of the projection π. Now $\pi_*(v)(f(u)) = v(\pi^* f(u)) = v(f(\pi(u))) = v(f(p))$ so we see that the vertical space consists of tangent vectors which vanish when applied to functions evaluated on the base. We define the **horizontal subspace** H_uP of the tangent space T_pP to be a complement of of V_uP in T_uP.

To construct the vertical space, take $A \in \mathfrak{g}$. We can exponentiate A to get a element in G, and use the right action $R_{\exp(tA)}u = u\exp(tA)$ to generate a curve through u in P. As $\exp(tA) \in G$, $\pi(u) = \pi(u\exp(tA)) = p$, so this curve lies within G_p. We then define a vector in the tangent space to P at u by

$$A^\# f(u) = \frac{d}{dt} f(u\exp(tA))|_{t=0}$$

for $f : P \to \mathbb{R}$ an arbitrary smooth function. We call $A^\#$ the **fundamental vector field** generated by A. The vector $A^\#$ is in fact tangent to G_p at u, and so $A^\# \in V_uP$. This gives us a map $\# : \mathfrak{g} \to V_uP$, which is a vector space isomorphism.

For example, consider the group $U(1)$ of unit complex numbers, $U(1) = \{e^{i\varphi} : 0 \le \varphi \le 2\pi\}$. An element of its Lie algebra $\mathfrak{u}(1)$ is a purely imaginary complex number, $\mathfrak{u}(1) = \{iA : A \in \mathbb{R}\}$. Then the fundamental vector field generated by iA is given by

$$(iA)^\# = A\frac{\partial}{\partial\varphi}$$

as then

$$A\frac{\partial}{\partial\varphi} f(e^{i\varphi}) = iAf' e^{i\varphi}$$

and

$$(iA)^{\#}f(e^{i\varphi}) = \frac{d}{dt}f(e^{i\varphi}\exp(itA))|_{t=0} = f'e^{i\varphi}\frac{d}{dt}\exp(itA))|_{t=0} = iAf'e^{i\varphi}$$

There remains the question of how precisely do we separate the tangent space into its vertical and horizontal components. The answer comes in the form of the following abstract definition and the somewhat more practical one that follows it:

A **connection** on a principal bundle P is a unique separation of the tangent space T_uP into vertical and horizontal components V_uP and H_uP such that
i) $T_uP = H_uP \oplus V_uP$
ii) a smooth vector field X on P is separated into smooth vector fields $X^H \in H_uP$ and $X^V \in V_uP$, with $X = X^H + X^V$.
iii) $H_{ug}P = R_{g*}H_uP$ for $u \in P$ and $g \in G$, i.e. horizontal subspaces H_uP and $H_{ug}P$ on the same fibre are related by a linear map induced by the right action.

A **connection one-form** is a Lie algebra valued 1-form, $\omega \in \mathfrak{g} \otimes T^*P$ which is a projection of T_uP onto the vertical component $V_uP \cong \mathfrak{g}$, satisfying
i) $\omega(A^{\#}) = A$
ii) $R_g^*\omega = \mathrm{Ad}_{g^{-1}}\omega$, i.e. for $X \in T_uP$, $R_g^*\omega_u(X) = \omega_{ug}(R_{g*}X) = g^{-1}\omega_u(X)g$.

We then define $H_uP = \ker\omega$.

The second condition in the definition of the connection one-form may seem rather mysterious, but it ensures that $H_{ug}P = R_{g*}H_uP$. For let $H_uP = \{X \in T_uP : \omega(X)\} = 0$, and take $X \in H_uP$ so that $R_{g*}X \in T_{ug}P$, then

$$\omega(R_{g*}(X)) = R_g^*\omega(X) = g^{-1}\omega(X)g = 0$$

hence $R_{g*}X \in H_{ug}$, and as R_{g*} is an invertible linear map any vector in $H_{ug}P$ is expressible in the form $R_{g*}X$ for some X in H_uP.

Locally, given an open covering $\{U_i\}$ of M and σ_i a local section defined on U_i, we can pull back ω to give a Lie algebra valued one-form $\mathcal{A}_i$ on U_i:

$$\mathcal{A}_i \equiv \sigma_i^*\omega \in \mathfrak{g} \otimes \Omega^1(U_i)$$

In fact the converse is true: given a Lie algebra valued one-form $\mathcal{A}_i$ on U_i and a section $\sigma_i : U_i \to \pi^{-1}(U_i)$, then there exists a connection one-form ω with $\mathcal{A}_i = \sigma_i^*\omega$. To construct this one-form, we define

$$\omega_i = g_i^{-1}\pi^*\mathcal{A}_i g_i + g_i^{-1}dg_i$$

where d is the exterior derivative on P and g_i is the canonical local trivialisation formed using σ_i (see section 1), i.e. we have $\phi^{-1}(u) = (p, g_i)$ for $u = \sigma_i(p)g_i$. Hence g_i can be thought of as the function on G that gives the group element $g_i(u)$ such that $u = \sigma_i(p)g_i(u)$.

Proof that $\sigma_i^*\omega_i = \mathcal{A}_i$: let $X \in T_pM$. Then $\sigma_{i*}X \in T_{\sigma_i(p)}P$, and if we note that $g_i = e$ at $\sigma_i(p)$ by the properties of the canonical local trivialisation we find

$$\begin{aligned}\sigma_i^*\omega_i|_p(X) = \omega_i|_{\sigma_i(p)}(\sigma_{i*}X) &= \pi^*\mathcal{A}_i(\sigma_{i*}X) + dg_i(\sigma_{i*}X)\\ &= \mathcal{A}_i(\pi_*\sigma_{i*}X) + dg_i(\sigma_{i*}X)\end{aligned}$$

but σ_i as a section satisfies $\pi \circ \sigma_i = \mathrm{id}$, hence $\pi_* \circ \sigma_{i*} = \mathrm{id}_*$; also as $g_i = e$ along $\sigma_{i*}X$ we have $dg_i(\sigma_{i*}X) = 0$, so we find $\sigma_i^*\omega_i(X) = \mathcal{A}_i(X)$ as wanted.

Proof that ω_i satisfies properties of a connection one-form: first we show that $\omega_i(A^{\#}) = A$. First note that if $A^{\#} \in V_uP$ then $\pi_*A = 0$ by definition. Hence we just have

$$\omega_i(A^{\#})(u) = (g_i^{-1}dg_i)(A^{\#})(u) = g_i^{-1}(u)A^{\#}(g_i(u)) = g_i^{-1}(u)\frac{d}{dt}g_i(u\exp(tA))\Big|_{t=0} = g_i^{-1}(u)g_i(u)\frac{d}{dt}\exp(tA)\Big|_{t=0} = A$$

Secondly, we want to show $R_h^*\omega = \mathrm{Ad}_{h^{-1}}\omega$ for $h \in G$. So let $X \in T_uP$, then we have

$$\begin{aligned}
R_h^*\omega_i|_u(X) = \omega_i|_{uh}(R_{h*}X) &= g_i^{-1}(uh)\mathcal{A}_i(\pi_* R_{h*}X)g_i(uh) + g_i^{-1}(uh)dg_i(uh)(R_{h*}X) \\
&= h^{-1}g_i^{-1}(u)\mathcal{A}_i(\pi_*X)g_i(u)h + h^{-1}g_i^{-1}(u)\frac{d}{dt}g_i(\gamma(t)h)\Big|_{t=0} \\
&= h^{-1}g_i^{-1}(u)\mathcal{A}_i(\pi_*X)g_i(u)h + h^{-1}g_i^{-1}(u)\frac{d}{dt}g_i(\gamma(t))\Big|_{t=0}h \\
&= h^{-1}g_i^{-1}(u)\mathcal{A}_i(\pi_*X)g_i(u)h + h^{-1}g_i^{-1}(u)dg_i(u)(X)h \\
&= h^{-1}\omega_i(X)h
\end{aligned}$$

where we have used that $\pi R_h X = \pi X$, and that $g_i(uh) = g_i(u)h$, as can be seen by considering that $\sigma_i(p)g_i(uh) = uh$ so $\sigma_i(p)g_i(u)h = uh$, and we also had that $\gamma(t)$ was a curve through $\gamma(0)$ whose tangent vector at u was X.

We now must ensure that on an intersection $U_i \cap U_j$ of open sets, we have $\omega_i = \omega_j$. First we note that if σ_i, σ_j are local sections over U_i, U_j and $X \in T_pM$ with $p \in U_i \cap U_j$ we have

$$\sigma_{j*}X = R_{t_{ij}*}(\sigma_{i*}X) + (t_{ij}^{-1}dt_{ij}(X))^{\#}$$

where t_{ij} is the transition function between the trivialisations over U_j and U_i.

Proof: We take a curve $\gamma : [0,1] \to M$ with $\gamma(0) = p$ and $\frac{d}{dt}\gamma\Big|_{t=0} = X$. Now we have $\sigma_j(p) = \sigma_i(p)t_{ij}(p)$, hence

$$\begin{aligned}
\sigma_{j*}X = \frac{d}{dt}\sigma_j(\gamma(t))\Big|_{t=0} &= \frac{d}{dt}[\sigma_i(\gamma(t))t_{ij}(\gamma(t))]\Big|_{t=0} \\
&= \frac{d}{dt}\sigma_i(\gamma(t))\Big|_{t=0}t_{ij}(p) + \sigma_i(p)\frac{d}{dt}t_{ij}(\gamma(t))\Big|_{t=0} \\
&= R_{t_{ij}*}(\sigma_{i*}X) + \sigma_j(p)t_{ij}^{-1}(p)\frac{d}{dt}t_{ij}(\gamma(t))\Big|_{t=0}
\end{aligned}$$

Now, $t_{ij}^{-1}(p)dt_{ij}(X) = t_{ij}^{-1}(p)\frac{d}{dt}t_{ij}(\gamma(t))\Big|_{t=0} = \frac{d}{dt}[t_{ij}^{-1}(p)t_{ij}(\gamma(t))\Big|_{t=0} \in T_e(G) \cong \mathfrak{g}$, so this must equal $(t_{ij}^{-1}dt_{ij}(X))^{\#}$ at $\sigma_j(p)$.

Let us now apply ω to the above expression. We have

$$\omega\sigma_{j*}X = \omega R_{t_{ij}*}(\sigma_{i*}X) + \omega(t_{ij}^{-1}dt_{ij}(X))^{\#}$$

$$\Rightarrow \sigma_j^*(\omega(X)) = R_{t_{ij}}^*\omega(\sigma_{i*}X) + t_{ij}^{-1}dt_{ij}(X)$$

and by property ii) of the connection one-form we obtain

$$\sigma_{j*}(\omega(X)) = t_{ij}^{-1}\sigma_i^*\omega(X)t_{ij} + t_{ij}^{-1}dt_{ij}(X)$$

or

$$\mathcal{A}_j = t_{ij}^{-1}\mathcal{A}_i t_{ij} + t_{ij}^{-1}dt_{ij}$$

which gives the necessary transformation property for the $\mathcal{A}_i$.

As an example, suppose we have two local sections σ_1 and σ_2, related by $\sigma_2(p) = \sigma_1(p)g(p)$ for some $g(p) \in G$, then the local forms are related by

$$\mathcal{A}_2 = g^{-1}\mathcal{A}_1 g + g^{-1}dg$$

which is an example of a **gauge transformation**.

4 Horizontal lifts and parallel transport

Given $\gamma : [0,1] \to M$ a curve through the base space M then we define a **horizontal lift** of γ to be a curve $\tilde{\gamma} : [0,1] \to P$ such that $\pi \circ \tilde{\gamma} = \gamma$ and the tangent vector to $\tilde{\gamma}$ is horizontal, i.e. an element of $H_{\gamma(t)}P$. From the definition of the connection one-form, if $\tilde{X}$ is tangent to the $\tilde{\gamma}$ then $\omega(\tilde{X}) = 0$. This gives an ordinary differential equation for the horizontal lift and so locally there exists a unique solution. Hence if we have $\gamma : [0,1] \to M$ and $u_0 \in \pi^{-1}(\gamma(0))$ then there exists a unique horizontal lift $\tilde{\gamma}(t)$ in P such that $\tilde{\gamma}(0) = u_0$.

Proof: let U_i a chart containing γ and choose a section σ_i over U_i. If the horizontal lift exists it can be expressed as $\tilde{\gamma}(t) = \sigma_i(\gamma(t))g_i(t)$, where $g_i(t) \equiv g_i(\gamma(t)) \in G$. Choose our section such that $\sigma_i(\gamma(0)) = \tilde{\gamma(0)}$, so $g_i(0) = e$. Let X be a tangent vector to $\gamma(t)$ at $t = 0$, then $\tilde{X} = \tilde{\gamma}_* X$ is tangent to $\tilde{\gamma}$ at u_0. Applying the above result relating trivialisations σ_i and σ_j with $\sigma_j = \tilde{\gamma}$ and $t_{ij} = g_i$ we find

$$\tilde{X} = R_{g_i(t)*}(\sigma_{i*}X) + (g_i^{-1}(t) dg_i(X))^{\#}$$

and applying ω,

$$\omega(\tilde{X}) = 0 = g_i^{-1}(t)\omega(\sigma_{i*}X)g_i(t) + g_i^{-1}(t)\frac{d}{dt}g_i(t)$$

so we obtain

$$\frac{d}{dt}g_i(t) = -\omega(\sigma_{i*}X)g_i(t)$$

or as $\omega(\sigma_{i*}X = \sigma_i^* \omega(X) = \mathcal{A}_i(X)$ we have locally

$$\frac{d}{dt}g_i(t) = -\mathcal{A}_i(X)g_i(t)$$

with formal solution for $g_i(0) = e$ and local coordinates x^μ

$$g_i(\gamma(t) = \mathcal{P}\exp\left(-\int_0^t \mathcal{A}_{i\mu}\frac{dx^\mu}{dt}dt\right) = \mathcal{P}\exp\left(-\int_{\gamma(0)}^{\gamma(t)} \mathcal{A}_{i\mu}(\gamma(t)dx^\mu\right)$$

where $\mathcal{P}$ is a path-ordering operator.

If $\tilde{\gamma}'(t)$ another horizontal lift of γ with $\tilde{\gamma}'(0) = \tilde{\gamma}(0)g$ then $\tilde{\gamma}'(t) = \tilde{\gamma}(t)'g$ for all t.

Given a curve $\gamma : [0,1] \to M$ and $u_0 \in \pi^{-1}(\gamma(0))$ then the unique horizontal lift gives a unique point $u_1 = \tilde{\gamma}(1) \in \pi^{-1}(\gamma(1))$, i.e. starting at u_0 we follow the horizontal lift to arrive at the unique endpoint u_1. This point u_1 is called the **parallel transport** of u_0 along γ. This defines a map $\Gamma(\gamma) : \pi^{-1}(\gamma(0)) \to \pi^{-1}(\gamma(1))$ which commutes with the right action. We can also define an inverse, $\Gamma^{-1}(\gamma) = \Gamma(\gamma^{-1})$ where $\gamma^{-1}(t) = \gamma(1-t)$, and a composition, $\Gamma(\beta) \circ \Gamma(\alpha) = \Gamma(\alpha * \beta)$ where $\alpha(1) = \beta(0)$ and $\alpha * \beta$ is the path given by $\alpha(2t)$ for $0 \le t < 1/2$ and $\beta(2t-1)$ for $1/2 \le t \le 1$.

It is clear that given two paths α and β with the same start and end points, and same initial data u_0, that the end points of the horizontal lifts may not be equal, $\tilde{\alpha}(1) \neq \tilde{\beta}(1)$. In particular if we consider a loop γ, i.e. $\gamma(0) = \gamma(1)$ then we may have $\tilde{\gamma}(0) \neq \tilde{\gamma(1)}$. Thus this loop defines an action $\tau_\gamma : \pi^{-1}(p) \to \pi^{-1}(p)$ on the fibre over $p = \gamma(0)$. This action is compatible with the right action, $\tau_\gamma(ug) = \tau_\gamma(u)g$.

Let $C_p(M)$ denote the space of all loops $\gamma(t)$ with $\gamma(0) = \gamma(1)$, then the **holonomy group** at u is

$$\Phi_u = \{g \in G : \tau_\gamma(u) = ug, \gamma \in C_p(M)\}$$

i.e. the set of elements in G corresponding to the transformations of u induced by the horizontal lifts of loops based at p. The identity element is defined from the constant loop $c : [0,1] \to p$, and composition and inverses follow from the above definitions.

5 Curvature

Recall the exterior derivative d which acts on r-forms to give $(r+1)$-forms. An r-form η is an alternating real-valued form acting on r vectors: $\eta : TM \wedge \cdots \wedge TM \to \mathbb{R}$. We can generalise this to vector valued

r-forms $\phi \in \Omega^r(P) \otimes V$, $\phi : TP \wedge \cdots \wedge TP \to V$, with V a k-dimensional vector space. Our general form of ϕ is $\phi = \sum_{a=1}^{k} \phi^a \otimes e_a$ with $\{e_a\}$ a basis for V and $\phi^a \in \Omega^r(P)$.

We define the **covariant derivative** of $\phi \in \Omega^r(P) \otimes V$ by

$$D\phi(X_1, \ldots, X_r) = d\phi(X_1^H, \ldots, X_r^H)$$

where X_i^H is the horizontal component of the tangent vector $X_i \in T_uP$ and $d\phi = d\phi^a \otimes e_a$.

The covariant derivative can be thought of as an extension of the usual derivative that takes into account the non-flat nature of the manifold. It leads to the idea of the **curvature two-form** Ω which is the covariant derivative of the connection one-form ω:

$$\Omega = D\omega \in \Omega^2(P) \otimes \mathfrak{g}$$

It satisfies

$$R_g^*\Omega = g^{-1}\Omega g$$

Proof: First note that $(R_{g*}X)^H = R_{g*}X^H$ and $dR_{g*} = R_{g*}d$. Then we have

$$\begin{aligned}
R_g^*\Omega(X,Y) = \Omega(R_{g*}X, R_{g*}Y) &= d\omega((R_{g*}X)^H, (R_{g*}Y)^H) \\
&= d\omega(R_{g*}X^H, R_{g*}Y^H) \\
&= dR_g^*\omega(X^H, Y^H) \\
&= d(g^{-1}\omega g)\omega(X^H, Y^H) \\
&= g^{-1}d\omega\omega(X^H, Y^H)g \\
&= g^{-1}\Omega(X,Y)g
\end{aligned}$$

where g is constant so $dg = 0$.

We can obtain a local form of the curvature by pulling back via a local section: $\mathcal{F} = \sigma^*\Omega$. This local form is related to the local connection form via

$$\mathcal{F} = d\mathcal{A} + \mathcal{A} \wedge \mathcal{A}$$

with

$$\mathcal{F}(X,Y) = d\mathcal{A}(X,Y) + [\mathcal{A}(X), \mathcal{A}(Y)]$$

for X, Y tangent vectors. In terms of local coordinates x^μ, $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}dx^\mu dx^{nu}$, with

$$\mathcal{F}_{\mu\nu} = \partial_\mu \mathcal{A}_\nu - \partial_\nu \mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\mu]$$

Note that in physics $\mathcal{A}_\mu = iA_\mu$ where $i^2 = -1$ and A_μ is called the gauge potential or gauge field. For the local curvature form we have $\mathcal{F}_{\mu\nu} = iF_{\mu\nu}$, with F representing, for instance, physically observable fields.

6 Monopole bundle

A **monopole** is a hypothetical particle carrying magnetic charge. They can be described using a principal $U(1)$ bundle over the sphere S^2, where $U(1)$ is the group of complex numbers of modulus 1 (and is isomorphic to the circle S^1). We cover the sphere with two coordinate patches:

$$U_N = \left\{(\theta, \phi) : 0 \leq \theta < \frac{\pi}{2} + \varepsilon, 0 \leq \phi < 2\pi\right\} \quad U_S = \left\{(\theta, \phi) : \frac{\pi}{2} - \varepsilon \leq \theta < \pi, 0 \leq \phi < 2\pi\right\}$$

Our trivialisations can be written as

$$\phi_N^{-1}(u) = (p, e^{i\alpha_N}) \in U_N \times U(1) \quad \phi_S^{-1}(u) = (p, e^{i\alpha_S}) \in U_S \times U(1)$$

for $\pi(u) \in p$. The intersection $U_N \cap U_S$ is an ε-neighbourhood of the equator and so homotopic to S^1. The transition function $t_{NS} \in U(1)$ can be written as $e^{i\Lambda_{NS}}$, with $e^{i\alpha_N} = e^{i\Lambda_{NS}} e^{i\alpha_S}$. Physically the magnetic monopole field $\vec{B} = \frac{g\vec{r}}{r^3}$ is given by the curl of a vector potential $\vec{A}$, $\vec{B} = \vec{\nabla} \times \vec{A}$ which motivates our definition of a connection.

It may be instructive to start off in Cartesian coordinates. Consider the vector potentials

$$\vec{A}_N = \left(-\frac{gy}{r(r+z)}, \frac{gx}{r(r+z)}, 0 \right) \qquad \vec{A}_S = \left(\frac{gy}{r(r-z)}, \frac{-gx}{r(r-z)}, 0 \right)$$

where $r = \sqrt{x^2 + y^2 + z^2}$. It can be directly verified that these yield a magnetic field $\vec{B} = \frac{g\vec{r}}{r^3}$.

Proof: First, note that $\frac{\partial r}{\partial x_i} = \frac{x_i}{r}$, for $x_i = x, y, z$. We need

$$\frac{\partial}{\partial z} \frac{1}{r(r+z)} = -\frac{1}{r^2(r+z)^2} \left(2r\frac{z}{r} + r + z\frac{z}{r} \right) = -\frac{1}{r^3(r+z)^2} (r+z)^2 = -\frac{1}{r^3}$$

and

$$\frac{\partial}{\partial x_i} \frac{x_i}{r(r+z)} = -\frac{x_i}{r^2(r+z)^2} \left(2r\frac{x_i}{r} + z\frac{x_i}{r} \right) + \frac{1}{r(r+z)} = -\frac{x_i^2}{r^3(r+z)^2} (2r+z) + \frac{1}{r(r+z)} =$$

for $x_i = x, y$. Then we find that

$$\left(\vec{\nabla} \times \vec{A_N} \right)_x = -\frac{\partial}{\partial z} \frac{gx}{r(r+z)} = \frac{gx}{r^3}$$

$$\left(\vec{\nabla} \times \vec{A_N} \right)_y = \frac{\partial}{\partial z} \frac{-gy}{r(r+z)} = \frac{gy}{r^3}$$

$$\begin{aligned} \left(\vec{\nabla} \times \vec{A_N} \right)_z = \frac{\partial}{\partial x} \frac{gx}{r(r+z)} + \frac{\partial}{\partial y} \frac{gy}{r(r+z)} &= \frac{2g}{r(r+z)} - \frac{1}{r^3(r+z)^2} (2r+z) (x^2 + y^2) \\ &= \frac{2g}{r(r+z)} - \frac{1}{r^3(r+z)^2} (2r+z) (r^2 - z^2) \end{aligned}$$

or

$$\left(\vec{\nabla} \times \vec{A_N} \right)_z = \frac{g}{r^3(r+z)^2} \left(2r^2(r+z) - (2r+z)(r-z)(r+z) \right) = \frac{g(r+z)}{r^3(r+z)^2} \left(2r^2 - 2r^2 + 2zr - zr + z^2 \right)$$

so

$$\left(\vec{\nabla} \times \vec{A_N} \right)_z = \frac{gz}{r^3}$$

The calculation for A_S is entirely similar.

Now let us write A_N as:

$$A_N = -\frac{gy}{r(r+z)} dx + \frac{gx}{r(r+z)} dy$$

and from $x = r \sin\theta \cos\phi$ and $y = r \sin\theta \sin\phi$ we have $dx = r\cos\theta\cos\phi\, d\theta - r\sin\theta\sin\phi\, d\phi$ and $dy = r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi$, hence with $z = r\cos\theta$

$$A_N = -\frac{g\sin\theta\sin\phi}{1+\cos\theta} (\cos\theta\cos\phi\, d\theta - \sin\theta\sin\phi\, d\phi) + \frac{g\sin\theta\cos\phi}{1+\cos\theta} (\cos\theta\sin\phi\, d\theta + \sin\theta\cos\phi\, d\phi)$$

$$\Rightarrow A_N = \frac{g\sin^2\theta(\sin^2\phi + \cos^2\phi)}{1+\cos\theta} d\phi = \frac{g(1-\cos^2\theta)}{1+\cos\theta} d\phi = g(1-\cos\theta)\, d\phi$$

and similarly

$$A_S = -g(1+\cos\theta)\, d\phi$$

This gives us our two local connection forms:

$$\mathcal{A}_N = ig(1-\cos\theta)\,d\phi \qquad \mathcal{A}_S = -ig(1+\cos\theta)\,d\phi$$

(Note that there is no group component, however representing an element of $U(1)$ by $e^{i\varphi}$ we see that under a gauge transformation these forms pick up a term $e^{-i\varphi}de^{i\varphi} = id\varphi$, so in fact we have

$$\mathcal{A}_N = ig(1-\cos\theta)\,d\phi + id\varphi$$

and similarly for $\mathcal{A}_S$. Observe the action of this on $\alpha\frac{\partial}{\partial\varphi}$, the fundamental vector field generated by $i\alpha \in \mathfrak{u}(1)$:

$$\mathcal{A}_N(i\alpha)^{\#} = i\alpha\frac{\partial}{\partial\varphi}d\varphi = i\alpha$$

and again similarly for $\mathcal{A}_S$.)

We now wish to find a transition function t_{NS} defined on $U_N \cap U_S$ such that the two potentials agree on the overlap. Under the transition t_{NS}, $\mathcal{A}_S \to t_{NS}^{-1}\mathcal{A}_S t_{NS} + t_{NS}^{-1}dt_{NS}$. Writing $t_{NS} = e^{i\Lambda_{NS}}$ and using the fact that $U(1)$ is abelian, we find

$$\mathcal{A}_S \to \mathcal{A}_S + id\Lambda_{NS}$$

and we want this to equal A_S on the overlap, i.e.

$$ig(1-\cos\theta)\,d\phi = -ig(1+\cos\theta)\,d\phi + id\Lambda_{NS}$$

$$\Rightarrow 2ig\,d\phi = id\Lambda_{NS} \Rightarrow \frac{d\Lambda_{NS}}{d\phi} = 2g$$

hence we find that

$$t_{NS} = e^{2ig\phi}$$

choosing the phase such that the constant of integration is zero. Now, we want the transition function to be single-valued, i.e. $e^{2ig\phi} = e^{2ig(\phi+2\pi)} = e^{4\pi ig}e^{2ig\phi}$, hence $4\pi g$ must be an integer multiple of 2π, so we find that

$$g = \frac{n}{2} \qquad n \in \mathbb{Z}$$

This also shows that the transition function is characterised by the integer $2g$ giving the number of times we wrap the southern equator around the northern, and so by the element $2g$ of the fundamental group of S^1. The quantity $2g$ also gives the monopole charge. Note that for $g = 0$ the transition function is just the identity and the bundle is in fact the trivial product $S^2 \times U(1)$.

We can also compute the local curvature form, $\mathcal{F} = \frac{1}{2}\mathcal{F}_{\mu\nu}dx^\mu \wedge dx^\nu$, where

$$\mathcal{F}_{\mu\nu} = \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu + [\mathcal{A}_\mu, \mathcal{A}_\nu]$$

As $U(1)$ is abelian the commutator vanishes, and we are left with

$$\mathcal{F} = \frac{1}{2}\left[\left(\frac{\partial\mathcal{A}_\theta}{\partial\phi} - \frac{\partial\mathcal{A}_\phi}{\partial\theta}\right)d\phi\wedge d\theta + \left(\frac{\partial\mathcal{A}_\phi}{\partial\theta} - \frac{\partial\mathcal{A}_\theta}{\partial\phi}\right)d\theta\wedge d\phi\right]$$

but $\mathcal{A}_\theta = 0$ and $\mathcal{A}_\phi = ig(1-\cos\theta) \Rightarrow \partial_\theta A_\phi = ig\sin\theta$ (for $\mathcal{A} = \mathcal{A}_N$), and we get

$$\mathcal{F} = ig\sin\theta\,d\theta\wedge d\phi$$

and in fact we obtain the same result for $\mathcal{A} = \mathcal{A}_S$, a consequence of the abelian nature of $U(1)$.

6.1 Monopole bundle as Hopf fibration

The Hopf fibration is a famous fibre bundle where the base space is S^2, the fibre is S^1 and the total space is S^3. It is in fact equivalent to the monopole bundle with $g = 1/2$ (i.e. the transition function is $e^{i\phi}$). It is convenient to view the spheres involved in terms of complex numbers:

$$S^1 = \{\lambda \in \mathbb{C} : |\lambda| = 1\}$$

$$S^3 = \{(z_1, z_2) : |z_1|^2 + |z_2|^2 = 1, z_1, z_2 \in \mathbb{C}\}$$

$$S^2 = \mathbb{C}P^1 = \{(z_1, z_2) : (z_1, z_2) \sim (\lambda z_1, \lambda z_2), z_1, z_2 \in \mathbb{C}, \lambda \in \mathbb{C} - 0\}$$

where $\mathbb{C}P^1$ is the complex projective line, i.e. the space of all complex lines through the origin in $\mathbb{C}^2$. It is isomorphic to $\mathbb{C}$ with a point at infinity attached which is isomorphic to the Riemann sphere (recall that the Riemann sphere is formed by identifying $\mathbb{C} \cong \mathbb{R}^2 \subset \mathbb{R}^3$ and projecting every complex number to the unit sphere, with the north pole representing the point at infinity).

Now viewing S^2 as the projective space $\mathbb{C}P^1$ means that when we project from S^3 to S^2 we will have that points in S^3 differing only by an overall scaling will end up at the same point in S^2. In particular we define the projection $\pi : S^3 \to S^2$ by $\pi^{-1}([z_1, z_2]) = \{(\lambda z_1, \lambda z_2) : |\lambda| = 1\} = S^1$, where the condition $|\lambda| = 1$ ensures that the preimage lies in S^3 (i.e. has norm one). What this is saying is that we when we project from $(z_1, z_2) \in S^3$ to the equivalence class of $(z_1, z_2) \in \mathbb{C}P^1 \cong S^2$ an entire set of points from S^3 parametrised by a unit complex number λ end up in the same equivalence class.

In general we would write

$$\pi^{-1}([z_1, z_2]) = \left\{ \frac{\lambda(z_1, z_2)}{\sqrt{|z_1|^2 + |z_2|^2}} : |\lambda| = 1 \right\} = S^1$$

Let us now take as two charts on the sphere the regions U_N and U_S on $\mathbb{C}P^1$, corresponding to complex numbers lying "within" in the Riemann sphere (i.e. inside the region bounded by the intersection of the Riemann sphere and the complex plane) and to complex numbers lying "outside" the Riemann sphere. Namely we have

$$U_N = \left\{ [z, 1] : |z| = \left| \frac{z_1}{z_2} \right| \geq 1 \right\} \subset \{[z_1, z_2] : z_2 \neq 0\}$$

$$U_S = \left\{ [1, z] : |z| = \left| \frac{z_2}{z_1} \right| \geq 1 \right\} \subset \{[z_1, z_2] : z_1 \neq 0\}$$

and local trivialisations defined by

$$\phi_N\left([z, 1], \lambda\right) = \left([z, 1], \frac{\lambda(z, 1)}{\sqrt{|z|^2 + 1}} \right)$$

$$\phi_S\left([1, z], \lambda\right) = \left([1, z], \frac{\lambda(1, z)}{\sqrt{1 + |z|^2}} \right)$$

(where we have explicitly included the base point on the right-hand-side). Now consider

$$([1, z], \lambda) \xrightarrow{\phi_S} \left([1, z], \frac{\lambda(1, z)}{\sqrt{1 + |z|^2}} \right) = \left([1, z], \frac{z}{|z|} \frac{\lambda(z^{-1}, 1)}{\sqrt{1 + |z^{-1}|^2}} \right) \xrightarrow{U_N} \left([z^{-1}, 1], \frac{z}{|z|} \frac{\lambda(1, z^{-1})}{\sqrt{1 + |z^{-1}|^2}} \right) \xrightarrow{\phi_N^{-1}} \left([z^{-1}, 1], \lambda \frac{z}{|z|} \right)$$

as on the intersection $U_N \cap U_S$ the point $[1, z] \in U_S$ corresponds to the point $[z^{-1}, 1] \in U_N$, so we see that $t_{NS} = \frac{z}{|z|}$ i.e. a unit complex number, so $t_{NS} = e^{i\phi}$, corresponding to the monopole bundle with unit monopole charge (recall that $S^1 \cong U(1)$).

7 Instanton bundle

One of the simplest non-abelian generalisations of the monopole bundle takes $SU(2)$ as the gauge group and S^4 as the base space. This corresponds to the description of **instantons**, which arise as energy-minimising solutions of the field equations arising in Yang-Mills theory (which is a non-abelian generalisation of electromagnetism using $SU(2)$). In reality instantons are defined on $\mathbb{R}^4$ but it is simpler to consider the one-point compactification of $\mathbb{R}^4$ which is S^4. As before we take two open sets

$$U_N = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 < R^2 + \varepsilon\}$$

$$U_S = \{(x, y, z, t) : x^2 + y^2 + z^2 + t^2 > R^2 - \varepsilon\}$$

with the intersection $U_N \cup U_S$ giving an ε-neighbourhood of S^3. The transition function is then a map from S^3 to $SU(2)$ but as an element of $SU(2)$ can be written

$$\begin{pmatrix} u & v \\ -\overline{v} & \overline{u} \end{pmatrix} \qquad |u|^2 + |v|^2 = 1$$

if we take $u = t + iz$, $v = y + ix$ we find that in fact $SU(2) \cong S^3$, and hence the transition function will correspond to an element of the homotopy group $\pi_3(S^3) \cong \mathbb{Z}$.

The transition function corresponding to $1 \in \pi_3(S^3)$ is the map

$$f : (x, y, z, t) \mapsto \frac{1}{R} \begin{pmatrix} t + iz & y + ix \\ -y + ix & t - iz \end{pmatrix} \qquad R = \sqrt{x^2 + y^2 + z^2 + t^2}$$

while that corresponding to $n \in \pi_3(S^3)$ is given by f^n.

7.1 Instanton bundle as Hopf fibration

There exists a corresponding Hopf fibration for the instanton bundle too, this time consisting of fibring S^3 over S^4 with total space S^7. The construction is practically identical to that for the monopole bundle except with quaternions instead of complex numbers. We view S^3 as the space consisting of unit quaternions

$$S^3 = \{q \in \mathbb{H} : |q| = 1\}$$

and S^7 as

$$S^7 = \{(q_1, q_2) \in \mathbb{H} : |q_1|^2 + |q_2|^2 = 1\}$$

while S^4 corresponds to the quaternionic projective line

$$S^4 \cong \mathbb{H}P^1 = \{(q_1, q_2) : (q_1, q_2) \sim (\lambda q_1, \lambda q_2), q_1, q_2 \in \mathbb{H}, \lambda \in \mathbb{H} - 0\}$$

Proceeding as before we have a projection $\pi : S^7 \to S^4$ with $\pi^{-1}([q_1, q_2]) = \{(\lambda q_1, \lambda q_2) : |\lambda| = 1\} \subset S^7$. Taking similar local charts and trivialisations we find that the transition map for this Hopf bundle is given by a unit quaternion, hence by an element of $S^3 \cong SU(2)$, and so by the unit element of $\pi_3(S^3)$.

References

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