# Quantum Field Theory - Useful Formulae and Feynman Rules *

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## Introduction

These are some notes which I originally intended to be a roughly 5 page list of all the formulae and tricks I needed for my quantum field theory exam. Somehow I have ended up with the 5 page list I wanted as well as 20+ pages trying to explain how to derive Feynman rules.

My conventions follow *Quantum Field Theory* by Mark Srednicki (though my illogical order of presentation certainly doesn't). My understanding of what actually is going on with path integrals is mostly based on *Quantum Field Theory in a Nutshell* by Tony Zee. Another useful book for understanding how free field path integrals correspond to Gaussian integrals is *Field Quantisation* by Greiner and Reinhardt. Small bits and pieces of some online notes (Advanced Quantum Field Theory lecture notes from Cambridge, Robert Clancy's Feynman rules notes from 2007-2008 in Trinity) contributed to a lesser extent.

The first section lists various useful relationships which you should already know. I then give the free Lagrangians and equations of motion for the most important fields we will encounter: real and complex scalar, spinor and massless vector. Section 3 contains identities useful for performing loop calculations (Feynman parameters, gamma functions and so on). In section 4 I list various facts that possibly come in handy when computing scattering amplitudes at tree level, including facts about gamma matrices and similar things.

Section 5 then contains a long, detailed and rambling[1] account of where Feynman rules come from, taking many things (the path integral formalism, LSZ reduction formula) for granted. À la Zee, we motivate the field theory constructions by exploring simple Gaussian integrals. The main aim is to describe how to write down propagator and vertex factors given some Lagrangian.

The final section collates various important quantum field theories and lists their Feynman rules.

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*Version 3, uploaded January 30, 2011.

[1] Not to mention probably highly inaccurate.

# 1 General

You should be completely familiar with manipulating objects such as the following:

- Einstein summation convention
- Spacetime ($c = 1$)

$$g^{\mu\nu} = g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad g_{\mu\rho}g^{\rho\nu} = \delta_\mu^\nu \qquad g_{\mu\nu}g^{\mu\nu} = g_\mu^\mu = 4 \tag{1.1}$$

$$x^\mu = (x^0, x^1, x^2, x^3) \qquad x_\mu = (-x_0, x_1, x_2, x_3) \qquad x^\mu = g^{\mu\nu}x_\nu \qquad x_\mu = g_{\mu\nu}x^\nu \tag{1.2}$$

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu} \qquad \partial^\mu = \frac{\partial}{\partial x_\mu} \qquad \partial_\mu x^\nu = \delta_\mu^\nu \quad \partial_\mu x_\nu = g_{\mu\nu} \tag{1.3}$$

- Four-vector products

$$ab \equiv a^\mu b_\mu = g_{\mu\nu}a^\mu b^\nu = -a^0 b^0 + \mathbf{a} \cdot \mathbf{b} \tag{1.4}$$

- Energy-momentum relations for an on-shell particle of mass $m$:

$$p^\mu = (E, \mathbf{p}) \qquad p_\mu p^\mu = -E^2 + |\mathbf{p}|^2 = -m^2 \tag{1.5}$$

- Totally anti-symmetric rank 4 tensor $\varepsilon_{\mu\nu\rho\sigma}$

$$\varepsilon^{0123} = +1 \qquad \varepsilon_{0123} = -1 \tag{1.6}$$

- Fourier transforms

$$\widetilde{\varphi}(k) = \int d^4x \, e^{ikx} \varphi(x) \qquad \varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{ikx} \widetilde{\varphi}(k) \tag{1.7}$$

$$\delta^4(x - x') = \int \frac{d^4k}{(2\pi)^4} e^{ik(x-x')} \tag{1.8}$$

- Classical fields. I use $A$ as a generic field index, so that $\varphi_A$ stands for any sort of field.

$$S = \int d^4x \, \mathcal{L}(\varphi_A, \partial_\mu \varphi_A) \qquad \delta S = 0 \Rightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi_A)} = \frac{\partial \mathcal{L}}{\partial \varphi_A} \tag{1.9}$$

$$\Pi^A = \frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi_A)} \qquad \mathcal{H} = \Pi^A \partial_0 \varphi_A - \mathcal{L} \tag{1.10}$$

- Units

$$\hbar = c = 1 \tag{1.11}$$

We measure everything in dimensions of mass:

$$[m] = +1 \qquad [x^\mu] = -1 \qquad [\partial^\mu] = +1 \tag{1.12}$$

As the path integral formalism involves integrating $e^{iS}$ the action must be dimensionless:

$$[S] = 0 \Rightarrow [\mathcal{L}] = +d \tag{1.13}$$

# 2 Fields

*"Oh my God - it's full of harmonic oscillators!"*

## 2.1 Scalar (Spin 0)

- A real scalar field $\varphi$ with free Lagrangian

$$\mathcal{L} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 \tag{2.1}$$

The equation of motion is the Klein-Gordon equation:

$$(-\partial_\mu\partial^\mu + m^2)\varphi = 0 \tag{2.2}$$

General solution:

$$\varphi(\mathbf{x}, t) = \int \widetilde{dk}\left(a(\mathbf{k})e^{ikx} + a^\dagger(\mathbf{k})e^{-ikx}\right) \tag{2.3}$$

with $k^0 = +\sqrt{|\mathbf{k}|^2 + m^2}$. Conjugate momentum:

$$\Pi = \partial_0\varphi \tag{2.4}$$

Canonical commutation relations:

$$[\varphi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) \tag{2.5}$$

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega\delta^3(\mathbf{k} - \mathbf{k}') \tag{2.6}$$

and others zero. Interpretation: $a^\dagger(\mathbf{k})$ is a creation operator for a single particle with momentum $\mathbf{k}$.

- A complex scalar field $\varphi$ with free Lagrangian

$$\mathcal{L} = -\partial_\mu\varphi^\dagger\partial^\mu\varphi - m\varphi^\dagger\varphi \tag{2.7}$$

It is convenient to treat $\varphi$ and its complex conjugate as independent. Both satisfy the Klein-Gordon equation

$$(-\partial_\mu \partial^\mu + m^2)\varphi = 0 \qquad (-\partial_\mu \partial^\mu + m^2)\varphi^\dagger = 0 \tag{2.8}$$

General solution:

$$\varphi(\vec{x}, t) = \int \widetilde{dk} \left[ a(\vec{k}) e^{ikx} + b^\dagger(\vec{k}) e^{-ikx} \right] \tag{2.9}$$

$$\varphi^\dagger(\vec{x}, t) = \int \widetilde{dk} \left[ a^\dagger(\vec{k}) e^{-ikx} + b(\vec{k}) e^{ikx} \right] \tag{2.10}$$

Conjugate momenta:

$$\Pi = \partial_0 \varphi^\dagger \qquad \Pi^\dagger = \partial_0 \varphi \tag{2.11}$$

Canonical commutation relations:

$$[\varphi(\mathbf{x}, t), \Pi(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) \qquad [\varphi^\dagger(\mathbf{x}, t), \Pi^\dagger(\mathbf{y}, t)] = i\delta^3(\mathbf{x} - \mathbf{y}) \tag{2.12}$$

$$[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega \delta^3(\mathbf{k} - \mathbf{k}') \qquad [b(\mathbf{k}), b^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega \delta^3(\mathbf{k} - \mathbf{k}') \tag{2.13}$$

and all others zero. Interpretation: two types of bosonic particle ($a$-type and $b$-type), which can be thought of as being positively and negatively charged.

## 2.2 Spinor (Spin 1/2)

Hopefully these expressions are familiar to you; we won't go into the messy details of spinors here.

- Two-component Weyl spinor field $\psi$ with Lagrangian

$$\mathcal{L} = i\psi^\dagger \overline{\sigma}^\mu \partial_\mu \psi - \frac{1}{2} m(\psi\psi + \psi^\dagger \psi^\dagger) \tag{2.14}$$

satisfies the equations

$$-i\overline{\sigma}^\mu \partial_\mu \psi + m\psi^\dagger = 0 \tag{2.15}$$

$$-i\sigma^\mu \partial_\mu \psi^\dagger + m\psi = 0 \tag{2.16}$$

In practice we can always recover a Weyl spinor by projecting from a Dirac spinor (see below).

- Four-component Dirac spinor field $\Psi$ with Lagrangian

$$\mathcal{L} = i\overline{\Psi} \partial\!\!\!/ \Psi - m\overline{\Psi}\Psi \qquad \partial\!\!\!/ \equiv \partial_\mu \gamma^\mu \tag{2.17}$$

satisfies the Dirac equation

$$(-i\partial\!\!\!/ + m)\Psi = 0 \tag{2.18}$$

General solution:

$$\Psi(x) = \sum_{s=\pm} \int \widetilde{dp} \left( b_s(\mathbf{p}) u_s(\mathbf{p}) e^{ipx} + d_s^\dagger(\mathbf{p}) v_s(\mathbf{p}) e^{-ipx} \right) \tag{2.19}$$

Here $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ are constant four-component spinors satisfying

$$(\not{p} + m) u_s(\mathbf{p}) = 0 \qquad (-\not{p} + m) v_s(\mathbf{p}) = 0 \tag{2.20}$$

Canonical anti-commutation relations:

$$\{\Psi_\alpha(\mathbf{x}, t), \overline{\Psi}_\beta(\mathbf{y}, t)\} = (\gamma^0)_{\alpha\beta} \delta^3(\mathbf{x} - \mathbf{y}) \tag{2.21}$$

$$\begin{aligned} \{b_s(\mathbf{p}), b_{s'}^\dagger(\mathbf{p}')\} &= (2\pi)^3 2\omega \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ss'} \\ \{d_s(\mathbf{p}), d_{s'}^\dagger(\mathbf{p}')\} &= (2\pi)^3 2\omega \delta^3(\mathbf{p} - \mathbf{p}') \delta_{ss'} \end{aligned} \tag{2.22}$$

and all others zero. Interpretation: we have two-types of fermionic particles which can have either spin-up or spin-down. These particles can be thought of as having positive and negative charge. The creation operator for a $b$-type particle with spin $s$ and momentum $\mathbf{p}$ is $b_s^\dagger(\mathbf{p})$.

- Four-component Majorana spinor field $\Psi$ with Lagrangian

$$\mathcal{L} = \frac{i}{2} \Psi^t \mathcal{C} \not{\partial} \Psi - \frac{1}{2} m \Psi^t \mathcal{C} \Psi \tag{2.23}$$

and satisfies the Dirac equation

$$(-i\not{\partial} + m)\Psi = 0 \tag{2.24}$$

- Analogy: Dirac field $\sim$ complex scalar field, Majorana field $\sim$ real scalar field. In particular, a Dirac field describes charged fermions while a Majorana field describes neutral fermions.

## 2.3 Vector (Spin 1)

- A massless vector field $A_\mu$ has field strength

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.25}$$

and Lagrangian

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \tag{2.26}$$

and satisfies the Maxwell equation

$$\partial_\nu F^{\mu\nu} = 0 \tag{2.27}$$

The field strength also satisfies the identity

$$\varepsilon_{\mu\nu\rho\sigma}\partial^\rho F^{\mu\nu} = 0 \tag{2.28}$$

The Lagrangian is invariant under gauge transformations

$$A^\mu \to A^\mu - \partial^\mu \Gamma \tag{2.29}$$

with $\Gamma$ some scalar field. This gauge invariance makes canonical quantisation trickier.

# 3 Loop integrals

- Feynman's trick:

$$\begin{aligned}\frac{1}{A_1}\dots\frac{1}{A_n} &= (n-1)!\int dx_1\dots dx_n \frac{\delta(1-x_1-\dots-x_n)}{(x_1A_1+\dots+x_nA_n)^n} \\ &= (n-1)!\int_0^1 dx_1 \int_0^{1-x_1} dx_2 \dots \int_0^{1-x_1-\dots-x_{n-1}} dx_n \, \frac{1}{(x_1A_1+\dots+x_nA_n)^n}\end{aligned} \tag{3.1}$$

- The all-important integral:

$$\int \frac{d^dq}{(2\pi)^d}\frac{(q^2)^a}{(q^2+D)^b} = i\frac{\Gamma\left(b-a-\frac{1}{2}d\right)\Gamma\left(a+\frac{1}{2}d\right)}{(4\pi)^{d/2}\Gamma\left(b\right)\Gamma\left(\frac{1}{2}d\right)}D^{-(b-a-d/2)} \tag{3.2}$$

- Gamma function:

$$\Gamma(s) = \int_0^\infty dt\, t^{s-1}e^{-t} \tag{3.3}$$

For $n \in \mathbb{N}$,

$$\Gamma(n+1) = n! \qquad \Gamma\left(n+\frac{1}{2}\right) = \frac{(2n-1)(2n-3)\dots 3\cdot 1}{2^n}\sqrt{\pi} \tag{3.4}$$

For $n \in \mathbb{N}$ and $x$ small,

$$\Gamma(-n+x) \approx \frac{(-1)^n}{n!}\left(\frac{1}{x} + \ln e^{-\gamma} + \sum_{k=1}^n \frac{1}{k}\right) \qquad \Gamma(x) \approx \frac{1}{x} + \ln e^{-\gamma} \tag{3.5}$$

- Useful integral relations:

$$\int d^dq\, \underbrace{q^\mu \dots q^\nu}_{\text{odd no.}} f(q^2) = 0 \qquad \int d^dq\, q^\mu q^\nu f(q^2) = \frac{1}{d}g^{\mu\nu}\int d^dq\, q^2 f(q^2) \tag{3.6}$$

In general, Lorentz invariance can be used to argue for the general structure of an integral with vector indices.

Figure 1: A one-loop correction to a tree.

# 4 Techniques for scattering amplitudes

## 4.1 Mandelstam variables

- For a process with two incoming particles of momentum $p_1, p_2$ and two outgoing particles of momentum $p'_1$, $p'_2$, the Mandelstam variables are

$$s = -(p_1 + p_2)^2 = -(p'_1 + p'_2)^2 \tag{4.1}$$

$$t = -(p_1 - p'_1)^2 = -(p_2 - p'_2)^2 \tag{4.2}$$

$$u = -(p_1 - p'_2)^2 = -(p_2 - p'_1)^2 \tag{4.3}$$

  They satisfy

$$s + t + u = m_1^2 + m_2^2 + m_1'^2 + m_2'^2 \tag{4.4}$$

- In the centre of mass frame, we have

$$\mathbf{p}_1 + \mathbf{p}_2 = 0 \qquad s = (E_1 + E_2)^2 \tag{4.5}$$

## 4.2 Essential facts about spinors

- The spinors $u_s(\mathbf{p})$ and $v_s(\mathbf{p})$ satisfy

$$(\not{p} + m)u_s(\mathbf{p}) = 0 \qquad (-\not{p} + m)v_s(\mathbf{p}) = 0 \tag{4.6}$$

- The gamma matrices satisfy

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu\nu} \tag{4.7}$$

  which tells us that

$$(\gamma^0)^2 = 1 \qquad (\gamma^i)^2 = -1 \qquad \gamma^\mu \gamma^\nu = -\gamma^\nu \gamma^\mu \quad \mu \neq \nu \tag{4.8}$$

- We also need

$$\gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = -\frac{i}{24}\varepsilon_{\mu\nu\rho\sigma}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma \tag{4.9}$$

$$\gamma_5^2 = 1 \qquad \gamma^\mu\gamma_5 = -\gamma_5\gamma^\mu \tag{4.10}$$

- Some important identities satisfied by the gamma matrices (in $d = 4$)

$$\gamma^\mu\gamma_\mu = -4 \qquad \gamma^\mu\gamma^\rho\gamma_\mu = 2\gamma^\rho \qquad \gamma^\mu\gamma^\rho\gamma^\sigma\gamma_\mu = 4g^{\rho\sigma} \qquad \gamma^\mu\gamma^\rho\gamma^\sigma\gamma^\nu\gamma_\mu = 2\gamma^\nu\gamma^\sigma\gamma^\rho \tag{4.11}$$

- Various traces:

$$\operatorname{tr} 1 = 4 \qquad \operatorname{tr}\gamma^\mu = 0 \qquad \operatorname{tr}\gamma_5 = 0 \qquad \operatorname{tr}\underbrace{\gamma^\mu \ldots \gamma^\nu}_{\text{odd no.}} = 0 \tag{4.12}$$

$$\operatorname{tr}\gamma^\mu\gamma^\nu = -4g^{\mu\nu} \qquad \operatorname{tr}\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = 4\left(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho}\right) \tag{4.13}$$

$$\operatorname{tr}\gamma_5\gamma^\mu\gamma^\nu = 0 \qquad \operatorname{tr}\gamma_5\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma = -4i\varepsilon^{\mu\nu\rho\sigma} \tag{4.14}$$

## 4.3 Spin sums

- Given a scattering amplitude $\mathcal{T}$ we compute the spin-averaged cross section $\langle|\mathcal{T}|^2\rangle$ by averaging over incoming spins and summing over final spins:

$$\langle|\mathcal{T}|^2\rangle = \underbrace{\frac{1}{2}\sum_{s_1}\cdots\frac{1}{2}\sum_{s_n}}_{n\,\text{incoming}}\underbrace{\sum_{s_1'}\cdots\sum_{s_n'}}_{n'\,\text{outgoing}}|\mathcal{T}|^2 \tag{4.15}$$

- Trace trick: generally $\mathcal{T}$ contains terms such as $\overline{u}_i A v_j$, for some matrix $A$, so that $\mathcal{T}^*$ contains terms $\overline{v}_j\overline{A}u_i$. Everything is contracted here (i.e. a scalar) so taking the trace does nothing and we can then use the cyclic property of the trace to obtain

$$\overline{u}_i A v_j \overline{v}_j \overline{A} u_i = \operatorname{tr}\left(\overline{u}_i A v_j \overline{v}_j \overline{A} u_i\right) = \operatorname{tr}\left(u_i\overline{u}_i A v_j \overline{v}_j \overline{A}\right) \tag{4.16}$$

  Before spin-averaging one should write $|\mathcal{T}|^2$ in terms of traces such as these, with all the spinors $u_i$ and $v_i$ standing to the left of their barred versions.

- Invaluable spin-sum replacements:

$$\sum_s u_s(\mathbf{p})\overline{u}_s(\mathbf{p}) = -\not{p} + m \qquad \sum_s v_s(\mathbf{p})\overline{v}_s(\mathbf{p}) = -\not{p} - m \tag{4.17}$$

## 4.4 Helicity sums

- In processes with external photons we must also average over incoming photon helicity and sum over outgoing photon helicity. We may use the replacement

$$\sum_{\lambda=\pm}\varepsilon_\lambda^\mu(\mathbf{k})\varepsilon_\lambda^{\nu*}(\mathbf{k}) \to g^{\mu\nu} \tag{4.18}$$

# 5 Feynman rules

The main aim of this section is to explain in a practical way how to find the Feynman rules for a given Lagrangian.

## 5.1 Propagators

If you are not interested in an enlightening build-up involving Gaussian integrals, and rather just want to learn how to find propagators, I encourage you to skip ahead...

### 5.1.1 Motivation

Our "motivation" is to consider the integral

$$\begin{aligned}\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}dx\,x^{2n}e^{-\alpha x^2/2} &= \sqrt{\frac{\alpha}{2\pi}}(-1)^n 2^n\frac{d^n}{d\alpha^n}\int_{-\infty}^{\infty}dx\,e^{-\alpha x^2/2}\\ &= \sqrt{\frac{\alpha}{2\pi}}(-1)^n 2^n\frac{d^n}{d\alpha^n}\sqrt{\frac{2\pi}{\alpha}} = \frac{(2n-1)\cdot(2n-3)\ldots 3\cdot 1}{\alpha^n}\end{aligned} \tag{5.1}$$

Here, in a certain sense, $1/\alpha$ can be interpreted as a "propagator." We can view this integral as an expectation value $\langle x\rangle$ with "partition function" $Z = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}dx e^{-\alpha x^2/2}$. The value of $\langle x^{2n}\rangle = \langle x\ldots x\rangle$ is given by the number of times we can pair up the distinct $x$ factors, with each pair multiplied by a factor of $1/\alpha$.

We can also evaluate this integral by introducing a "source" $J$:

$$\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}dx\,e^{-\alpha x^2/2+Jx} = \sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}dx\,e^{-\alpha(x-J/\alpha)^2/2}e^{J^2\alpha/2} = e^{J^2/2\alpha} \tag{5.2}$$

so

$$\begin{aligned}\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}dx\,x^{2n}e^{-\alpha x^2/2} &= \sqrt{\frac{\alpha}{2\pi}}\frac{d^{2n}}{dJ^{2n}}\int_{-\infty}^{\infty}dx\,e^{-\alpha x^2/2+Jx}\Big|_{J=0}\\ &= \frac{d^{2n}}{dJ^{2n}}e^{J^2/2\alpha}\Big|_{J=0}\end{aligned} \tag{5.3}$$

It's easiest to evaluate this using the power series expansion of $e^{J^2/2\alpha}$. Clearly the differentiation picks out the $n^{th}$ term, $\frac{1}{n!}(J^2/2\alpha)^n$, and we obtain

$$\sqrt{\frac{\alpha}{2\pi}}\int_{-\infty}^{\infty}dx\,x^{2n}e^{-\alpha x^2/2} = \frac{1}{\alpha^n}\frac{2n!}{2^n n!} = \frac{(2n-1)(2n-3)\ldots 3\cdot 1}{\alpha^n} \tag{5.4}$$

as before.

We can also consider the multi-dimensional version:

$$\int dx_1\ldots dx_m e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x}} \tag{5.5}$$

with $A$ a symmetric positive-definite $m \times m$ matrix. This integral can be performed by diagonalising $A$ with an orthogonal matrix $U$: $\mathbf{x} \to \mathbf{y} = U\mathbf{x}$, $A \to UAU^t$. The product $\mathbf{x} \cdot A\mathbf{x} \equiv x_i A_{ij} x_j \to y_i^2 \lambda_i$, where $\lambda_i$ are the eigenvalues of $A$. Hence we obtain

$$\int dx_1 \ldots dx_m e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x}} = \sqrt{\frac{(2\pi)^m}{\lambda_1 \ldots \lambda_m}} = \frac{(2\pi)^{m/2}}{(\det A)^{1/2}} \tag{5.6}$$

We can again introduce a source $\mathbf{J}$ giving us the integral

$$\int dx_1 \ldots dx_m e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x}+\mathbf{J}\cdot\mathbf{x}} \tag{5.7}$$

We "complete the square" using the fact that $A$ is symmetric and hence too is its inverse:

$$-\frac{1}{2}\mathbf{x}^t A\mathbf{x} + \mathbf{J}^t\mathbf{x} = -\frac{1}{2}(\mathbf{x}^t - \mathbf{J}^t A^{-1})A(\mathbf{x} - A^{-1}\mathbf{J}) + \frac{1}{2}\mathbf{J}^t A^{-1}\mathbf{J} \tag{5.8}$$

We immediately find

$$\int dx_1 \ldots dx_m e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x}+\mathbf{J}\cdot\mathbf{x}} = \frac{(2\pi)^{m/2}}{(\det A)^{1/2}} e^{\frac{1}{2}\mathbf{J}^t A^{-1}\mathbf{J}} \tag{5.9}$$

We can now evaluate

$$\frac{(\det A)^{1/2}}{(2\pi)^{m/2}} \int dx_1 \ldots dx_m \, x_{i_1} \ldots x_{i_{2n}} e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x}} = \frac{(\det A)^{1/2}}{(2\pi)^{m/2}} \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_{2n}}} \int dx_1 \ldots dx_m e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x}+\mathbf{J}\cdot\mathbf{x}} \tag{5.10}$$

We need to work out

$$\left. \frac{\partial}{\partial J_{i_1}} \cdots \frac{\partial}{\partial J_{i_{2n}}} e^{\frac{1}{2}\mathbf{J}^t A^{-1}\mathbf{J}} \right|_{\mathbf{J}=0} \tag{5.11}$$

Let's write $\mathbf{J}^t A^{-1}\mathbf{J} = J_i A^{-1}_{ik} J_k$. As $A^{-1}$ is symmetric differentiation with respect to $J_{i_{2n}}$ brings down a factor of $A^{-1}_{i_{2n}k} J_k$. The next differentiation hits this term, giving $A^{-1}_{i_{2n}i_{2n-1}}$, and the exponential again, leaving a prefactor of $A^{-1}_{i_{2n}k} J_k A^{-1}_{i_{2n-1}l} J_l$. Then the next differentiation brings down a $A^{-1}_{i_{2n-2}k} J_k$ when it hits either exponential, and in the second term produces $A^{-1}_{i_{2n}i_{2n-2}} A^{-1}_{i_{2n-1}l} J_l$ and $A^{-1}_{i_{2n}k} A^{-1}_{i_{2n-1}i_{2n-2}}$. Continuing in this fashion we see that we will end up with all conceivable pairings $A^{-1}_{ij}$ (note also that if we had an odd number of differentiations there would always be a $J_i$ left over to cancel everything). The final result is

$$\langle x_{i_1} \ldots x_{i_{2n}} \rangle \equiv \frac{(\det A)^{1/2}}{(2\pi)^{m/2}} \int dx_1 \ldots dx_m \, x_{i_1} \ldots x_{i_{2n}} e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x}} = \sum_{\text{pairings}} A^{-1}_{i_1 i_2} \ldots A^{-1}_{i_{2n-1}i_{2n-2}} \tag{5.12}$$

Finally we have something we can attach a firm graphical interpretation too: we join up the $2n$ points $x_{i_r}$ in all possible ways with lines, with the line joining $x_i$ and $x_j$ weighted by a "propagator" factor $A^{-1}_{ij}$. It is nice to say that $A^{-1}$ is a "Green's function" for the matrix $A$ at this point.

We should also discuss the complex-valued case - however we merely refer you to Greiner's book and quote the formulae:

$$\int dz_1 dz_1^\dagger \dots dz_m dz_m^\dagger e^{-\mathbf{z}^\dagger A \mathbf{z}} = \frac{(2\pi)^m}{(\det A)} \tag{5.13}$$

$$\int dz_1 dz_1^\dagger \dots dz_m dz_m^\dagger e^{-\mathbf{z}^\dagger A \mathbf{z} + \mathbf{J}^\dagger \mathbf{z} + \mathbf{z}^\dagger \mathbf{J}} = \frac{(2\pi)^m}{(\det A)} e^{\mathbf{J}^\dagger A^{-1} \mathbf{J}} \tag{5.14}$$

The key point to note is the disappearance of the factor of 1/2 in the exponentials. These formulae are relevant for complex scalar fields and Dirac spinor fields, although we won't really discuss why.

### 5.1.2 The simplest field theory example

Graduating now to the example of a real scalar field we have the partition function

$$Z_0(J) = \int \mathcal{D}\varphi \exp\left( i \int d^4x \left( -\frac{1}{2}\varphi(-\partial_\mu \partial^\mu + m^2)\varphi + J\varphi \right) \right) \tag{5.15}$$

We assume our integration measure is normalised so that $Z_0(0) = 1$. Either by extrapolating from our previous results or discretising our field theory we can convince ourselves that the operator $-\partial_\mu \partial^\mu + m^2$ is the equivalent of the matrix $A$ (or the number $\alpha$), and that we will need its inverse (Green's function, or *propagator*). We also note that we now have a factor of $i$: but we'll ignore all issues of convergence and analytical continuation and insert this into our Gaussian integral formulae and treat it no differently to any other number.

The easiest way to invert the differential operator is to pass to Fourier space, so that $(-\partial_\mu \partial_\mu + m^2)\varphi(x) \to (k^2 + m^2)\widetilde{\varphi}(k)$, which tells us that the inverse we are looking for is $\widetilde{\Delta}(k) = 1/(k^2 + m^2)$, or

$$\Delta(x - x') = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-x')}}{k^2 + m^2 - i\varepsilon} \tag{5.16}$$

(the $i\varepsilon$ is inserted to allow the use of contour integration, and in practice we will always be interested in taking the $\varepsilon \to 0$ limit). It can be easily checked that $(-\partial_\mu \partial^\mu + m^2)\Delta(x - x') = \delta^4(x - x')$.

To evaluate $Z(J)$ explicitly it is also convenient to write everything in terms of Fourier transforms. (This section isn't finished. Go read Srednicki chapter 8.)

## 5.2 Finding propagators

If we write the equation of motion satisfied by some (free) field $\varphi_A$ as $D_{AB}\varphi_B$ then the propagator $\Delta_{AB}$ for that field is the Green's function for the differential operator $D_{AB}$:

$$D_{AB}\Delta_{BC}(x - x') = \delta_{AC}\delta^4(x - x') \tag{5.17}$$

In practice one obtains $\Delta_{BC}$ by writing this defining equation in momentum space.

We will be most interested in obtaining the form of the propagator (and accompanying Feynman rule phase factor) directly from the Lagrangian. We will now explain how in general one can achieve this.

Obviously one looks at only the part of the Lagrangian corresponding to the free field (consisting of derivative and mass terms). Integration by parts allows us to write it in the form

$$\mathcal{L}_{free} = -\frac{1}{2}\varphi_A D_{AB} \varphi_B \tag{5.18}$$

for real fields, or

$$\mathcal{L}_{free} = -\varphi_A^{\dagger} D_{AB} \varphi_B \tag{5.19}$$

for complex fields (this is very schematic, for instance † refers to any sort of conjugation used). Writing the above in momentum space (using the shortcut $\partial_\mu \to ik_\mu$) allows for the propagator to be easily read off. Finally, we note that in the path integral formalism the Lagrangian will be multiplied by a factor of $i$: the propagator hence comes with a factor of $1/i$.

We will now look at some examples.

### 5.2.1 Real scalar field

We have

$$\mathcal{L}_{free} = -\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 = -\frac{1}{2}\varphi(-\partial_\mu\partial^\mu + m^2)\varphi \tag{5.20}$$

ignoring boundary terms. In momentum space

$$(-\partial_\mu\partial^\mu + m^2)\varphi \to (k^2 + m^2)\varphi \tag{5.21}$$

and the propagator is

$$\frac{1}{k^2 + m^2 - i\varepsilon} \tag{5.22}$$

This enters Feynman diagrams as

$$\frac{1}{i}\frac{1}{k^2 + m^2 - i\varepsilon} \tag{5.23}$$

### 5.2.2 Complex scalar field

We have

$$\mathcal{L}_{free} = -\partial_\mu\varphi^\dagger\partial^\mu\varphi - m^2\varphi^2 = -\varphi^\dagger(-\partial_\mu\partial^\mu + m^2)\varphi \tag{5.24}$$

We see that the propagator has the same form as that for a real scalar field.

### 5.2.3 Dirac spinor

We have

$$\mathcal{L} = -\overline{\Psi}(-i\partial\!\!\!/ + m)\Psi \tag{5.25}$$

Passing to momentum space,

$$(-i\partial\!\!\!/ + m)\Psi(x) \to (p\!\!\!/ + m)\widetilde{\Psi}(p) \tag{5.26}$$

The inverse of $p\!\!\!/ + m$ can be seen to be

$$\widetilde{S}(p) = \frac{-p\!\!\!/ + m}{p^2 + m^2 - i\varepsilon} \tag{5.27}$$

This is the Dirac propagator: it enters Feynman diagrams as

$$\frac{1}{i}\frac{-p\!\!\!/ + m}{p^2 + m^2 - i\varepsilon} \tag{5.28}$$

### 5.2.4 Electromagnetic field

This is a bit subtle, because of gauge freedom. We have

$$\begin{aligned}\mathcal{L}_{free} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} &= -\frac{1}{2}\partial^\mu A^\nu \partial_\mu A_\nu + \frac{1}{2}\partial^\mu A^\nu \partial_\nu A_\mu \\ &= -\frac{1}{2}A_\mu(-g^{\mu\nu}\partial_\sigma\partial^\sigma + \partial^\mu\partial^\nu)A_\nu\end{aligned} \tag{5.29}$$

Passing to momentum space

$$(-g^{\mu\nu}\partial_\sigma\partial^\sigma + \partial^\mu\partial^\nu)A_\nu(x) \to (k^2 g^{\mu\nu} - k^\mu k^\nu)\widetilde{A}_\nu(x) \tag{5.30}$$

The operator $P^{\mu\nu} = g^{\mu\nu} - k^\mu k^\nu / k^2$ is a projector onto the subspace of momentum space orthogonal to $k^\nu$. As $P^{\mu\nu}k_\nu = 0$ it has a zero eigenvalue, and so is not invertible on the whole of momentum space. On the subspace it projects onto, $P^{\mu\nu}$ is the identity. We therefore have to impose a condition restricting to this subspace. This condition is clearly that $k_\mu \widetilde{A}^\mu(k) = 0$, which in real space corresponds to Lorenz gauge, $\partial_\mu A^\mu = 0$. Having imposed Lorenz gauge we can invert $k^2 P^{\mu\nu}$ to find the propagator:

$$\widetilde{\Delta}^{\mu\nu}(k) = \frac{1}{k^2 - i\varepsilon}P^{\mu\nu} \tag{5.31}$$

This can be further simplified by noting that the $k^\nu$ term in $P^{\mu\nu}$ won't contribute, as it will always contract with $\widetilde{A}_\nu$ in Lorenz gauge or else with $\widetilde{J}_\nu$. In the latter case current conservation means $\partial_\mu J^\mu = 0$ and so $k_\mu \widetilde{J}^\mu = 0$. We therefore find the photon propagator in Feynman gauge:

$$\widetilde{\Delta}^{\mu\nu}(k) = \frac{g^{\mu\nu}}{k^2 - i\varepsilon} \tag{5.32}$$

This enters Feynman diagrams as

$$\frac{1}{i}\frac{g^{\mu\nu}}{k^2 - i\varepsilon} \tag{5.33}$$

## 5.3 Interactions

### 5.3.1 Motivation

For free field theory, all our integrals are essentially Gaussian, and so solvable. In an interacting field theory we will not be able to compute the path integrals explicitly: however we can develop a perturbative approach that will work wonders.

Again we motivate our discussion by starting with a simpler example. We consider the integral

$$Z(J) = \int_{-\infty}^{\infty} dx \, e^{-\alpha x^2/2 + g x^3/3! + Jx} \tag{5.34}$$

This cannot be computed explicitly but we can write it as

$$Z(J) = \exp\left(\frac{g}{3!}\left(\frac{\partial}{\partial J}\right)^3\right) \int_{-\infty}^{\infty} dx \, e^{-\alpha x^2/2 + Jx} = \sqrt{\frac{2\pi}{\alpha}} \exp\left(\frac{g}{3!}\left(\frac{\partial}{\partial J}\right)^3\right) e^{J^2/2\alpha} \tag{5.35}$$

The overall factor of $\sqrt{\frac{2\pi}{\alpha}}$ can be ignored. We now expand both exponentials to generate a joint power series in $g$ and $J$:

$$\begin{aligned} &\left(1 + \frac{\lambda}{3!}\left(\frac{\partial}{\partial J}\right)^3 + \frac{1}{2}\frac{\lambda^2}{3!^2}\left(\frac{\partial}{\partial J}\right)^6 + \dots\right)\left(1 + \frac{1}{2\alpha}J^2 + \frac{1}{2}\frac{1}{4\alpha^4}J^4 + \frac{1}{3!}\frac{1}{8\alpha^3}J^6 + \dots\right) \\ &= 1 + \frac{1}{2}\frac{J^2}{\alpha} + \frac{1}{2^3}\frac{J^4}{\alpha^2} + \dots + \frac{1}{2}g\frac{J}{\alpha^2} + \frac{5}{12}g\frac{J^3}{\alpha^3} + \dots \end{aligned} \tag{5.36}$$

We can interpret every term graphically in the following way: each factor of $1/\alpha$ corresponds to a line. The sources $J$ correspond to blobs at the end of a line, while the coupling $g$ gives a vertex joining three lines. Some examples are shown in figure 2. The numerical factors in front of each term are known as symmetry factors of the diagram and correspond to ways we can rearrange lines and sources without changing the diagram.

At this stage the graphical interpretation may seem a little ad hoc - how are we supposed to know the composition of the rightmost diagram in 2, for example? To better see where the rules for the diagrams come from, we need to move on to more complicated integrals.

Before doing so we note that the correlation functions $\langle x^n \rangle$ can now be written as

$$\langle x^n \rangle = \left(\frac{\partial}{\partial J}\right)^n Z(J)\Big|_{J=0} \tag{5.37}$$

and is given by simply "removing" all the $J$s from any term with $n$ $J$ factors. Later on we will restrict this by looking only at *connected* diagrams.

The logical next step is to consider

$$Z(J) = \int dx_1 \dots dx_m e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x} + \frac{g}{3!}\mathbf{x}^3 + \mathbf{J}\cdot\mathbf{x}} \tag{5.38}$$

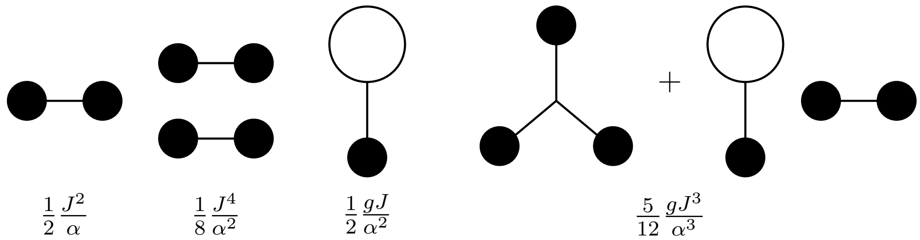


Figure 2: Example diagrams for expansion of one-dimensional integral. The symmetry factors for the constituent parts of the rightmost diagram are $\frac{1}{3!}$ for the subdiagram with 3 sources, and $\frac{1}{2}\frac{1}{2}$ for the disconnected subdiagram, which should be thought of as a product (not a sum) of its connected components.

where by $\mathbf{x}^3$ we mean $(x_1)^3 + \cdots + (x_m)^3$ (I couldn't think of a better notation). This time around we can write

$$Z(J) = \exp\left(\frac{g}{3!}\sum_i \left(\frac{\partial}{\partial J_i}\right)^3\right) \int dx_1 \ldots dx_m dx_m e^{-\frac{1}{2}\mathbf{x}\cdot A\mathbf{x} + \mathbf{J}\cdot\mathbf{x}} \tag{5.39}$$

Up to an overall scaling we find

$$Z(J) \propto \exp\left(\frac{g}{3!}\sum_i \left(\frac{\partial}{\partial J_i}\right)^3\right) \exp\left(\frac{1}{2}\mathbf{J}\cdot A^{-1}\mathbf{J}\right) \tag{5.40}$$

Expanding in a power series we have

$$\begin{aligned}
&\left(1 + \frac{g}{3!}\sum_i \left(\frac{\partial}{\partial J_i}\right)^3 + \frac{1}{2}\frac{g^2}{3!^2}\sum_i \left(\frac{\partial}{\partial J_i}\right)^6 + \ldots\right)\left(1 + \frac{1}{2}\sum_{i,k} J_i A^{-1}_{ik} J_k + \frac{1}{2}\frac{1}{4}\sum_{i,k,l,m} J_i A^{-1}_{ik} J_k J_l A^{-1}_{lm} J_m + \ldots\right) \\
&= 1 + \frac{1}{2}\sum_{i,k} J_i A^{-1}_{ik} J_k + \frac{1}{8}\sum_{i,k,l,m} J_i A^{-1}_{ik} J_k J_l A^{-1}_{lm} J_m + \ldots \\
&\quad \cdots + \frac{1}{2}\sum_{i,k} J_i A^{-1}_{ik} A^{-1}_{kk} + \frac{g}{4}\sum_{i,k,l,m} J_i A^{-1}_{ik} A^{-1}_{kk} J_l A^{-1}_{lm} J_m + \frac{g}{6}\sum_{i,k,l,m} J_i A^{-1}_{im} J_k A^{-1}_{km} J_l A^{-1}_{lm} + \ldots
\end{aligned} \tag{5.41}$$

Our graphical analogy then becomes: the sum $\sum_i J_i$ is represented by a blob at the end of a line, while a sum $g\sum_k$ corresponds to a vertex joining three lines (it should be clear that a derivative $(\frac{\partial}{\partial J_i})^3$ gives three Kronecker deltas when it hits a term in the right-hand power series, and effectively "joins" three $A^{-1}_{jk}$ indices together). Sources and vertices are joined together by propagators $A^{-1}_{ik}$. Interpreting $i$ as indexing the positions of a lattice, we can think of $J_i$ as representing particle creation or annihilation at $i$, and $A^{-1}_{ik}$ representing particle propagation from $i$ to $k$. The sums over the indices then signify that these processes can happen anywhere on the lattice.

**Exercise:** Verify that the terms shown in equation (5.41) do indeed emerge as shown, and compare their graphical representation with figure 2.

After this example it is almost trivial to finally generalise to a quantum field theory. For a real scalar field, we have

$$Z(J) = \int \mathcal{D}\varphi \exp\left(i \int d^4x \left(-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{g}{3!}\varphi^3 + J\varphi\right)\right) \tag{5.42}$$

which we can write as

$$\begin{aligned} Z(J) &= \exp\left(i\frac{g}{3!}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right) \int \mathcal{D}\varphi \exp\left(i \int d^4x \left(-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi - \frac{1}{2}m^2\varphi^2 + J\varphi\right)\right) \\ &= \exp\left(i\frac{g}{3!}\int d^4x \left(\frac{1}{i}\frac{\delta}{\delta J(x)}\right)^3\right) \exp\left(\frac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right) \end{aligned} \tag{5.43}$$

We see that in place of sums we now have integrals, and the derivatives with respect to $J_i$ have been replaced by functional derivatives with respect to $J(x)$. Expanding both exponentials we obtain a power series in $J(x)$ and $g$. The only slight difference is that we now have various factors of $i$: to work these out, we note that the term of order $g^V$ containing $P$ propagators will have $E = 2P - 3V$ sources, and carry a factor of $i^V i^{-3V} i^P = i^{E+V-P}$.

The graphical representation of our expansion then consists of sources $i\int d^4xJ(x)$, vertices $ig\int d^4x$ and propagators $\frac{1}{i}\Delta(x-y)$, along with accompanying numerical factors.

In practice, we are only interested in connected diagrams. To see this, let $D$ denote some general diagram appearing in the expansion, which can possibly be decomposed into a product of connected subdiagrams $C_I$, each of which occurs $n_I$ times:

$$D = \prod_I \frac{1}{n_I!}C_I \tag{5.44}$$

For instance, we have previously encountered (see the rightmost diagram in figure 2) the diagram corresponding to

$$i^{1+3-3}\frac{g}{4}\int d^4xd^4yd^4zd^4wJ(x)\Delta(x-y)\Delta(y-y)J(z)\Delta(z-w)J(w) \tag{5.45}$$

which can be written as a product of

$$i^{1+1-2}\frac{g}{2}\int d^4xd^4yJ(x)\Delta(x-y)\Delta(y-y) \tag{5.46}$$

and

$$i^{2-1}\frac{1}{2}\int d^4zd^4wJ(z)\Delta(z-w)J(w) \tag{5.47}$$

Note that $C_I$ includes the symmetry factor for the subdiagram and $n_I$ is an additional symmetry factor corresponding to interchanges of entire subdiagrams. Clearly $Z(J)$ is given by the sum of all diagrams $D$:

$$Z(J) = \sum_I \prod_I \frac{1}{n_I!} C_I = \prod_I \sum_I \frac{1}{n_I!} C_I = \prod_I \exp(C_I) = \exp\left(\sum_I C_I\right) \tag{5.48}$$

and hence in fact by the exponential of the sum of all connected diagrams. We denote this by $iW(J) \equiv \sum_I C_I$. Also, to enforce $Z(J=0)=1$, we divide by the exponential of the sum of all connected diagrams with no sources, equivalent to leaving those out of the sum altogether.

Of course, we are really interested not in $Z(J)$ itself but in the (time-ordered) correlation functions we can use it to measure:

$$\langle 0|T\varphi(x_1)\dots\varphi(x_n)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\dots\frac{1}{i}\frac{\delta}{\delta J(x_n)} Z(J)\bigg|_{J=0} \tag{5.49}$$

Note that

$$\langle 0|T\varphi(x_1)\dots\varphi(x_n)|0\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\dots\frac{1}{i}\frac{\delta}{\delta J(x_n)} iW(J)\bigg|_{J=0} + \text{disconnected terms} \tag{5.50}$$

The disconnected terms are of the form $\left(\frac{1}{i}\frac{\delta}{\delta J(x_{i_1})}\dots\frac{1}{i}\frac{\delta}{\delta J(x_{i_j})} iW(J)\right)\dots\left(\frac{1}{i}\frac{\delta}{\delta J(x_{i_r})}\dots\frac{1}{i}\frac{\delta}{\delta J(x_{i_s})} iW(J)\right)$, and do not constitute true contributions to whatever $n$-particle amplitude we are interested in. Thus we define the connected correlation function by

$$\langle 0|T\varphi(x_1)\dots\varphi(x_n)|0\rangle_C = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\dots\frac{1}{i}\frac{\delta}{\delta J(x_n)} iW(J)\bigg|_{J=0} \tag{5.51}$$

Obviously to find this we take all connected diagrams with $n$ sources, and then remove these sources by the $n$ functional derivatives (the effect is to replace $\int d^4x J(x)$ with $\delta(x-x_i)$). These derivatives match up to the sources in $n!$ different ways, some of which may be identical.

### 5.3.2 LSZ reduction and the view from momentum space

The formula which allows us to turn connected correlation functions into actual amplitudes is the LSZ reduction formula. For a process with $n$ incoming particles and $n'$ outgoing, we have

$$\begin{aligned}\langle f|i\rangle = i^{n+n'} \int d^4x_1 e^{ik_1x_1}(-\partial_1^\mu\partial_{1\mu}+m^2)\dots \\ \dots\int d^4x_{1'} e^{-ik_{1'}x_{1'}}(-\partial_{1'}^\mu\partial_{1'\mu}+m^2)\dots\langle 0|T\varphi(x_1)\dots\varphi(x_{1'})\dots|0\rangle_C\end{aligned} \tag{5.52}$$

Obviously we are glossing over all the details that go into the derivation and validity of this expression (see Srednicki chapter 5). Note as well that it the precise form of the formula will be different for spinor and vector fields.

What happens when we substitute in for $\langle 0|T\varphi(x_1)\ldots\varphi(x_{1'})\ldots|0\rangle_C$? Each external particle comes with a propagator $\frac{1}{i}\Delta(x_i - y)$. The factors of $i$ cancel out the $i^{n+n'}$, and the Klein-Gordon operators act on each propagator to give delta functions, $\delta^4(x_i - y)$. We then integrate over all $\int d^4x_1 \ldots \int d^4x_{1'}$ using these delta functions. We are left with integrals $ig\int d^4y$ over each vertex; we also convert the internal propagators into momentum integrals using

$$\frac{1}{i}\Delta(y-z) = \frac{1}{i}\int \frac{d^4k}{(2\pi)^4}\frac{e^{ik(y-z)}}{k^2+m^2-i\varepsilon} \tag{5.53}$$

We can gather together all the terms of the form $e^{iy\sum k}$ where the sum over $k$s may include internal propagator momenta or external particle momenta. Clearly integrating over the remaining spatial variables generates further delta functions which have the effect of conserving momentum at each vertex. We can use these delta functions to fix the momentum on some but not necessarily all of the internal lines: we may be left with a certain number of loop momenta which we must still integrate over.

In fact for a diagram with $I$ internal lines and $V$ vertices we have $L = I - V + 1$ loops (follows from Euler's formula $v - e + f = 2$, look it up!). It follows we can integrate over $V$ vertices and $I - L = V - 1$ momenta, generating an overall factor of $(2\pi)^{4V}/(2\pi)^{4(V-1)} = (2\pi)^4$ from the resulting delta functions. As we will also be left with an overall delta function which conserves momentum, we can finally define the amplitude $\mathcal{T}$ for the process via

$$\langle f|i\rangle = (2\pi)^4\delta^4(k_{in} - k_{out})i\mathcal{T} \tag{5.54}$$

The value of $i\mathcal{T}$ is then given by the momentum space Feynman rules, which are clearly generated by this process. The general idea is of course valid for more complicated theories than the scalar field theory we were using as an example. The major difference will be that the factor corresponding to external lines will be changed for fields with higher spin, as incoming and outgoing particles will enter the LSZ formula with eg appropriate spinor or vector factors.

## 5.4 Finding interaction vertices

### 5.4.1 Interactions without derivatives

We now want to determine how to find the value of an arbitrary interaction vertex, given some interaction Lagrangian for any number of scalar, spinor and vector fields. We note that the general procedure is to replace each field with a functional derivative with respect to the appropriate source. These derivatives then act on sources in the exponential expansion for the free field partition function, to produce a power series in the interaction vertex. It is clear that if a field $\varphi_A$ occurs $n$ times in the interaction term then the derivative $(\delta/\delta J_A)^n$ will produce a factor of $n!$ from all the possible ways it can hit sources in the expansion.

Our first rule then is that for a combination of $r$ arbitrary fields interacting via

$$\lambda\varphi_{A_1}^{n_1}\dots\varphi_{A_r}^{n_r} \tag{5.55}$$

(with appropriate contractions to give a Lorentz scalar) we will obtain a vertex factor of

$$i\lambda n_1!\dots n_r! \tag{5.56}$$

The factor of $i$ follows directly from our path integrals. As the coupling may in fact carry indices we should consider interactions including terms like

$$\lambda_{AB}\varphi^A\varphi^B \tag{5.57}$$

This would contribute a vertex factor of

$$i\lambda_{AB} \tag{5.58}$$

which would have to contract with the indices on a propagator or external line (so whether or not we write the vertex factor as carrying upper or lower indices is actually unimportant. Also, there would be a factor of 2 if $\varphi^A$ and $\varphi^B$ are the same field.) Generalisation of this example should be obvious.

### 5.4.2 Examples

There aren't really any non-trivial examples. All scalar field interactions of the form

$$\frac{g_n}{n!}\varphi^n \tag{5.59}$$

contribute

$$ig_n \tag{5.60}$$

For complex fields we must remember that we treat the field and its conjugate as independent, hence for a complex scalar field $\varphi$ the interaction

$$\lambda(\varphi^\dagger\varphi)^2 \tag{5.61}$$

contributes

$$i2!2!\lambda \tag{5.62}$$

As an example of an interaction with multiple indices, we can consider that between a Dirac spinor field $\Psi$ and a massless vector field $A^\mu$:

$$e\overline{\Psi}\gamma_\mu A^\mu\Psi \tag{5.63}$$

In Feynman diagrams this contributes

$$ie\gamma^\mu \tag{5.64}$$

with the vector and spinor indices of $\gamma^\mu$ contracted suitably.

Finally, consider the following interaction between a massless vector field $A^\mu$ and a complex scalar field $\varphi$:

$$-e^2 A^\mu A_\mu \varphi^\dagger \varphi = -e^2 g_{\mu\nu} A^\mu A^\nu \varphi^\dagger \varphi \tag{5.65}$$

We have two vector fields, giving a factor of 2!, but we treat $\varphi$ and $\varphi^\dagger$ as independent. The vertex factor appearing in Feynman diagrams is

$$-2ie^2 g_{\mu\nu} \tag{5.66}$$

and the vector indices must contract with indices on either the vector field propagator or its external lines.

### 5.4.3 Interactions with derivatives

In the case that the interaction term includes a derivative acting on one of the fields we should be a little bit more careful. If we trace through the derivation of momentum space Feynman rules from the LSZ formula we see that

$$\partial_\mu \varphi_A(x) \to \partial_\mu \frac{\delta}{\delta J_A(x)} \to \partial_\mu \Delta(x-y) \tag{5.67}$$

Now, when we compute a connected correlation function this will be either an internal propagator, or one connecting an external line with an internal vertex. In the former case, we write the propagator as a Fourier transform and the derivative gives us a factor of $ik$. If it is a propagator corresponding to an external line it gets transformed into a delta function by the wave operators in the LSZ formula, leaving us with

$$\int d^4x_i e^{ik_i x_i} \partial_\mu \delta^4(x_i - x) \tag{5.68}$$

(for convenience assume all particles are ingoing). Now $\partial_\mu$ is a derivative with respect to $x$, but we can write

$$\frac{\partial}{\partial x^\mu} \delta^4(x_i - x) = -\frac{\partial}{\partial x_{i\mu}} \delta^4(x_i - x) \tag{5.69}$$

and integrate by parts to obtain a factor of $ik_1$.

We see that going to momentum space Feynman rules effectively means making the replacement

$$\partial_\mu \varphi \to ik_\mu \tag{5.70}$$

(suggested immediately by the Fourier transform of $\varphi$ anyway). We conclude that we should make this replacement to obtain momentum space Feynman rules for interactions with derivatives. The only thing to be careful for is that we must take into account that the possibilities of the derivative hitting any of the lines coming into that vertex, for the same reason that we previously generated factorial terms. Also note that as for each derivative we will have a distinguished line in the vertex, we will not have the same combinatorial factors as before. This is developed in the first example we are now going to look at.

### 5.4.4 Examples

Consider a theory of one real scalar field with interaction term

$$g\varphi\partial^\mu\varphi\partial_\mu\varphi \tag{5.71}$$

We make the replacements $\partial_\mu\varphi \to ik$ for a vertex joining three lines of momenta $k_1, k_2, k_3$, all treated as incoming. If there were no derivative terms we would generate a 3! corresponding to the 3! ways of arranging the lines coming in to the vertex. Instead we now distinct possibilities where the derivatives give us $(ik_1)(ik_2)$, $(ik_1)(ik_3)$ or $(ik_2)(ik_3)$. Each of these cases is multiplied by 2! as we can freely swap the two lines acted on by the derivatives. Thus,

$$g\varphi\partial^\mu\varphi\partial_\mu\varphi \to -2g(k_1k_2 + k_1k_3 + k_2k_3) \tag{5.72}$$

As the momenta are incoming, we have $k_1 + k_2 + k_3 = 0$. This gives a vertex factor

$$ig(k_1^2 + k_2^2 + k_3^2) \tag{5.73}$$

As a second example, the counterterms for a real scalar field include the term

$$-\frac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi \tag{5.74}$$

which is equivalent under an integration by parts to

$$+\frac{1}{2}(Z_\varphi - 1)\varphi\partial_\mu\partial^\mu\varphi \tag{5.75}$$

This joins two lines; by momentum conservation each must have momentum $k$ and so we obtain a 2! and a momentum space factor in Feynman diagrams of

$$-i(Z_\varphi - 1)k^2 \tag{5.76}$$

For a complex scalar field the equivalent counterterm is

$$+\frac{1}{2}(Z_\varphi - 1)\varphi^\dagger\partial_\mu\partial^\mu\varphi \tag{5.77}$$

The derivatives act on the $\varphi$ field, which we treat as being different from its conjugate, so that the Feynman diagram factor is again

$$-i(Z_\varphi - 1)k^2 \tag{5.78}$$

Similarly the counterterm for a Dirac spinor is

$$i(Z_\Psi - 1)\overline{\Psi}\not{\partial}\Psi \tag{5.79}$$

which produces a momentum space factor in Feynman diagrams of

$$-i(Z_\Psi - 1)\not{p} \tag{5.80}$$

# 6 Theories

We now want to describe the Lagrangians and Feynman rules for a number of different theories.

## 6.1 General features of momentum space Feynman rules

A given (connected) diagram is composed of external lines, internal lines (propagators) and vertices. The value of a diagram is given by the product of the appropriate factors corresponding to each constituent part.

External lines correspond to internal and outgoing particles, which are on-shell. Momentum is always conserved at vertices (we think of momentum as flowing along the lines of the diagram - draw arrows to guide this process). A diagram with $n$ loops will have $n$ loop momenta whose values are not fixed by momentum conservation (these correspond to virtual particles). We integrate over each of these momenta $l_i$ with measure $d^d l_i/(2\pi)^d$. Loop diagrams may be accompanied by symmetry factors, which we should divide by.

## 6.2 $\varphi^3$-theory

The Lagrangian including counterterms is

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 + \frac{1}{6}Z_g g\varphi^3 + \mathcal{L}_{ct} \\ \mathcal{L}_{ct} &= -\frac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}(Z_m - 1)m^2\varphi^2 + Y\varphi \end{aligned} \tag{6.1}$$

Feynman diagrams are composed of:

1. External lines with value 1
2. Internal lines with momentum $k$ and value
$$\frac{1}{i}\frac{1}{k^2 + m^2 - i\varepsilon} \tag{6.2}$$
3. Vertices joining three-lines, with value
$$iZ_g g \tag{6.3}$$
4. Counterterm vertices (marked on diagrams as a "×" through a line) joining two lines. By momentum conservation both lines will have the same momentum $k$, and the value of the vertex is
$$-i(Z_\varphi - 1)k^2 - i(Z_m - 1)m^2 \tag{6.4}$$
5. The $Y$ counterterm is added to remove tadpole diagrams (consisting of a loop joined to the rest of the diagram by a single line) and can be thought of having already accomplished this.

The mass dimension of the coupling constant is $[g] = (6 - d)/2$ which is dimensionless in $d = 6$.

## 6.3 $\varphi^4$-theory

The Lagrangian including counterterms is

$$\begin{aligned}\mathcal{L} &= -\frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}m^2\varphi^2 - \frac{1}{24}Z_g g\varphi^4 + \mathcal{L}_{ct} \\ \mathcal{L}_{ct} &= -\frac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}(Z_m - 1)m^2\varphi^2\end{aligned} \tag{6.5}$$

The Feynman rules are identical to $\varphi^3$-theory except the interaction vertex now joins four lines.

The mass dimension of the coupling constant is $[g] = 4 - d$ which is dimensionless in $d = 4$.

## 6.4 Complex scalar field

The Lagrangian including counterterms is

$$\begin{aligned}\mathcal{L} &= -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\varphi - \frac{1}{4}Z_g g(\varphi^\dagger\varphi)^2 + \mathcal{L}_{ct} \\ \mathcal{L}_{ct} &= -(Z_\varphi - 1)\partial^\mu\varphi^\dagger\partial_\mu\varphi - (Z_m - 1)m^2\varphi^\dagger\varphi\end{aligned} \tag{6.6}$$

To distinguish the two-types of particles in this theory we label all momenta lines with arrows and "conserve directions of arrows" at vertices. Feynman diagrams then consist of

1. External lines:
   - For each incoming particle with momentum $k$, a line with an arrow pointing towards a vertex, labelled with $+k$.
   - For each outgoing particle with momentum $k$, a line with an arrow pointing away from a vertex, labelled with $+k$.
   - For each incoming anti-particle with momentum $k$, a line with an arrow pointing away from a vertex, labelled with $-k$.
   - For each outgoing anti-particle with momentum $k$, a line with an arrow pointing towards a vertex, labelled with $-k$.
2. Internal lines, with value
$$\frac{1}{i}\frac{1}{k^2 + m^2 - i\varepsilon} \tag{6.7}$$
3. Vertices joining four lines, two with arrows pointing towards the vertex and two with arrows pointing away from the vertex, with value
$$-iZ_g g \tag{6.8}$$
4. Counterterm vertices (marked on diagrams as a "×" through a line) joining two lines. By momentum conservation both lines will have the same momentum $k$, and the value of the vertex is
$$-i(Z_\varphi - 1)k^2 - i(Z_m - 1)m^2 \tag{6.9}$$

## 6.5 Pseudo-scalar Yukawa theory

This theory describes the coupling of a Dirac field to a pseudo-scalar field. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi - \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}M^2\varphi^2 + iZ_g g\varphi\overline{\Psi}\gamma_5\Psi - \frac{1}{4!}Z_\lambda\lambda\varphi^4 + \mathcal{L}_{ct} \\ \mathcal{L}_{ct} &= i(Z_\Psi - 1)\overline{\Psi}\partial\!\!\!/\Psi - (Z_m - 1)m\overline{\Psi}\Psi - \frac{1}{2}(Z_\varphi - 1)\partial^\mu\varphi\partial_\mu\varphi - \frac{1}{2}(Z_M - 1)M^2\varphi^2\end{aligned} \tag{6.10}$$

To distinguish the two-types of fermions (electrons and positrons) we label all momenta lines with arrows and conserve arrows at vertices. We represent spinor propagators with solid lines and scalar propagators with dashed lines. Feynman diagrams consist of:

1. External lines

   - External scalars: for incoming scalars draw an arrow pointing towards a vertex, for outgoing scalars draw an arrow pointing away from a vertex, value 1.
   - For each incoming electron with momentum $p$ and spin $s$ a line with an arrow pointing towards a vertex, labelled with $+p$. Value:
   $$u_s(\mathbf{p}) \tag{6.11}$$
   - For each outgoing electron with momentum $p'$ and spin $s'$ a line with an arrow pointing away from a vertex, labelled with $+p'$. Value:
   $$\overline{u}_{s'}(\mathbf{p}') \tag{6.12}$$
   - For each incoming positron with momentum $p$ and spin $s$ a line with an arrow pointing away from a vertex, labelled with $-p$. Value:
   $$\overline{v}_s(\mathbf{p}) \tag{6.13}$$
   - For each outgoing positron with momentum $p'$ and spin $s'$ a line with an arrow towards a vertex, labelled with $-p'$. Value:
   $$v_{s'}(\mathbf{p}') \tag{6.14}$$

2. For each internal scalar line with momentum $k$:
$$\frac{1}{i}\frac{1}{k^2 + M^2 - i\varepsilon} \tag{6.15}$$

3. For each internal fermionic line with momentum $p$:
$$\frac{1}{i}\frac{-p\!\!\!/ + m}{p^2 + m^2 - i\varepsilon} \tag{6.16}$$

4. For each fermionic loop, a factor of
$$-1 \tag{6.17}$$

5. Vertices joining two spinor and one scalar line: for the spinor lines, there must always be one arrow pointing towards the vertex and one arrow pointing away. The value of the vertex is
$$-Z_g g\gamma_5 \tag{6.18}$$

6. Vertices joining four scalar lines, value:
$$-iZ_\lambda\lambda \tag{6.19}$$

7. All spinor indices must be contracted. This is done along each fermionic line, starting with a $\overline{u}$ or $\overline{v}$ and contracting backwards against the arrows, writing down each term as you come to it. Fermionic loops give a trace of spinor factors.

8. Overall signs of two or more contributing diagrams with external spinor lines are determined by drawing all diagrams in a standard form with left spinor external lines labelled in the same order. Choosing one diagram as a reference with sign +1 the sign of all other diagrams is determined by the ordering of the labels on the right spinor external lines (even/odd permutations give $\pm 1$).

9. Spinor counterterms, joining two spinor lines and as usual marked with a "×" through the lines, with value:
$$-i(Z_\Psi - 1)\not{p} - i(Z_m - 1)m \tag{6.20}$$

10. Scalar counterterms as before, joining two scalar lines, marked with a "×", value:
$$-i(Z_\varphi - 1)k^2 - i(Z_M - 1)M^2 \tag{6.21}$$

## 6.6 Quantum electrodynamics

This theory describes the interaction of a Dirac field with a massless vector field. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= i\overline{\Psi}\not{\partial}\Psi - m\overline{\Psi}\Psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + Z_1 e\overline{\Psi}A_\mu\gamma^\mu\Psi + \mathcal{L}_{ct} \\ \mathcal{L}_{ct} &= i(Z_2 - 1)\overline{\Psi}\not{\partial}\Psi - (Z_m - 1)m\overline{\Psi}\Psi - \frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu}\end{aligned} \tag{6.22}$$

We represent spinors propagators by solid lines, with the usual arrows, and photon propagators by wavy lines (suggestive of the wavy nature of light).

1. External lines

- For each incoming electron with momentum $p$ and spin $s$ a line with an arrow pointing towards a vertex, labelled with $+p$. Value:

$$u_s(\mathbf{p}) \tag{6.23}$$

- For each outgoing electron with momentum $p'$ and spin $s'$ a line with an arrow pointing away from a vertex, labelled with $+p'$. Value:

$$\overline{u}_{s'}(\mathbf{p}') \tag{6.24}$$

- For each incoming positron with momentum $p$ and spin $s$ a line with an arrow pointing away from a vertex, labelled with $-p$. Value:

$$\overline{v}_s(\mathbf{p}) \tag{6.25}$$

- For each outgoing positron with momentum $p'$ and spin $s'$ a line with an arrow towards a vertex, labelled with $-p'$. Value:

$$v_{s'}(\mathbf{p}') \tag{6.26}$$

- For each incoming photon with momentum $k$ and helicity $\lambda$, a wavy line with an arrow pointing towards a vertex, value:

$$\varepsilon_\lambda^{\mu *}(\mathbf{k}) \tag{6.27}$$

- For each outgoing photon with momentum $k'$ and helicity $\lambda'$, a wavy line with an arrow pointing away from a vertex, value:

$$\varepsilon_{\lambda'}^{\mu}(\mathbf{k}') \tag{6.28}$$

2. For each internal fermionic line with momentum $p$:

$$\frac{1}{i}\frac{-\not{p}+m}{p^2+m^2-i\varepsilon} \tag{6.29}$$

3. For each internal photon line with momentum $q$:

$$\frac{1}{i}\frac{g^{\mu\nu}}{q^2-i\varepsilon} \tag{6.30}$$

4. Vertices joining two spinor lines and one photon line, value

$$ieZ_1\gamma^\mu \tag{6.31}$$

5. For each spinor loop, a factor

$$-1 \tag{6.32}$$

6. Spinor indices must be fully contracted; this is achieved as in Yukawa theory by contracting backwards along each spinor line.

7. Vector indices must be fully contracted. The index on the $\gamma^\mu$ in each vertex is contracted with either the $g_{\mu\nu}$ in an attached photon propagator, if the vertex is attached to an internal photon line, or with a photon polarisation vector, if the vertex is attached to an external photon line.

8. Overall sign of a diagram determined by drawing in standard order and looking at permutations of external spinor lines on the right, as before.

9. Spinor counterterms, joining two spinor lines and marked with a "$\times$", value:

$$-i(Z_2 - 1)\not{p} - i(Z_m - 1)m \tag{6.33}$$

10. Photon counterterms, joining two photon lines and marked with a "$\times$", value:

$$-i(Z_3 - 1)(k^2 g^{\mu\nu} - k^\mu k^\nu) \tag{6.34}$$

## 6.7 Scalar electrodynamics

This theory describes a complex scalar field interacting with a massless vector field. The Lagrangian is

$$\begin{aligned}\mathcal{L} &= -\partial^\mu\varphi^\dagger\partial_\mu\varphi - m^2\varphi^\dagger\phi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + iZ_1 e A^\mu\left(\partial_\mu\varphi^\dagger\varphi - \varphi^\dagger\partial_\mu\varphi\right) - Z_4 e^2 A^\mu A_\mu \varphi^\dagger\varphi - \frac{1}{4}Z_\lambda\lambda(\varphi^\dagger\varphi) + \mathcal{L}_{ct}\\ \mathcal{L}_{ct} &= -(Z_2 - 1)\partial^\mu\varphi^\dagger\partial_\mu\varphi - (Z_m - 1)m^2\varphi^\dagger\varphi - \frac{1}{4}(Z_3 - 1)F_{\mu\nu}F^{\mu\nu}\end{aligned} \tag{6.35}$$

The funny looking interaction terms make more sense when written (for a bare Lagrangian) in terms of the covariant derivative $D_\mu\varphi = (\partial_\mu - ieA_\mu)\varphi$: we can then combine the electromagnetic interaction and the scalar kinetic terms into $-(D^\mu\varphi)^\dagger D_\mu\varphi$.

We will refer to the two types of charged scalar particle present in the theory as "selectrons" and "spositrons."[2]

1. External lines

   - For each incoming selectron with momentum $k$ a line with an arrow pointing towards from a vertex, labelled with $k$.
   - For each outgoing selectron with momentum $k'$ a line with an arrow pointing away from a vertex, labelled with $+k'$. Value:
   - For each incoming spositron with momentum $k$ a line with an arrow pointing away from a vertex, labelled with $-k$.

[2]Supersymmetry Street is brought to you by the letter 'S'.

- For each outgoing spositron with momentum $k'$ a line with an arrow towards a vertex, labelled with $-k'$.
- For each incoming photon with momentum $k$ and helicity $\lambda$, a wavy line with an arrow pointing towards a vertex, value:

$$\varepsilon_{\lambda}^{\mu *}(\mathbf{k}) \tag{6.36}$$

- For each outgoing photon with momentum $k'$ and helicity $\lambda'$, a wavy line with an arrow pointing away from a vertex, value:

$$\varepsilon_{\lambda'}^{\mu}(\mathbf{k}') \tag{6.37}$$

2. For each internal scalar line with momentum $k$:

$$\frac{1}{i}\frac{1}{k^2+m^2-i\varepsilon} \tag{6.38}$$

3. For each internal photon line with momentum $q$:

$$\frac{1}{i}\frac{g^{\mu\nu}}{q^2-i\varepsilon} \tag{6.39}$$

4. A vertex joining one photon line and two scalar lines, with one scalar arrow pointing towards the vertex and one pointing away, with value:

$$iZ_1 e(k+k')^{\mu} \tag{6.40}$$

5. A vertex joining two photons lines and two scalar lines, with one scalar arrow pointing towards the vertex and one pointing away, with value:

$$-2iZ_4 e^2 g^{\mu\nu} \tag{6.41}$$

6. A vertex joining four scalar lines, two with arrows pointing towards the vertex and two with arrows pointing away, with value:

$$-iZ_{\lambda}\lambda \tag{6.42}$$

7. Vector indices must be fully contracted. The index on each vertex is contracted with either a photon propagator on an attached internal line or with the photon polarisation on an attached external line.