Some K-theory examples

The purpose of these notes is to compute K-groups of various spaces and outline some useful methods for Ma448: K-theory and Solitons, given by Dr Sergey Cherkis in 2008-09. Throughout our vector bundles are complex and our topological spaces are compact Hausdorff. Corrections/suggestions to cblair[at]maths.tcd.ie.

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1 K-theory

We begin by defining the essential objects in K-theory: the unreduced and reduced K-groups of vector bundles over compact Hausdorff spaces. In doing so we need the following two equivalence relations: firstly we say that two vector bundles E and E' are **stably isomorphic** denoted by $E \approx_S E'$ if they become isomorphic upon addition of a trivial bundle ε^n: $E \oplus \varepsilon^n \approx E' \oplus \varepsilon^n$. Secondly we have a relation $\sim$ where $E \sim E'$ if $E \oplus \varepsilon^n \approx E' \oplus \varepsilon^m$ for some m, n.

Unreduced K-groups: The unreduced K-group of a compact space X is the group $K(X)$ of virtual pairs $E_1 - E_2$ of vector bundles over X, with $E_1 - E_1' = E_2 - E_2'$ if $E_1 \oplus E_2' \approx_S E_2 \oplus E_2'$. The group operation is addition: $(E_1 - E_1') + (E_2 - E_2') = E_1 \oplus E_2 - E_1' \oplus E_2'$, the identity is any pair of the form $E - E$ and the inverse of $E - E'$ is $E' - E$. As we can add a vector bundle to $E - E'$ such that E' becomes trivial, every element of $K(X)$ can be represented in the form $E - \varepsilon^n$.

Reduced K-groups: The reduced K-group of a compact space X is the group $\widetilde{K}(X)$ of vector bundles E over X under the $\sim$-equivalence relation. The group operation is addition, the identity element is the equivalence class of ε^0, and existence of inverses follows as these are vector bundles over a compact space.

These groups are related quite simply: $\widetilde{K}(X)$ is the kernel of the map restricting $K(X)$ to a point. We also have a natural homomorpism sending $E - \varepsilon^n \in K(X)$ to the equivalence class of E in $\widetilde{K}(X)$. This is a surjective map and its kernel consists of elements $E \sim \varepsilon^0$ which means that $E \approx_s \varepsilon^m$ for some m and so the kernel is the subgroup $\{\varepsilon^m - \varepsilon^n\} \approx \mathbb{Z}$ of $K(X)$, hence $K(X) \approx \widetilde{K}(X) \oplus \mathbb{Z}$.

2 Useful tools

2.1 General topological and vector bundle constructions

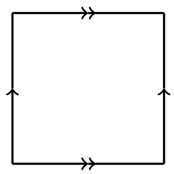
We first define some frequently occurring topological constructions.

Wedge product: The wedge product $X \vee Y$ of two topological spaces X and Y is the space formed by taking the disjoint union of X and Y and identifying one point on X with a point on Y: $X \vee Y = X \sqcup_{pt} Y = X \sqcup Y/(x_0 \sim y_0)$.

Example: The wedge product of two circles is a figure of eight.

Smash product: The smash product $X \wedge Y$ of two topological spaces X and Y is the space formed by taking the product of X and Y and then quotienting by the wedge product of X and Y, $X \wedge Y = X \times Y/X \vee Y = X \times Y/\{x_0\} \times Y \sqcup X \times \{y_0\}$.

Example: The smash product of two circles is a sphere. To see this we first note that $S_1 \times S_1$ gives the torus:



and then quotienting by $S_1 \vee S_1$ means identifying the edges of the square in the above diagram, and so we obtain a sphere.

Example: It holds in general that $S^m \wedge S^n = S^{m+n}$. We can prove this by considering S^n as a cell complex consisting of an n-disc D^n and a point x_0 with attaching morphism sending $\partial D^n \to x_0$. Then $S^n \times S^m$ is a cell complex whose cells are the products of the cells in S^n and S^m: $S^{n+m} = D^n \times D^m \cup D^n \times y_0 \cup x_0 \times D^m \cup x_0 \times y_0 \cup x_0 \times y_0$, and in the middle of this we see we have $S^n \vee S^m = D^n \times y_0 \cup x_0 \times D^m$, so quotienting by the wedge gives the space $D^n \times D^m \cup pt = D^{n+m} \times pt = S^{m+n}$ with attaching morphisms as before.

Cone: The cone CX over X is the space formed by taking the direct product of X and the interval $I = [0, 1]$ and collapsing one end to a point: $CX = X \times I/(X \times \{1\})$. Note that CX is contractible.

Suspension: The suspension SX of a space X is the space formed by taking the union of two copies of the cone over X, or equivalently the space formed by attaching I both "above" and "below" X and then collapsing X to a point, which can also be written as $SX = X \times I/(X \times \{0\} \sqcup X \times \{1\})$.

Example: The suspension of a zero sphere $S^0 = \{x_0, x_1\}$ consists of two lines (one over each point in S^0) joined at 0 and 1, giving a circle. Similarly, the suspension of S^1 is a cylinder with the top and bottom circles collapsed to points, giving the sphere S^2. In fact $SS^n = S^{n+1}$ in general.

Reduced suspension: The reduced suspension ΣX is the suspension of X quotiented by $\{x_0\} \times I$ for some $x_0 \in X$, $\Sigma X = X \times I/(X \times \{0\} \cup X \times \{1\} \cup \{x_0\} \times I)$. It is homotopically equivalent to the unreduced suspension.

Example: The reduced suspension of any space X in fact equals the smash product of X with S^1. To see this we view S^1 as an interval with the endpoints identified, $S^1 = I/\partial I = I/(\{0\} \sim \{1\})$, and write $X \wedge S^1 = X \times S^1/X \vee S^1 = X \times I/X \vee S^1 \sqcup X \times \partial I = X \times I/X \times \{0\} \sqcup X \times \{1\} \sqcup \{x_0\} \times I$ where x_0 was the point on x_0 identified with the point $0 \sim 1$ on S^1 in the wedge product.

Example: The reduced suspension of a wedge product is the wedge product of the reduced suspensions of the two spaces involved: $\Sigma(X \vee Y) = \Sigma X \vee \Sigma Y$. This follows as $(X \vee Y) \times I = X \times I \sqcup Y \times I/(\{x_0\} \times I \sim \{y_0\} \times I)$ and then quotienting by $X \vee Y \times \{0\} \sqcup X \vee Y \times \{1\} \sqcup (x_0, y_0) \times I$ gives $\Sigma X \vee \Sigma Y$.

Example: Denote by X_+ the space X with a point adjoined, then $\Sigma X_+ = \Sigma X \vee S^1$. This is as $\Sigma(X \sqcup pt) = X \times I \sqcup pt \times I/(X \sqcup pt) \times \{0\} \sqcup (X \sqcup pt) \times \{1\} \sqcup (x_0 \sqcup pt) \times I$ so the first two quotients give the suspension of X with an extra line attached (resulting from collapsing the $pt \times I$ factor to the space at $pt \times 0$ and $pt \times 1$), while the next quotient collapses the two ends of this line to the same point, giving $\Sigma X \vee S^1$.

***n*-fold suspension:** We define the n-fold suspension (or reduced suspension) iteratively: $S^n X = \underbrace{SS\ldots S}_{n} X$.

Example: The n-fold reduced suspension $\Sigma^n X$ equals the smash product of an n-sphere S^n with X: $\Sigma^n X = S^n \wedge X$.

We turn now to vector bundles.

Pullback bundles: Given two spaces X and Y with a map $f : X \to Y$ and a vector bundle $E \to Y$ we can form a pullback bundle $f^*(E)$ using the definition $f^*(E) = \{(x,v) : x \in X, v \in E, f(x) = p(v)\}$, i.e. the fibre over the point x in X is the fibre over the point $f(x)$ in Y.

Example: Consider the diagonal map $\Delta : X \to X \times X, x \mapsto (x,x)$ and let E be a vector bundle over $X \times X$. Then Δ induces a pullback bundle $\Delta^*(E)$ with the fibre over $x \in X$ corresponding to the fibre over $(x,x) \in X \times X$.

Example: Let f_2 be the map from the circle to itself defined by $f_2(z) = z^2$, for $z \in S^1$. We define the Mobius bundle $M \to S^1$ to be the quotient of $I \times \mathbb{R}$ by the identification $(0,t) \sim (1,-t)$, with the projection onto I giving a vector bundle over the circle (as we have $0 \sim 1$). We can also view the Mobius bundle as the bundle $S^1 \times \mathbb{R}$ with the identification $(z,t) \sim (-z,-t)$. Then the pullback is $f_2^*(M) = \{(z,v) \in S^1 \times M : f_2(z) = z^2 = p(v)\}$. Now, the map f_2 winds the circle around itself twice, so that each point z in M has as a preimage two points in S^1 (one coming from the first "half" of the circle and the other coming from the second "half"). As the Mobius bundle has one twist (over the point $0 \sim 1$) the pullback $f_2^* M$ has two twists (one over $0 \sim 1$ and one in the middle). These two twists cancel out to give the trivial bundle $S^1 \times \mathbb{R}$.

Example: In general for the map $f_n : z \mapsto z^n$ we have that the pullback of the Mobius bundle is the trivial bundle for even n (an even number of twists) and the Mobius bundle again for odd n (an odd number of twists).

2.2 Ring structures, exact sequences and Bott periodicity

The **tensor product** of two vector bundles $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ consists of disjoint union of the tensor products of the fibres $p_1^{-1}(x) \otimes p_2^{-1}(x)$ with topology defined in a certain way. In a more concrete realisation, if $g^1_{\alpha\beta}$ and $g^2_{\alpha\beta}$ are transition functions for E_1 and E_2 then $E_1 \otimes E_2$ is the vector bundle whose transition functions are the tensor products of the transition functions of E_1 and E_2, $g^1_{\alpha\beta} \otimes g^2_{\alpha\beta}$.

Example: The tensor product of two trivial bundles ε^n and ε^m over X is the trivial bundle ε^{nm} over X. Note that the dimension of the tensor product is the product of the dimensions.

The tensor product gives us a natural **ring structure** on $K(X)$, with multiplication defined by $(E_1 - E_1')(E_2 - E_2') = E_1 \otimes E_2 + E_1' \otimes E_2' - E_1 \otimes E_2' - E_1' \otimes E_2$. The multiplicative identity is the trivial line bundle. The reduced group $\widetilde{K}(X)$ is also a ring.

A map $f : X \to Y$ induces a map $f^* : K(Y) \to K(X)$ such that $E - E' \in K(Y)$ is mapped to $f^*(E) - f^*(E') \in K(X)$. This map is a ring homomorphism, as is the induced map on reduced groups (modulo some technicalities about base-pointed spaces).

External product: The external product $\mu : K(X) \otimes K(Y) \to K(X \times Y)$ is defined by $\mu(a \otimes b) = p_1^*(a) p_2^*(b)$ where $a \in K(X)$, $b \in K(Y)$ and p_1 and p_2 are the projections from $X \times Y$ onto X and Y respectively, and the multiplication on the right-hand side is the usual ring multiplication in $K(X \times Y)$. We will also write $a * b \equiv \mu(a \otimes b)$.

External product theorem: The external product $\mu : K(X) \otimes K(S^2) \to K(X \times S^2)$ is an isomorphism.

Example: It is important to gain a good understanding of $K(S^2)$. Note that the sphere is isomorphic to the complex projective line, $S^2 \approx \mathbb{C}P^1$, and that the canonical line bundle H over $\mathbb{C}P^1$ satisfies $H \otimes H \oplus 1 = H \oplus H$, which in $K(S^2)$ can be written $H^2 + 1 = 2H$ or $(H-1)^2 = 0$. This gives us a natural ring homomorphism $\mathbb{Z}[H]/(H-1)^2 \to K(S^2)$ where $\mathbb{Z}[H]/(H-1)^2$ is the ring of polynomials in H with integer coefficients, modulo the relation $(H-1)^2 = 0$. This means we can define a homomorphism $\mu' : K(X) \otimes \mathbb{Z}[H]/(H-1)^2 \to K(X) \otimes K(S^2) \to K(X \times S^2)$ with the second map being the external product. In fact we can prove that this is a ring isomorphism (which also proves the external product theorem). If we take $X = pt$ we find that $K(S^2) \approx \mathbb{Z}[H]/(H-1)^2$ as a ring, and $K(S^2) \approx \mathbb{Z} \oplus \mathbb{Z}$ as a group. It is generated by 1 and H, or equivalently by 1 and $(H-1)$, as we can write a general element as $n+mH=(n+m)+m(H-1)$. As $\widetilde{K}(S^2)$ is the kernel of the restriction $K(S^2)\to K(pt)$ and $K(pt)$ is clearly generated by the trivial bundle 1 we see that $\widetilde{K}(S^2)=\mathbb{Z}$, with generator $(H-1)$ and trivial multiplication.

To display the full power of the external product theorem we must first investigate **exact sequences** of K-groups. Recall that a sequence of groups and homomorphisms

$$G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} \dots \xrightarrow{f_{p-1}} G_p$$

is exact if for all $1<i<p$, $\ker f_i = \operatorname{im} f_{i-1}$ (i.e. everything that gets mapped into G_i by f_i gets mapped to the identity in G_{i+1} by the next map). If the sequence

$$0\to G_1 \xrightarrow{f_1} G_2$$

is exact then f_1 is injective (and conversely); if the sequence

$$G_1 \xrightarrow{f_1} G_2 \to 0$$

is exact then f_1 is surjective (and conversely), and if the sequence

$$0\to G_1 \xrightarrow{f_1} G_2 \xrightarrow{f_2} G_3 \to 0$$

is exact and there exists a homomorphism $g: G_3\to G_2$ such that $f_2\circ g = id: G_3\to G_3$ then the sequence is said to be split exact and $G_2\cong G_1\oplus G_3$.

Proposition: If X is compact Hausdorff and $A\subset X$ a closed subspace then the inclusion map $i:A\to X$ and the quotient map $q: X\to X/A$ induce an exact sequnce

$$\widetilde{K}(X/A) \xrightarrow{q^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A)$$

Proposition: If A is contractible then the quotient map $q: X\to X/A$ induces a bijection between isomorphism classes of n-dimensional bundles over X/A and isomorphism classes of n-dimensional bundles over X.

The two preceding propositions allow us to construct a long exact sequence of K-groups. We start with the inclusion $A\hookrightarrow X$ and add spaces by at each step forming the union of the preceding space with the cone of the space two steps back. We then also quotient out by the most recently attached cone, giving us the following sequence of inclusions (horizontal maps) and quotients (vertical maps):

$$\begin{array}{ccccccccc} A\hookrightarrow X\hookrightarrow & X\cup CA & \hookrightarrow & (X\cup CA)\cup CX & \hookrightarrow & \Big((X\cup CA)\cup CX\Big)\cup(X\cup CA) & \hookrightarrow & \dots \\ & \downarrow & & \downarrow & & \downarrow & & \\ & X/A & & SA & & SX & & \end{array}$$

Cones are contractible, so the vertical maps induce isomorphisms of reduced K-groups. So for instance we have the inclusion $A\to X\to X\cup CA$, but the quotient $X\cup CA\to X/A$ induces an isomorphism between $\widetilde{K}(X\cup CA)$ and $\widetilde{K}(X/A)$ so this gives an exact sequence $\widetilde{K}(X/A)\xrightarrow{q^*}\widetilde{K}(X)\xrightarrow{i^*}\widetilde{K}(A)$. This sequence is then extended using the inclusion $X\cup CA\to(X\cup CA)\cup CX$, where $\widetilde{K}(X\cup CA)\approx\widetilde{K}(X/A)$ and $\widetilde{K}((X\cup CA)\cup CX)\approx\widetilde{K}(SA)$, and so on, giving us the following long exact sequence:

$$\dots\to\widetilde{K}(S(X/A))\to\widetilde{K}(SX)\to\widetilde{K}(SA)\to\widetilde{K}(X/A)\to\widetilde{K}(X)\to\widetilde{K}(A)$$

We can immediately put this sequence to use.

Example: Consider a wedge product $X = A \vee B$ and the long exact sequence in terms of X and A:

$$\cdots \to \widetilde{K}(SB) \to \widetilde{K}(SX) \to \widetilde{K}(SA) \to \widetilde{K}(B) \to \widetilde{K}(X) \to \widetilde{K}(A)$$

In particular let us consider the last three terms:

$$\widetilde{K}(B) \xrightarrow{q^*} \widetilde{K}(X) \xrightarrow{i^*} \widetilde{K}(A)$$

where $i : A \to A \vee B$ is the inclusion of A in X and $q : A \vee B \to B$ is the quotient map collapsing A to a point. Now let q' be the quotient map collapsing B to a point, then the composition $q' \circ i$ is the identity on A and so induces the identity $i^* \circ q'^*$ on $\widetilde{K}(A)$, implying that i^* is surjective and q'^* is injective. Similarly, let j be the inclusion of B in X, then the composition $q \circ j$ is the identity on B and so induces the identity $j^* \circ q^*$ on $\widetilde{K}(B)$, meaning that q^* is injective. Thus we get a split exact sequence, and find that

$$\widetilde{K}(A \vee B) \cong \widetilde{K}(A) \oplus \widetilde{K}(B)$$

Example: Consider the smash product $X \wedge Y = X \times Y / X \vee Y$ and the long exact sequence for $X \times Y$ and $X \vee Y$.

$$\cdots \to \widetilde{K}(S(X \wedge Y)) \to \widetilde{K}(S(X \times Y)) \to \widetilde{K}(S(X \vee Y)) \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X \vee Y)$$

Now the suspension is homotopically equivalent to the reduced suspension so using this and the preceding example we have

$$\widetilde{K}(S(X \vee Y)) \approx \widetilde{K}(\Sigma(X \vee Y)) \approx \widetilde{K}(\Sigma X \vee \Sigma Y) \approx \widetilde{K}(\Sigma X) \oplus \widetilde{K}(\Sigma Y) \approx \widetilde{K}(SX) \oplus \widetilde{K}(SY)$$

and also

$$\widetilde{K}(X \vee Y) \approx \widetilde{K}(X) \oplus \widetilde{K}(Y)$$

This means we have $\widetilde{K}(X \times Y) \xrightarrow{i^*} \widetilde{K}(X \vee Y) \approx \widetilde{K}(X) \oplus \widetilde{K}(Y)$. Now we can show this map is surjective, reasoning as follows. Let q' and q be the projections from $X \vee Y$ onto X and Y (using the notation of the previous example). Then define projections p_1 and p_2 from $X \times Y$ to X and Y respectively, then using the inclusion $i : X \vee Y \to X \times Y$ we see that projecting from $X \vee Y$ onto X or Y is the same as first including $X \vee Y$ into $X \times Y$ and then projecting out the factors, i.e.

$$q' = p_1 \circ i \qquad q = p_2 \circ i$$

Now in fact the induced map $q'^* \oplus q^* : \widetilde{K}(X) \oplus \widetilde{K}(Y) \to \widetilde{K}(X \vee Y)$ is an isomorphism (again from the previous example), and can also be written

$$q'^* \oplus q^* = i^* p_1^* \oplus i^* p_2^* = i^*(p_1^* \oplus p_2^*)$$

hence i^* is a surjection and $p_1^* \oplus p_2^*$ together with the suspended maps $Sp_1^* \oplus Sp_2^*$ gives us a splitting of the exact sequence such that

$$0 \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y) \to 0$$

and so

$$\widetilde{K}(X \times Y) \approx \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$$

and also

$$\widetilde{K}(S(X \times Y)) \approx \widetilde{K}(S(X \wedge Y)) \oplus \widetilde{K}(SX) \oplus \widetilde{K}(SY)$$

Reduced external product: We can use this last example to obtain a version of the external product for reduced K-groups. Suppose $a \in \widetilde{K}(X) = \ker(K(X) \to K(x_0))$ and $b \in \widetilde{K}(Y) = \ker(K(Y) \to K(y_0))$.

Then their external product is $a * b = \mu(a \otimes b) = p_1^*(a)p_2^*(b) \in K(X \times Y)$, with $p_1^*(a)$ restricting to zero in $K(x_0 \times Y)$ and $p_2^*(a)$ restricting to zero in $K(X \times y_0)$, i.e. $p_1^*(a)p_2^*(b)$ is zero in $K(X \vee Y)$, and $a * b \in \widetilde{K}(X \times Y)$ as it is in the kernel of $K(X \times Y) \to K(x_0 \times y_0)$. From the short exact sequence $0 \to \widetilde{K}(X \wedge Y) \to \widetilde{K}(X \times Y) \to \widetilde{K}(X) \oplus \widetilde{K}(Y) \to 0$ we see that $a * b$ is in the kernel of the map from $\widetilde{K}(X \times Y) \to \widetilde{K}(X \vee Y) \approx \widetilde{K}(X) \oplus \widetilde{K}(Y)$ and so in the image of the preceding map: it follows that $a * b$ pulls back to a unique element of $\widetilde{K}(X \wedge Y)$. Hence we have a reduced external product

$$\widetilde{K}(X) \otimes \widetilde{K}(Y) \to \widetilde{K}(X \wedge Y)$$

This can also be seen by writing $K(X) = \widetilde{K}(X) \oplus \mathbb{Z}$, $K(Y) = \widetilde{K}(Y) \oplus \mathbb{Z}$ and considering

$$\begin{array}{ccccc} K(X) \otimes K(Y) & \approx & \widetilde{K}(X) \otimes \widetilde{K}(Y) & \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \\ \downarrow & & \downarrow & \\ K(X \times Y) & \approx & \widetilde{K}(X \wedge Y) & \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y) \oplus \mathbb{Z} \end{array}$$

using that $G \otimes \mathbb{Z} \cong G$ for an abelian group G, and so we see that the by restricting the unreduced external product to $\widetilde{K}(X) \otimes \widetilde{K}(Y)$ we get the reduced external product.

Bott periodicity: Now, as $S^n \wedge X \approx \Sigma^n X$ and the full suspension S^n is related to the reduced suspension Σ^n by a quotient map of a collapsible subspace we have an isomorphism $\widetilde{K}(S^n X) \approx \widetilde{K}(S^n \wedge X)$. Now, from the external product theorem we know that $K(X) \otimes K(S^2) \to K(X \times S^2)$ is a ring isomorphism, and so the restriction of this to reduced groups $\widetilde{K}(X) \otimes \widetilde{K}(S^2) \to \widetilde{K}(X \wedge S^2) = \widetilde{K}(S^2 X)$ is an isomorphism. In fact we can go further. Consider the map $\beta : \widetilde{K}(X) \to \widetilde{K}(S^2 X)$ defined by $\beta(a) = (H-1) * a$ where $a \in \widetilde{K}(X)$ and $(H-1)$ is the canonical line bundle over $S^2 \approx \mathbb{C}P^1$. This is a composition $\widetilde{K}(X) \to \widetilde{K}(S^2) \otimes \widetilde{K}(X) \to \widetilde{K}(S^2 X)$. The first map in the composition is $a \mapsto (H-1) \otimes a$ and is an isomorphism as $(H-1)$ generates $\widetilde{K}(S^2)$, and the second map is an isomorphism from the external product theorem, hence we have **Bott periodicity:**

$$\widetilde{K}(X) \approx \widetilde{K}(S^2 X)$$

3 Examples

3.1 Point

A vector bundle over a point is just a single copy of $\mathbb{C}^n$ and so $K(pt) = \{\mathbb{C}^n - \mathbb{C}^p\} \equiv \{n - p\} \cong \mathbb{Z}$. The multiplication on this group is the usual multiplication on $\mathbb{Z}$, for if we represent the equivalence class of $(n-p)$ simply by $m \equiv \mathbb{C}^m - \mathbb{C}^0$ where $m = n - p$ then the tensor product of bundles gives $m \cdot r \equiv \mathbb{C}^m \otimes \mathbb{C}^r = \mathbb{C}^{mr} = mr$. The reduced group is the kernel of the restriction $K(pt) \to K(pt)$ which is the identity, and so $\widetilde{K}(pt) = 0$.

3.2 S^0

A vector bundle over $S^0 = \{x_0, x_1\}$ consists of one copy of $\mathbb{C}^m$ over x_0 and one copy of $\mathbb{C}^p$ over x_1 (note $m \neq n$ in general as S^0 is not connected). It follows that $K(S^0) = \{m - n, p - q\} \cong \mathbb{Z} \oplus \mathbb{Z}$ where $m - n$ represents the equivalence class of $\mathbb{C}^m - \mathbb{C}^n$. The ring structure is the usual multiplication on Z on each factor.

The reduced group $\widetilde{K}(S^0)$ is the kernel of the restriction $K(S^0) \to K(x_0)$, i.e. the kernel of the map sending $\{\mathbb{C}^m - \mathbb{C}^n, \mathbb{C}^p - \mathbb{C}^q\}$ to $\{\mathbb{C}^m - \mathbb{C}^n\}$. The kernel consists of those elements with $m = n$, hence $\widetilde{K}(S^0) = \{0, \mathbb{C}^p - \mathbb{C}^q\} \equiv \{p - q\} \cong \mathbb{Z}$, and the ring structure is the usual multiplication on $\mathbb{Z}$.

3.3 S^1

All complex bundles over S^1 are trivial, hence $K(S^1) = \{\mathbb{C}^m - \mathbb{C}^n\} \cong \mathbb{Z}$ as before with usual multiplication, and $\widetilde{K}(S^1) = 0$.

3.4 S^2

The sphere S^2 was discussed previously, however we may as well repeat the results. We have that $K(S^2) \approx \mathbb{Z} \oplus \mathbb{Z}$ as a group and $K(S^2) \approx \mathbb{Z}[H]/(H-1)^2$ as a ring. We can write this as $\{n + m(H-1) : m, n \in \mathbb{Z}\} \equiv (n, m)$, and the ring structure is given explicitly by $(n+m(H-1))(p+q(H-1)) = nm+(mp+nq)(H-1)$, so in short, $(n, m)(p, q) = (n, p, mp+nq)$. The reduced group $\widetilde{K}(S^2)$ is the kernel of the restriction $(n, m) \to n$ and so $\widetilde{K}(S^2) = \mathbb{Z}$, generated by $(H-1)$ and with trivial multiplication.

3.5 S^n

From Bott periodicity it follows that $\widetilde{K}(S^{2n+1}) = 0$ and $\widetilde{K}(S^{2n}) = \mathbb{Z}$ (generated by the n-fold reduced external product $(H-1) * \cdots * (H-1)$, so multiplication is trivial). Then for odd-dimensional spheres we have $K(S^{2n+1}) = \mathbb{Z}$ with usual multiplication, while for even-dimensional spheres we have $K(S^{2n}) = \mathbb{Z} \oplus \mathbb{Z}$ with the same ring structure as S^2.

3.6 Torus

The torus $T^2 = S^1 \times S^1$. We know from the long exact sequence for a pair $(X \times Y, X \vee Y)$ that $\widetilde{K}(X \times Y) \approx \widetilde{K}(X \wedge Y) \oplus \widetilde{K}(X) \oplus \widetilde{K}(Y)$, so with $X = Y = S^1$ we have

$$\widetilde{K}(T^2) \approx \widetilde{K}(S^2) \oplus \widetilde{K}(S^1) \oplus \widetilde{K}(S^1) = \widetilde{K}(S^2) = \mathbb{Z}$$

as $S^1 \wedge S^1 = S^2$ and $\widetilde{K}(S^1) = 0$. In fact the above isomorphism is a ring isomorphism, so the ring structure on $\widetilde{K}(T^2)$ is that on $\widetilde{K}(S^2)$, i.e. trivial multiplication. We then have $K(T^2) = \widetilde{K}(T^2) \oplus \mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$, with the same ring structure as $K(S^2)$.

Note that this example shows that the torus and the sphere S^2 have the same K and $\widetilde{K}(K)$ groups - to distinguish between them in K-theory we must introduce additional K-groups, which we will do in the next section.

3.7 Wedge of spheres

Consider a wedge of spheres $S^n \vee S^m$. We have

$$\widetilde{K}(S^n \vee S^m) = \widetilde{K}(S^n) \oplus \widetilde{K}(S^m)$$

so that

$$\widetilde{K}(S^n \vee S^m) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} & n, m \text{ even} \\ \mathbb{Z} & \text{one of } n, m \text{ even} \\ 0 & n, m \text{ odd} \end{cases}$$

with ring structure being trivial in all cases (i.e. $(p, q)(r, s) = (0, 0)$ if both n and m even and $pq = 0$ if only one even). We also have

$$K(S^n \vee S^m) = \begin{cases} \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} & n, m \text{ even} \\ \mathbb{Z} \oplus \mathbb{Z} & \text{one of } n, m \text{ even} \\ \mathbb{Z} & n, m \text{ odd} \end{cases}$$

If both n, m odd then the ring structure is the usual multiplication on $\mathbb{Z}$, if one is even and one is odd then the ring structure is the same as in S^2, and if both are even then the ring structure is given by $\mathbb{Z}[\alpha, \beta]/(\alpha^2, \beta^2)$ where α and β are the generators of $\widetilde{K}(S^n)$ and $\widetilde{K}(S^m)$.

4 Cohomology

A **cohomology theory** is a theory assigning to a topological space X a sequence $h^p(X)$ of abelian groups, satisfying certain axioms known as the Eilenberg-Steenrod axioms. A **generalised cohomology theory** drops the dimension axiom, which states that $h^p(pt) = 0$ for $p \neq 0$. If we define $\widetilde{K}^0(X) \equiv \widetilde{K}(X), \widetilde{K}^{-1}(X) = \widetilde{K}(SX)$ and in general $\widetilde{K}^{-p}(X) = \widetilde{K}(S^p X)$, then we can view K-theory as a cohomology theory. We can also introduce relative groups by letting $\widetilde{K}^{-p}(X, A) = \widetilde{K}(S^p(X/A))$ Owing to Bott periodicity we have $\widetilde{K}^{-2} = \widetilde{K}^0(X)$, and from the long exact sequence derived above we have the following 6-term exact sequence:

$$\begin{array}{ccccc} \widetilde{K}^0(X,A) & \longrightarrow & \widetilde{K}^0(X) & \longrightarrow & \widetilde{K}^0(A) \\ \uparrow & & & & \downarrow \\ \widetilde{K}^{-1}(X,A) & \longleftarrow & \widetilde{K}^{-1}(X) & \longleftarrow & \widetilde{K}^{-1}(A) \end{array}$$

It is useful to accumulate all the information we can find from K-groups into a single object: $\widetilde{K}^*(X) = \widetilde{K}^0(X) \oplus \widetilde{K}^{-1}(X)$. From the external product we have an induced multiplication $\widetilde{K}^*(X) \otimes \widetilde{K}^*(Y) \to \widetilde{K}^*(X \wedge Y)$, and we can use this to construct a product on $\widetilde{K}^*(X)$ by composing the external product with the map induced by the diagonal mapping $\Delta : X \to X \wedge X, x \mapsto (x, x)$. The result is a composition

$$\widetilde{K}^*(X) \otimes \widetilde{K}^*(X) \longrightarrow \widetilde{K}^*(X \wedge X) \xrightarrow{\Delta^*} \widetilde{K}^*(X)$$

which makes $\widetilde{K}^*(X)$ into a ring.

Example: The above multiplication when restricted to the $\widetilde{K}^0(X)$ factor in $\widetilde{K}^*(X)$ is just the usual multiplication on $\widetilde{K}(X)$. To see this, note that this multiplication is $\widetilde{K}(X) \otimes \widetilde{K}(X) \xrightarrow{\mu} \widetilde{K}(X \wedge X) \xrightarrow{\Delta^*} \widetilde{K}(X)$ where the first map is $(a, b) \mapsto p_1^*(a)p_2^*(b)$ with p_1 and p_2 being the projections onto the first and second X factors in $X \wedge X$. The multiplication here is the usual product in $\widetilde{K}(X \wedge X)$ and the fibre over the point $(x_1, x_2) \in X \wedge X$ is the tensor product of the fibres over $x_1 \in X$ and $x_2 \in X$ (as p_1^* pulls back the fibre over $x_1 \in X$ to $\{x_1\} \wedge X$ and similarly for p_2^*). Finally the diagonal map pulls back the fibre over $(x, x) \in X \wedge X$ to the point x in X - and hence with respect to the given composition means that the fibre over $x \in X$ consists of the tensor product of the fibre in a over x with the fibre in b over x, and this is just the usual tensor product of a and b.

The relative form of the external product is a map $\widetilde{K}^*(X, A) \otimes \widetilde{K}^*(Y, B) \to \widetilde{K}^*(X/A \wedge Y/B)$. When composed with the relativised diagonal map $X/(A \cup B) \to X/A \wedge X/B$ we get a product $\widetilde{K}^*(X, A) \otimes \widetilde{K}^*(X, B) \to \widetilde{K}^*(X, A \cup B)$.

Example: If X is a union of two contractible subspaces, $X = A \cup B$, then we have $\widetilde{K}(X) \approx \widetilde{K}(X/A)$ and $\widetilde{K}(X) \approx \widetilde{K}(X/A)$, and as a result the product $\widetilde{K}^*(X) \otimes \widetilde{K}^*(X) \to \widetilde{K}^*(X)$ can be written as a composition $\widetilde{K}^*(X, A) \otimes \widetilde{K}^*(X, B) \to \widetilde{K}^*(X, A \cup B) \to \widetilde{K}^*(X)$, but the middle group is $\widetilde{K}^*(X, A \cup B) \approx \widetilde{K}^*(X, X) = 0$ and so the multiplication on $\widetilde{K}^*(X)$ is trivial.

Example: In general if X can be written as the n-fold union of contractible subspaces then all n-fold products in $\widetilde{K}^*(X)$ are trivial.

Let us now note a neat way to recover the unreduced K-groups given the reduced K-groups. We define $K^{-p}(X) \equiv \widetilde{K}^{-p}(X_+)$ where X_+ is X with a point adjoined. This means that

$$K^0(X) = \widetilde{K}^0(X_+) = \text{Ker}\,(K(X_+) \to K(+) = K(X)$$

so this definition agrees with our previous definition of $K(X)$ and $\widetilde{K}(X)$, and

$$K^{-1}(X) = \widetilde{K}^{-1}(X_+) = \widetilde{K}(SX_+) = \widetilde{K}(\Sigma X_+) = \widetilde{K}(\Sigma X \vee S^1) = \widetilde{K}(\Sigma X) \oplus \widetilde{K}(S^1) = \widetilde{K}(SX) = \widetilde{K}^{-1}(X)$$

For a pair (X, A) with $A \neq \emptyset$ then we define $K^{-p}(X, A) = \widetilde{K}^{-p}(X, A)$, and can write the same 6-term exact sequence for unreduced groups. We also have a product $K^*(X) \otimes K^*(Y) \to K(X \times Y)$ and by composing with the diagonal map $\Delta : X \to X \times X$ we obtain a ring structure on $K^*(X)$.

Note that from this definition we find that $K^{-2}(pt) = \widetilde{K}^{-2}(pt \sqcup pt) = \widetilde{K}^{-2}(S^0) = \widetilde{K}(S^2 S^0) = \widetilde{K}(S^2) = \mathbb{Z}$ so K-theory is a generalised cohomology theory.

Example: For even-dimensional spheres, $\widetilde{K}^{-1}(S^{2n}) = \widetilde{K}(S^{2n+1}) = 0$, while for odd-dimensional spheres $\widetilde{K}^{-1}(S^{2n+1}) = \widetilde{K}(S^{2n+2}) = \mathbb{Z}$, so for all spheres, $\widetilde{K}^*(S^n) = \mathbb{Z}$. It follows that $K^{-1}(S^{2n}) = 0$ and $K^{-1}S^{2n+1} = \mathbb{Z}$ so that $K^*(S^n) = \mathbb{Z} \oplus \mathbb{Z}$ for all spheres.

Example: For the torus T^2 we look at the split exact sequence $0 \to \widetilde{K}(S(X \wedge Y)) \to \widetilde{K}(S(X \times Y)) \to \widetilde{K}(SX) \oplus \widetilde{K}(SY) \to 0$ with $X = Y = S^1$, so that $\widetilde{K}^{-1}(T^2) = \widetilde{K}(SS^2) \oplus \widetilde{K}(SS^1) \oplus \widetilde{K}(SS^2) = \widetilde{K}(S^2) \oplus \widetilde{K}(S^2) = \mathbb{Z} \oplus \mathbb{Z}$ so that $\widetilde{K}^*(T^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$, and also $K^{-1}(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ so $K^*(T^2) = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Thus we see that the torus and sphere S^2 do indeed have distinct K-groups when looked at from this viewpoint.

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