Ma3423/3424 Topics in Complex Analysis

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Some notes for the Complex Analysis course; also includes solutions to some exercises from the book by Palka.

1 Power series

Power series If f is holomorphic in $\Delta_R(a)$ then we can write

$$f(z) = \sum_{n \geq 0} c_n (z-a)^n \qquad c_n = \frac{1}{2\pi i} \int_{\partial \Delta_r(a)} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}} \quad 0 < r < R \tag{1.1}$$

The *radius of convergence* is R: the power series converges inside $\Delta_R(a)$ and diverges outside it. We may differentiate termwise:

$$f'(z) = \sum_{n \geq 1} c_n n (z-a)^{n-1} \tag{1.2}$$

which gives us the expression

$$f^{(k)}(a) = \frac{k!}{2\pi i} \int_{\partial \Delta_r(a)} \frac{f(\zeta) d\zeta}{(\zeta - a)^{k+1}} \tag{1.3}$$

which also tells us that $c_k = \frac{1}{k!} f^{(k)}(a)$.

Cauchy estimates Take the above formula for $f^{(k)}(a)$, and bound it using $|\zeta - a| = r$ and $M = \sup_{\Delta_R(a)} |f|$:

$$|f^{(k)}(a)| \leq \frac{k! M}{2\pi i} \int_{\partial \Delta_r(a)} \frac{d\zeta}{r} \leq \frac{k! M}{r} \tag{1.4}$$

Hence we find the *Cauchy estimate*:

$$|f^{(k)}(a)| \leq \frac{k! M}{R} \tag{1.5}$$

Liouville's Theorem *If f is holomorphic and bounded in $\mathbb{C}$ then f is constant.*

Proof of Liouville's Theorem As f is holomorphic in $\mathbb{C}$ it is holomorphic in any disc $\Delta_R(0)$. Arguing as for the Cauchy estimates and letting $M = \sup_{\mathbb{C}} f(z)$ (which exists as f is bounded) we have

$$|f^{(k)}(0)| \leq \frac{k! M}{R} \tag{1.6}$$

This holds for all R, letting $R \to \infty$ we have that $|f^{(k)}| \to 0$ for all $k > 0$, so $f^{(k)}(0) = 0$ for all $k > 0$ and from the Taylor series

$$f(z) = \sum_{n \geq 0} \frac{f^{(n)}(0)}{n!} z^n = f(0) \tag{1.7}$$

we see that $f(z)$ is constant. $\square$

Application of Cauchy Estimates (Palka 8.29) Let $f(z)$ be holomorphic in $\mathbb{C}$, and $|f(z)| \leq c|z|^\lambda + d$ where c, λ and d are positive real numbers. We will show that $f(z)$ is necessarily a polynomial of degree at most λ. In a disc $\Delta_R(0)$ the Cauchy estimates give

$$|f^{(n)}(0)| \leq \frac{n!}{R^n}\left(cR^\lambda + d\right) = n!\left(cR^{\lambda-n} + dR^{-n}\right) \tag{1.8}$$

This holds for any disc $\Delta_R(0)$, letting R go to infinity we see that if $n > \lambda$ then $|f^{(n)}| \to 0$. Considering the Taylor series at the origin

$$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(0) z^n = \sum_{n=0}^{m} \frac{1}{n!} f^{(n)}(0) z^n \tag{1.9}$$

where $m \leq \lambda$, so we see that $f(z)$ is a polynomial of degree at most λ.

Application of Liouville's Theorem (Palka 8.32) Let $f(z)$ be holomorphic in $\mathbb{C}$, and $|f(z)| \leq me^{\alpha x}$ for all $z = x + iy$, where m and α are positive real numbers. We wll show that f is necessarily of the form $f(z) = Ae^{\alpha z}$ for some constant A. Consider the function $f(z)/e^{\alpha z}$. This is holomorphic in $\mathbb{C}$ (as $e^{\alpha z} \neq 0$ for all complex numbers z). We also have $|f(z)/e^{\alpha z}| = |f(z)|/|e^{\alpha z}| \leq me^{\alpha x}/e^{\alpha x} = m$ using the fact that $|e^z| = |e^x| \cdot |e^{iy}| = |e^x|$. Hence $f(z)/e^{\alpha z}$ is holomorphic and bounded, and so constant. We conclude that $f(z) = Ae^{\alpha z}$.

Application of Liouville's Theorem (Palka 8.33) Let $f(z)$ be holomorphic in $\mathbb{C}$, and $f \not\equiv$ constant. Then $f(\mathbb{C})$ is dense. If this were not so then there would exist some point w_0 and some $\varepsilon > 0$ such that $f(z) \notin \Delta_\varepsilon(w_0)$ for all $z \in \mathbb{C}$, i.e. $|f(z) - w_0| > \varepsilon$. Define $g(z) = 1/(f(z) - w_0)$, then $g(z)$ is holomorphic in $\mathbb{C}$, and $|g(z)| = 1/|f(z) - w_0| < 1/\varepsilon$ so that g is bounded. Hence by Liouville $g(z)$ is constant, and so $f(z) = w_0 + 1/g(z)$ is constant too, a contradiction.

Fundamental Theorem of Algebra *If $P(z) = a_n z^n + \cdots + a_1 z_1 + a_0$ a polynomial of positive degree $n \neq 0$ then $\exists z_0 \in \mathbb{C}$ such that $P(z_0) = 0$.*

Proof of Fundamental Theorem of Algebra Now, $P(z)$ is holomorphic in $\mathbb{C}$. Let $f(z) = 1/P(z)$ and assume $P(z) \neq 0\ \forall z$ so that $f(z)$ is holomorphic in $\mathbb{C}$. We will show that $f(z)$ is bounded, hence constant, implying that $P(z)$ is constant, and so obtaining a contradiction. Consider

$$\begin{aligned} |P(z)| &= |a_n z^n| \cdot \left|1 + \frac{a_{n-1}}{a_n}\frac{z^{n-1}}{z^n} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\right| \\ &\geq |a_n z^n| \cdot \left|1 - \left|\frac{a_{n-1}}{a_n}\frac{z^{n-1}}{z^n} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}\right|\right| \end{aligned} \tag{1.10}$$

Now $\exists R$ such that $\forall |z| \geq R$, $|\frac{a_{n-1}}{a_n}\frac{z^{n-1}}{z^n} + \cdots + \frac{a_0}{a_n}\frac{1}{z^n}| < 1/2$, so that $|P(z)| \geq \frac{1}{2}|a_n z^n| \geq \frac{1}{2}|a_n|R^n$, and $|f(z)| \leq 2/|a_n|R^n$. Also, for all $|z| < R$ we use the fact that f is holomorphic and hence bounded on compact sets to conclude that $|f|$ is bounded for all $z \in \mathbb{C}$, and so constant by Liouville's theorem. $\square$

Cauchy-Hadamard formula for radius of convergence The radius of convergence of a power series is given by $R = (\overline{\lim}_{n\to\infty}|c_n|^{1/n})^{-1}$, where $\overline{\lim} f \equiv \limsup f$. To see this is true, consider first $|z| < R$. Then there exists r_1, r_2 such that $|z| < r_1 < r_2 < R$. Now, $\overline{\lim}_{n\to\infty}|c_n|^{1/n} = 1/R > 1/r_2$, so there exists N such that for all $n \geq N$, $|c_n|^{1/n} < 1/r_2$ so $|c_n| < 1/r_2^n$, and $|c_n z^n| < (r_1/r_2)^n$. But $r_1/r_2 < 1$, so $\sum_n (r_1/r_2)^n$ is convergent and hence so is the power series $\sum_n c_n z^n$ (by the comparison test).

Now consider $|z| > R$. Then there exists R_1 such that $|z| > R_1 > R$. Now, $\overline{\lim}_{n\to\infty}|c_n|^{1/n} = 1/R > 1/R_1$, so there exists a subsequence (n_k) such that $|c_{n_k}|^{1/n_k} > 1/R_1$, so $|c_{n_k}| > 1/R_1^{n_k}$, and $|c_{n_k} z^{n_k}| > (R_1/R_1)^{n_k} = 1$. Hence by the nth term test the power series diverges.

Uniform and compact convergence A sequence of functions $f_n \to f$ *uniformly* on a set K if $\forall \varepsilon > 0 \exists N$ such that $\forall n \geq N$, $|f_n(z) - f(z)| < \varepsilon$, $\forall z \in K$. We say that $f_n \to f$ *compactly* on K if it converges uniformly on every compact subset.

Lemma *Let $\Omega = \{|z| < R\}$, then $f_n \to f$ uniformly on every $\{|z| \leq r\}$, $r < R$ if and only if $f_n \to f$ uniformly on all compact $K \subset \Omega$.*

Proof of lemma Suppose $f_n \to f$ uniformly on every compact $K \subset \Omega$, then it follows automatically that $f_n \to f$ uniformly on every $\{|z| \leq r\}$, $r < R$, as these are compact sets.

Conversely, let $f_n \to f$ uniformly on every $\{|z| \leq r\}$, $r < R$. Let $K \subset \Omega$ be a compact set. Then for all $z_0 \in K$ consider the discs $\{|z| < |z_0| + \varepsilon\}$, where we have chosen $\varepsilon > 0$ such that $|z_0| + \varepsilon < R$ (possible as Ω is open). These discs give a covering of K, as K is compact we can choose a finite subcovering. In fact pick the point $z_0' \in K$ which gives the disc $\Delta_{z_0'} = \{|z| < |z_0'| + \varepsilon\}$ of maximum radius. Then $K \subset \Delta_{z_0'} \subset \Omega$, and as $f_n \to f$ uniformly on discs, $f_n \to f$ uniformly on K. $\square$

Weierstrass' Theorem *Let f_n be holomorphic in Ω $\forall n$, and $f_n \to f$ uniformly on every compact subset $K \subset \Omega$. Then f is holomorphic in Ω and $f_n' \to f'$ uniformly on compact subsets.*

Proof of Weierstrass' Theorem The first part of the proof uses Morera's theorem, if $\int_{\partial\Delta} f dz = 0$ for all triangles $\overline{\Delta} \subset \Omega$, then f holomorphic in Ω. As f_n is holomorphic in Ω, $\int_{\partial\Delta} f_n dz = 0$ for all such triangles. These triangles are compact subsets, hence $f_n \to f$ uniformly on each Δ, i.e. for all $\varepsilon > 0$ there exists N such that $\forall n \geq N$, $|f_n(z) - f(z)| < \varepsilon$ for all $z \in \Delta$. Now consider $\int_{\partial\Delta}(f - f_n)dz \leq \max_{\partial\Delta} |f - f_n| \cdot \text{length}\,(\partial\Delta)$. For $n \to \infty$, $\max_{\partial D} |f - f_n| \to 0$, so $\int_{\partial\Delta} f_n dz \to \int_{\partial\Delta} f dz$, and so $\int_{\partial\Delta} f dz = 0$. As a uniform limit f is continuous on each compact set, and f is continuous on Ω. Then by Morera f is holomorphic on Ω.

The second part of the proof looks more involved, so I'll do it later. $\square$

Laurent Series The *Laurent series expansion* of a function $f(z)$ which is holomorphic in $r < |z - a| < R$ is

$$f(z) = \sum_{n\in\mathbb{Z}} c_n(z-a)^n \qquad c_n = \frac{1}{2\pi i} \int_{\partial B_\rho(a)} \frac{f(\zeta)d\zeta}{(\zeta - a)^{n+1}} r < \rho < R \tag{1.11}$$

We can write this series as

$$\sum_{n\in\mathbb{Z}} c_n(z-a)^n = \sum_{n<0} c_n(z-a)^n + \sum_{n\geq 0} c_n(z-a)^n \tag{1.12}$$

where the sum over negative integers constitutes the *principal part* of the Laurent series. The Laurent series will converge if both these series converge. Now, R is the radius of convergence of $\sum_{n\geq 0} c_n(z-a)^n$, while we can write $\sum_{n<0} c_n(z-a)^n = \sum_{m>0} c_{-m} 1/(z-a)^m$, which converges for $|1/(z-a)| < \widetilde{R} \equiv 1/r$, and hence converges for $|z - a| > r$.

2 Zeros

Order of zero We define the *order of zero* or *order of vanishing* of a function $f(z)$ at z_0 to be

$$\text{ord}_f(z_0) = \min\{n | f^{(n)}(z_0) \neq 0\} \tag{2.1}$$

An equivalent definition using power series is $\text{ord}_f(z_0) = \min\{n | c_n \neq 0\}$, where we have expanded in a power series about z_0. Note that if $f(z_0) \neq 0$, then $\text{ord}_f(z_0) = 0$. The *multiplicity* of f at z_0 is the zero order of $f(z) - f(z_0)$.

Examples of multiplicities (Palka 5.1) i) $f(z) = e^{z(\cos z - 1)}$ at $z = 0$: we have $f(0) = 1$. Expanding in a power series about 0, $f(z) = 1 + z(\cos z - 1) + \frac{1}{2}z^2(\cos z - 1) + \cdots - 1 = -\frac{1}{2}z^3 + \dots$ so the lowest non-vanishing coefficient is that multiplying z^3, hence the multiplicity is 3.

ii) $f(z) = z^{\mathrm{Log}\, z}$ at $z = 1$: we have $f(1) = 1$. Use $\mathrm{Log} f(z) = (\mathrm{Log} z)^2$ and differentiate, hence $1/f(z)f'(z) = 2\mathrm{Log}\, z/z \Rightarrow f'(z) = 2f(z)\mathrm{Log} z/z$ and $f'(1) = 0$. Differentiating again, $f''(z) = 2f'(z)\mathrm{Log}\, z/z + 2f(z)/z^2 - 2f(z)\mathrm{Log}\, z/z^2$ and now $f''(1) = 2$, hence the multiplicity is 2.

v) $f(z) = (1 + z^2 - e^{z^2})^3$ at $z = 0$: $f(0) = 0$. Expanding in a power series, $1 + z^2 - e^{z^2} = -z^4/2 + \dots$ and so the multiplicity at 0 is 12.

Factorisation lemma *Let f be holomorphic in Ω and $z_0 \in \Omega$. Let $d = \mathrm{ord}_f(z_0) < \infty$, then $f(z) = (z - z_0)^d g(z)$, where $g(z)$ holomorphic in Ω and $g(z_0) \neq 0$.*

Proof of factorisation lemma In a disc Δ centred at z_0, we can expand f in a power series:

$$\begin{aligned} f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n &= c_d (z - z_0)^d + \cdots + c_n (z - z_0)^n + \dots \\ &= (z - z_0)^d \underbrace{(c_d + c_{d+1}(z - z_0) + \dots)}_{\equiv g(z)} \end{aligned} \tag{2.2}$$

and we see that $g(z_0) = c_d \neq 0$. □

Corollary *Let f holomorphic in Ω, and $z_0 \in \Omega$, $d = \mathrm{ord}_f(z_0) < \infty$, then $f(z) = (h(z))^d$ in some neighbourhood of z_0, with $h(z)$ holomorphic in the same neighbourhood.*

Proof of corollary Let $h(z) = (z - z_0)(g(z))^{1/d}$. To see that we can take a branch of the dth root, consider a ray R through zero such that $g(z_0) \notin R$, and let $U = \mathbb{C} \backslash R$, then $z^{1/d}$ has a branch $\varphi(w)$, $w \neq R$, $\varphi(w)^d = w$. Now $g(z)$ continuous at z_0 implies that $g(z) \notin R$ for z in some neighbourhood of z_0, so that $g(z)^{1/d} = \varphi(g(z))$ a branch of $g(z)^{1/d}$. □

Identity Principle *Let f, g holomorphic in Ω, Ω connected, and $f(z) = g(z)$ for all $z \in A \subset \Omega$ where A has limit points in Ω, then $f \equiv g$ in Ω. (Note that to say A has limit points in Ω is to say that there exists a sequence $z_n \to z_0$, $z_n \neq z_0$, $z_n \in A$, $z_0 \in \Omega$).*

Proof of Identity Principle Consider $h = f - g$. Suppose z_0 a limit point of A, and let $d = \mathrm{ord}_h(z_0) < \infty$, so that $h(z) = (z - z_0)^d \widetilde{g}(z)$, where $\widetilde{g}(z_0) \neq 0$ and $\widetilde{g}(z)$ has no zeros near z_0. This implies that z_0 is an isolated zero of h, and so z_0 not a limit point of $h|_A = 0$. This is a contradiction and implies that $d = \infty$ and so $h \equiv 0$ in some neighbourhood of z_0.

Now consider the set $U = \{a \in \Omega | h \equiv 0 \text{ in some neighbourhood of } a\}$, which is obviously open. If z_0 is in the limit set of U then $z_0 \in U$ by the first part of the proof (i.e. if z_0 a limit point of U (where $h \equiv 0$) then $h \equiv 0$ in a neighbourhood of z_0). This means that U is closed; hence U is open-closed and non-empty which implies that $U = \Omega$ (as otherwise we could partition Ω into U and $\Omega \backslash U$; the former is open and the latter is the complement of U, which is closed, in Ω and hence open, but by hypothesis Ω is connected so this is a contradiction) and hence $h \equiv 0$ on Ω. □

3 Singularities

Singularity A *singularity* of a function f is any point a where it is not defined or not holomorphic. A singularity is called *isolated* if f is holomorphic in some $\Delta_\varepsilon(a) \backslash \{a\}$. An isolated singularity is called *removable* if f has a holomorphic extension $\widetilde{f}$ which is holomorphic at z_0 and $\widetilde{f}|_{\Delta_\varepsilon(a) \backslash \{a\}} = f$.

Somewhat surprising (to me at least) example of removable isolated singularity The function $f(z) = z/z$ is not defined at $z = 0$, but admits the holomorphic extension $f(z) = 1$.

Example of a non-isolated singularity The function $f(z) = 1/\sin(1/z)$ has singularities when $1/z = \pi k$, $k \in \mathbb{Z}$, i.e. for $z = 1/(\pi k)$, $k \in \mathbb{Z}$. The singularity at $z = 0$ is in fact not isolated, as we can always find another singularity arbitrarily close to 0.

Riemann extension theorem *If f has an isolated singularity at $a \in \Omega$ and f is bounded in Ω, then the singularity is removable.*

Proof of Riemann extension theorem Consider the Laurent series expansion $f(z) = \sum_{n \in \mathbb{Z}} c_n (z-a)^n$ in $\Delta_\varepsilon(a) \backslash \{a\}$, where

$$c_n = \frac{1}{2\pi i} \int_{\partial \Delta_\delta(a)} \frac{f(\zeta) d\zeta}{(\zeta - a)^{n+1}} \Rightarrow |c_n| \leq \frac{1}{2\pi} \frac{\sup_{\partial \Delta_\delta(a)} |f|}{\delta^{n+1}} 2\pi\delta \tag{3.1}$$

For $n < 0$ $|c_n| \to 0$ as $\delta \to 0$ (using that $|f|$ is bounded, hence $\sup |f|/\delta^n \to 0$). Hence $c_n = 0$ for all $n < 0$ and f has the form $f(z) = \sum_{n \geq 0} c_n (z-a)^n$ for all $z \in a$ in $\Delta_\varepsilon(a)$. This defines $\widetilde{f}(z)$ a holomorphic extension of f which can be extended to $z = a$, and so the singularity is removable. $\square$

Pole An isolated singularity a is called a *pole* is $\lim_{z \to a} f(z) = \infty$.

Theorem *Let f has an isolated singularity at a, then f has a pole at a $\Leftrightarrow$ the Laurent series of f about a has a non-zero and finite principal part.*

Proof of theorem ($\Rightarrow$) Let f have a pole at a, and consider $g(z) = 1/f(z)$, then $g(z)$ is holomorphic in some $\Delta_\delta(a) \backslash \{a\}$ and $g(z) \to 0$ as $z \to a$ so g is bounded (for suitable δ) and by the Riemann extension theorem g has a holomorphic extension $\widetilde{g}(z)$. We can factorise $\widetilde{g}(z) = (z-a)^d \varphi(z)$, $\varphi(a) \neq 0$, so $f(z) = (1/(z-a)^d)(1/\varphi(z))$. Now $1/\varphi(z)$ is holomorphic in some neighbourhood of a so we can write $1/\varphi(z) = \sum_{n \geq 0} c_n (z-a)^n$, and hence $f(z) \sum_{n \geq 0} c_n (z-a)^{n-d} = \sum_{m \geq -d} c_{m+d} (z-a)^m$.

($\Leftarrow$) Let the Laurent series of f at a by $f(z) = \sum_{n \geq -d} c_n (z-a)^n$. with $d > 0$. Then clearly $\lim_{z \to a} f(z) = \infty$ as we can write $f(z) = \sum_{n \geq -d} c_n (z-a)^{n-d}/(z-a)^d$ and $\lim_{z \to a} \sum_{n \geq -d} c_n (z-a)^{n-d} = c_{-d} \neq 0$. $\square$

Pole order If f has a pole at a with $f(z) = \sum_{n \geq -d} c_n (z-a)^n$ then the *order of the pole* at a is d.

Combining singularities (Palka 5.20) Suppose that f has a pole at a of order d_1 and g has a pole at a of order d_2. We can expand them both in Laurent series as $f(z) = \sum_{n \geq d_1} c_n (z-a)^n$ and $g(z) = \sum_{n \geq -d_2} c'_n (z-a)^n$. Consider the sum $f + g$. Clearly we could have $c_n = -c'_n$ for some (or all) $-d \leq n < 0$, so that either $f + g$ wil have a pole of order less than $\max(d_1, d_2)$ or it will have a removable singularity at a. Next consider the product fg. Multiplying the Laurent series together we see that we will always have a leading term $c_{-d_1} c_{d_2} (z-a)^{-d_1-d_2}$ and thus a pole of order $d_1 + d_2$ at a. Finally consider the quotient g/f. This is holomorphic in some $\Delta_\varepsilon(a) \backslash \{a\}$. Write $g(z) = (z-a)^{-d_2} \varphi(z)$ and $f(z) = (z-a)^{-d_1} \psi(z)$ where $\varphi(a) \neq 0$ and $\psi(a) \neq 0$. Hence $g/f = (z-a)^{d_1-d_2} \varphi(z)/\psi(z)$, which has a pole of order $d_2 - d_1$ at a if $d_1 < d_2$ and a removable singularity at a if $d_1 \geq d_2$.

Application of the Riemann extension theorem and Liouville's theorem (Palka 5.27) Let f and g be holomorphic in $\mathbb{C}$ and let $|g(z)| \leq |f(z)|$. We will show that $g(z) = cf(z)$ for some constant c. Consider the quotient $g(z)/f(z)$ (note that if $f \equiv 0$ then $g \equiv 0$ and the result holds trivially, we assume $f \not\equiv 0$ from now on). This is holomorphic except for points where $f(z)$ is zero (these are isolated as $f(z) \not\equiv 0$ is holomorphic). As $|g(z)/f(z)| \leq 1$ we see that $g(z)/f(z)$ is bounded and so by the Riemann extension theorem can be extended to a holomorphic function at any possible singularities. The extended function is bounded and holomorphic in $\mathbb{C}$ and so constant by Liouville's theorem. We conclude that $g(z) = cf(z)$.

Meromorphic A function f is called *meromorphic* in Ω if it is holomorphic in $\Omega\backslash A$ where $A \subset \Omega$ discrete and f has a pole at all points $a \in A$. We denote by $\mathcal{M}(\Omega)$ the space of functions which are meromorphic in Ω.

Examples of meromorphic functions All rational functions (ratios of two polynomials) are meromorphic on $\mathbb{C}$. The function $f(z) = 1/\sin(1/z)$ is meromorphic on $\mathbb{C}\backslash\{0\}$.

Meromorphic functions on Ω form a ring

Essential singularity A singularity of a function f is called *essential* if it is neither removable nor a pole. We have that f has an essential singularity at $a \Leftrightarrow$ the principal part of the Laurent series expansion of f about a is infinite.

Casoratti-Weierstrass Theorem *If f has an essential singularity at a then $f(\Delta_\varepsilon(a)\backslash\{a\})$ is dense in $\mathbb{C}$ for all small ε.*

Proof of Casoratti-Weierstass Theorem If $f(\Delta_\varepsilon(a)\backslash\{a\})$ is not dense in $\mathbb{C}$ then there exists $w_0 \in \mathbb{C}$ and $\delta > 0$ such that $|f(z) - w_0| \geq \delta$ for all $z \in \Delta_\varepsilon(a)\backslash\{a\}$. Define $g(z) = 1/(f(z) - w_0)$, then $g(z)$ has a singularity at a but $|g(z)| \leq 1/\delta$, so g is bounded and this singularity is removable. Also $1/g$ must have an isolated singularity at a, which is either a pole or a removable singularity, and so the singularity of $f(z) = \frac{1}{g(z)} + w_0$ at a is either a pole or a removable singularity, contradicting the fact that a is an essential singularity. $\square$

Great Picard Theorem *Another remarkable result: if f has an essential singularity at a then $\mathbb{C}\backslash f(\Delta_\varepsilon(a)\backslash\{a\})$ has at most one point.*

Example of essential singularity The functions $f(z) = e^{1/z}$, $g(z) = \sin 1/z$ and $h(z) = \cos 1/z$ all have essential singularities at 0, as can be seen from writing out the power series definitions of these functions and noting that they give a Laurent series with infinite principal part.

4 Argument Principle

Theorem *Let $f \in \mathcal{M}(\Omega)$, $\overline{D} \subset \Omega$ a simple region, $\gamma = \partial D$, and let f have no poles or zeros on γ. Then*

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_k \mathrm{ord}_f(a_k) = \text{number of zeros - number of poles of } f \text{ in } \Omega \tag{4.1}$$

where a_k is a zero or pole.

Proof of theorem We use the residue theorem which tells us that

$$\frac{1}{2\pi i}\int_\gamma \frac{f'(z)}{f(z)}dz = \sum_{a_k} \mathrm{Res}_{a_k} \frac{f'(z)}{f(z)} \tag{4.2}$$

Now, for a a pole or zero we can expand $f(z) = (z-a)^d\varphi(z)$ where $d = \mathrm{ord}_f(a)$ is either the zero order or minus the pole order at a. Hence $f'(z) = d(z-a)^{d-1}\varphi(z) + (z-a)\varphi'(z)$ and so

$$\frac{f'(z)}{f(z)} = \frac{d}{z-a} + \frac{\varphi'(z)}{\varphi(z)} \tag{4.3}$$

which has residue at $d = \mathrm{ord}_f(a)$ at a. $\square$

Rouché's Theorem *Let $F, f \in \mathcal{M}(\Omega)$, $D \subset \Omega$ a simple region and $\overline{D} \subset \Omega$, $\gamma = \partial D$ and $|f(z)| < |F(z)|$ on γ, and let f and F have no poles on γ. Denote by n_F the number of zeros minus the number of poles of F, then $n_{F+f} = n_F$.*

Proof of Rouché's Theorem Note that $|F(z)| > |f(z)| \geq 0$ so F has no zeros on γ; similarly, $|F + f| \geq |F| - |f| > 0$ so $F + f$ has no zeros on γ. Define

$$\varphi(t) = \frac{1}{2\pi i} \int_\gamma \frac{F'(z) + tf'(z)}{F(z) + tf(z)} = n_{F+tf} \qquad 0 \leq t \leq 1 \tag{4.4}$$

so that $\varphi(0) = n_F$ and $\varphi(1) = n_{F+f}$. We have $|F + tf| \geq |F| - t|f| > 0$, so $F + tf$ has no zeros on γ, and hence $\varphi(t) \in \mathbb{N}$ for all t. As the integrand in the definition of $\varphi(t)$ is continuous in both z and t, $\varphi(t)$ is also continuous and hence it must be constant (as it is valued in the natural numbers), hence the result. $\square$

Open Mapping Theorem *Let f holomorphic in Ω, Ω connected and $f \not\equiv$ constant, then $f(\Omega)$ is open.*

Proof of Open Mapping Theorem Let $w_0 \in f(\Omega)$, so that $w_0 = f(z_0)$ for some $z_0 \in \Omega$. As $f \not\equiv$ constant we have that $f \not\equiv w_0$ in any neighbourhood of z_0, so that $\mathrm{ord}_{f-w_0}(z_0) = d \neq \infty$. We can factorise $f(z) - w_0 = (z - z_0)^d \varphi(z)$ with $\varphi(z_0) \neq 0$. We now want to count the number of solutions of $f(z) = w$ for w near w_0.

Now, $f(z_0) = w_0$ implies that $f(z) - w_0$ has a zero of order d at z_0. Let $D = \Delta_\varepsilon(z_0)$ be such that $\overline{D} \subset \Omega$ and $\varphi(z) \neq 0$ on $\overline{D}$. Then $f(z) - w_0$ has no zeros on ∂D and d zeros in D. Now as $f(z) - w_0$ is continuous, $|f(z) - w_0|$ has a minimum on ∂D, so there exists $\delta > 0$ such that $|f(z) - w_0| \geq \delta$ for $z \in \partial D$. Take $u \in \mathbb{C}$ such that $|u| < d$ and set $\tilde{f}(z) = u$ for all $z \in \mathbb{C}$. Then $|\tilde{f}| < |f(z) - w_0|$ on ∂D, and by Rouché's theorem, $f(z) - w_0$ and $f(z) - w_0 + \tilde{f}(z)$ have the same number of zeros. Hence $f(z) = w_0 - u$ has a solution for all u such that $|u| < \delta$, and so $\Delta_\delta(w_0) \subset f(\Omega)$ and so $f(\Omega)$ is open. $\square$

Univalence Theorem *Let f holomorphic in Ω, $z_0 \in \Omega$ then f injective in some neighbourhood of z_0 $\Leftrightarrow f'(z_0) \neq 0$.*

Proof of Univalence Theorem ($\Leftarrow$) Suppose that $f'(z_0) \neq 0$. Let $w_0 = f(z_0)$, then $f(z) - w_0 = (z - z_0)\varphi(z)$ with $\varphi(z_0) \neq 0$, and $f(z) = w \Leftrightarrow (z - z_0)\varphi(z) + w_0 - w = 0$ for w near w_0. Take $\Delta_\varepsilon(z_0)$ such that $\varphi(z) \neq 0$ on $\overline{\Delta_\varepsilon(z_0)}$, then $(z - z_0)\varphi(z)$ has one zero in $\Delta_\varepsilon(z_0)$. Now $|f(z)| < |(z - z_0)\varphi(z)|$ on $\partial\Delta_\varepsilon(z_0)$ and $|(z - z_0)\varphi(z)| \neq 0$ on $\partial\Delta_\varepsilon(z_0)$. Denote $\delta = \min_{\partial\Delta_\varepsilon(z_0)} |(z - z_0)\varphi(z)| > 0$ and take w such that $|w - w_0| < \delta$. Then $f(z) = w$ has one solution in $\partial\Delta_\varepsilon(z_0)$ (by Rouché's theorem for $(z - z_0)\varphi(z)$ and $w - w_0$). Choose a neighbourhood U of z_0 such that if $z \in U$ then $|f(z) - w_0| < \delta$ by continuity, then f is injective in U.

($\Rightarrow$) Let f be injective in some neighbourhood of z_0, so that then $f \not\equiv$ constant in that neighbourhood. Suppose that $f'(z_0) = 0$ so that $\mathrm{ord}_{f(z)-w_0}(z_0) = d > 1$ and we can factorise $f(z) - w_0 = \psi(z)^d$. Choose U a neighbourhood of z_0 such that f injective in U, ψ holomorphic in U and $\psi \not\equiv$ constant. Then $\psi(U)$ is open and $\psi(z_0) = 0$. Hence there exists $\Delta_\varepsilon(0) \subset \psi(U)$. Take $\tau_1 = \varepsilon/2$ and $\tau_2 = \alpha_d \varepsilon/2$ where $\alpha_d \neq 1$ and $\alpha_d^d = 1$. Then $\tau_1 \neq \tau_2 \in \Delta_\varepsilon(0) \subset \psi(U)$ but $\tau_1^d = \tau_2^d$, and so there exists $z_1, z_2 \in U$ such that $\psi(z_1) = \tau_1$, $\psi(z_2) = \tau_2$. Now $z_1 \neq z_2$ but $f(z_1) = \psi(z_1)^d = \psi(z_2)^d = f(z_2)$ and so f is not injective, a contradiction. $\square$

Inverse Function Theorem *Let f holomorphic in Ω, f injective then $f(\Omega)$ open, then $f^{-1} : f(\Omega) \to \Omega$ holomorphic.*

Proof of Inverse Function Theorem That $f(\Omega)$ is open follows from the Open Mapping theorem. As f is injective there exists an inverse $f^{-1} : f(\Omega) \to \Omega$. Now f^{-1} is continuous if for all U open $(f^{-1})^{-1}(U) = f(U)$ open, which is true. We will use the Univalence Theorem to show that $\lim f^{-1}(w) - f^{-1}(w_0)/(w - w_0)$ exists. Now, this equals $\lim_{w\to w_0}(z - z_0)/(f(z) - f(w_0)$, and we know that $\lim_{z\to z_0}(f(z) - f(z_0))/(z - z_0)$ exists and is non-zero (by the Univalence Theorem). We also need $w \to w_0$ as $z \to z_0$ but this holds by continuity, hence

$$\lim_{w\to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0} = \frac{1}{\lim_{z\to z_0} \frac{f(z)-f(z_0)}{z-z_0}} = \frac{1}{f'(z_0)} \tag{4.5}$$

exists, and f^{-1} is $\mathbb{C}$-differentiable and so f^{-1} is holomorphic. □

5 Riemann Mapping Theorem

Lemma

Proof of lemma

Riemann Mapping Theorem *If $\Omega \subset \mathbb{C}$, $\Omega \neq \mathbb{C}$, then there exists a biholomorphic map $f : \Omega \to \Delta$.*

Outline of proof of Riemann Mapping Theorem

6 Möbius transformations and automorphisms

Riemann sphere We obtain the *Riemann sphere* by adding the *point at infinity* to $\mathbb{C}$. The resulting space is denoted $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We have the following rules for manipulating the point at infinity: $1/0 = \infty$, $1/\infty = 0$.

The terminology “Riemann sphere” comes from the fact that we can in fact represent every point in $\overline{\mathbb{C}}$ as a point on the unit sphere S^2. If we embed the complex plane within $\mathbb{R}^3$ we obtain the point $(x, y, z) \in S^2$ corresponding to the point $z \in \overline{\mathbb{C}}$ by projecting from z to the north pole of the unit sphere at the origin: the point where this projection intersects the sphere gives the coordinates of the point as an element of the latter. Note that ∞ corresponds to the north pole of the sphere.

The map

$$\tau(x, y, t) = \frac{x + iy}{1 - t} \tag{6.1}$$

maps a point (x, y, t) on the Riemann sphere to a point in $\overline{\mathbb{C}}$. The inverse mapping, for a point $z = x + iy$, is

$$\tau^{-1}(z) = \Big(2x/(1 + |z|^2), 2y/(1 + |z|^2), (1 - |z|^2)/(1 + |z|^2)\Big) \tag{6.2}$$

establishing that τ is an isomorphism (and in fact a homeomorphism).

The topology on $\overline{\mathbb{C}}$ can be thought of as being induced by τ. The open sets in $\overline{\mathbb{C}}$ are the open sets in $\mathbb{C}$ and the complements of compact subsets of $\mathbb{C}$ (i.e. discs around ∞).

Holomorphic/meromorphic at ∞ To be able to define concepts of holomorphy etc. at ∞ we will need the inversion

$$\sigma(z) = \frac{1}{z} \tag{6.3}$$

which maps $\overline{\mathbb{C}}$ to $\overline{\mathbb{C}}$ and in particular swaps ∞ and 0. This can essentially be thought of as a change of chart on the Riemann sphere. On $\overline{\mathbb{C}}$ we have the following:

- $z_0 \in \mathbb{C}$, $f(z_0) \in \mathbb{C}$: usual holomorphy/meromorphy.

- $z_0 = \infty$, $f(z_0) \in \mathbb{C}$: f holomorphic/meromorphic at $\infty \Leftrightarrow f \circ \sigma$ holomorphic/meromorphic at 0.
- $z_0 \in \mathbb{C}$, $f(z_0) = \infty$: f holomorphic/meromorphic at $\infty \Leftrightarrow \sigma \circ f$ holomorphic/meromorphic at 0.
- $z_0 = \infty$, $f(z_0) = \infty$: f holomorphic/meromorphic at $\infty \Leftrightarrow \sigma \circ f \circ \sigma$ holomorphic/meromorphic at 0.

Theorem *Any meromorphic function on $\overline{\mathbb{C}}$ is rational.*

Proof of theorem As f is meromorphic it has isolated poles, so for all $a \in \overline{\mathbb{C}}$ there exists an open set $U_a \subset \overline{\mathbb{C}}$ such that $a \in U_a$ and f is holomorphic in $U_a \backslash \{a\}$. We have $\overline{\mathbb{C}} = \bigcup_{a \in \overline{\mathbb{C}}} U_a$ and as $\overline{\mathbb{C}}$ is compact (recall it is isomorphic to S^2) there exists a finite subcover $\overline{\mathbb{C}} = U_{a_1} \cup \cdots \cup U_{a_N}$. This (somewhat subtly) shows that f has $n \leq N$ poles. We now induct on the number of poles.

If $n = 0$ then f is holomorphic in $\overline{\mathbb{C}}$ and so is a map $f : \overline{\mathbb{C}} \to \mathbb{C}$, hence $\lim_{z \to \infty} f(z) = f(\infty) \in \mathbb{C}$. Thus f is bounded in a neighbourhood of ∞, i.e. on $\{|z| \geq R\}$. As a continuous function on a compact set f is also bounded on $\{|z| \leq R\}$ and so is globally bounded. By Liouville's theorem the restriction of f to $\mathbb{C}$ is constant, and by continuity f is constant on $\overline{\mathbb{C}}$, and so rational.

Now let f have n poles. If f has a pole at ∞ then $f \circ \sigma$ has a pole at 0. In this case it suffices to show that $f \circ \sigma$ is rational, as then $f \circ \sigma \circ \sigma = f$ is also rational. Hence we can suppose that f has a pole at $a \in \mathbb{C}$. In some $\Delta_\varepsilon(a)$ we have the Laurent series expansion $f(z) = \sum_{k \in \mathbb{Z}} c_k (z-a)^k$ with finite principal part $p(z) = \sum_{k<0} c_k (z-a)^k$. Clearly $p(z)$ is rational and holomorphic in $\mathbb{C} \backslash \{a\}$. If f has poles at $a, a_1, \ldots, a_{n-1}$ then $f - p$ has poles $a_1, \ldots, a_n$, as near a $f(z) - p(z) = \sum_{k \geq 0} c_k (z-a)^k$, a convergent power series, and hence $f - p$ has no pole at a. So $f - p$ has fewer poles than f and so by the induction hypothesis is rational. As p is rational it follows that f is rational. □

Möbius transformation A *Möbius transformation* is a mapping of the form

$$f(z) = \frac{az + b}{cz + d} \tag{6.4}$$

where $a, b, c, d \in \mathbb{C}$ satisfy $ad - bc \neq 0$. Note that $f(z)$ has a pole at $z = -d/c$ and so is meromorphic on $\mathbb{C}$. Extending the definition of $f(z)$ from $\mathbb{C}$ to $\overline{\mathbb{C}}$ we obtain a meromorphic function on the extended complex plane.

Möbius transformations and matrices It is clear that we can associate a Möbius transformation to each two-by-two invertible complex matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \leftrightarrow f_A(z) = \frac{az + b}{cz + d} \tag{6.5}$$

There is not a uniquely defined correspondence, as we see that scaling A by any non-zero complex number λ does not change the Möbius transformation f_A.

Group structure of Möbius transformations

Automorphism An *automorphism* of Ω is a biholomorphic map $f : \Omega \to \Omega$.

Automorphisms of complex plane $\mathbb{C}$ We claim that $\mathrm{Aut}\,(\mathbb{C})$ consists of affine linear transformations, $f(z) = az + b$, $a, b \in \mathbb{C}$, $a \neq 0$. It is clear that any such transformation maps $\mathbb{C}$ biholomorphically to itself. To see that these exhaust all such maps, suppose that $f(z)$ is a biholomorphic map of $\mathbb{C}$ to itself. Such a map has a power series expansion valid in the entire plane, $f(z) = \sum_n c_n z^n$. As $f(z)$ is not identically constant, $a_n \neq 0$ for some $n \geq 1$. We then have a singularity as $z \to \infty$ which is either a pole or an essential singularity. If essential, then by the Casoratti-Weierstrass theorem, the image of any disc V around ∞ is dense in $\mathbb{C}$. In particular if we consider a disc V around ∞ and a disc U around 0 then $f(U) \cap f(V) \neq \emptyset$. A point in the intersection then has a preimage in U and a preimage in V, contradicting the injectivity of f. We conclude that the singularity is a pole, so that the power series expansion must terminate after finitely many terms. Suppose that $f(z) = \sum_{n \geq 0}^{m} c_n z^n$. If $m > 1$ then $f'(z)$ is a polynomial of degree at least one, and so has a root in $\mathbb{C}$. But injectivity requires that $f'(z) \neq 0$ for all $z \in \mathbb{C}$. We conclude that f must be of the form $f(z) = az + b$. □

Automorphisms of extended complex plane $\overline{\mathbb{C}}$ We claim that $\mathrm{Aut}\,(\overline{\mathbb{C}})$ consists of all Möbius transformations. It is clear that any such transformations maps $\overline{\mathbb{C}}$ biholomorphically to itself (we can explicitly write down the inverse $f_{A^{-1}}$ for any Möbius transformation f_A). Now suppose that f is an arbitrary biholomorphic map of $\overline{\mathbb{C}}$. If $f(\infty) = \infty$, set $g(z) = f(z)$. If $f(\infty) = w_0 \neq \infty$, set $g(z) = 1/(f(z) - w_0)$ so that $g(\infty) = \infty$. In either case $g(z)$ gives a biholomorphic map of $\mathbb{C}$ to itself, and so is of the form $g(z) = az + b$, which is Möbius. Then either $f(z) = g(z)$ or $1/g(z) + w_0$, and is Möbius.

Automorphisms of unit disc Δ We claim that $\mathrm{Aut}\,\Delta$ consists of maps $\varphi_{a,\theta}(z) = e^{i\theta} \frac{az-1}{\bar{a}z+1}$. We already know such maps are biholomorphic mappings. Now let $f(z)$ be a biholomorphic map of the disc onto itself. Suppose that $f(0) = a$, and define $g = \varphi_a \circ f$ so that $g(0) = 0$. By the Schwarz lemma, $|g(z)| \leq |z|$. But g is also biholomorphic, and its inverse g^{-1} also sends 0 to 0, so $|g^{-1}(w)| \leq |w|$, i.e. $z \leq |g(z)|$, so $|g(z)| = |z|$ and g is a rotation.

Now let $f^{-1}(0) = b$ and let $h = f \circ \varphi_b$ so that $h(0) = 0$. By the same arguments, $|h(z)| = |z|$ so that $h(z)$ is a rotation. We also see that $f = h \circ \varphi_b^{-1} = h \circ \varphi_b$ which is of the desired form.

Automorphisms of upper half-plane H