Course 231: Equations of Mathematical Physics

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These notes cover the 2007-2008 Methods course given by Dr. Conor Houghton, up until the end of Hilary term (excluding power series).

Contents

Part I
Vector Calculus

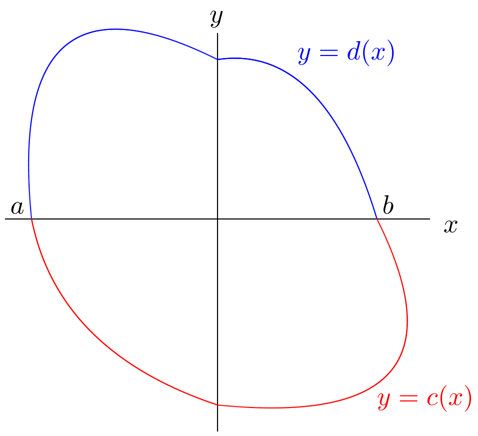
1 2 and 3-Dimensional Integration

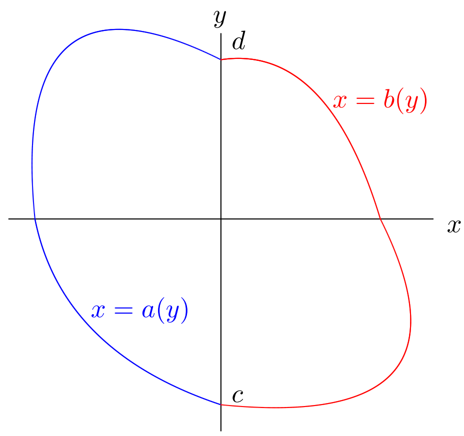
1.1 2-Dimensional Integration

We wish to integrate a scalar field $\phi(x, y)$ over a 2-dimensional region R

$$\iint_R dA\, \phi(x, y)$$

To do so we write the integral as a double integral, iterated either first in the y and then the x direction, or vice versa.





If the top boundary of R can be written as $y = d(x)$ and the bottom as $y = c(x)$ then we can write

$$\iint_R dA\, \phi(x, y) = \int_a^b dx \int_{c(x)}^{d(x)} dy\, \phi(x, y)$$

i.e. we integrate first over y from $c(x)$ to $d(x)$, and then over x from a to b . Similarly in the other direction,

$$\int\int_R dA\,\phi(x,y) = \int_c^d dy \int_{a(y)}^{b(y)} dx\,\phi(x,y)$$

1.2 3-Dimensional Integration

The idea in 3-dimensional integration is similar; we write the top surface as $z = f(x,y)$ and the bottom surface as $z = e(x,y)$, and treat the projection of the region in the xy plane as in the 2-dimensional case, hence

$$\int\int\int_R dV\,\phi(x,y,z) = \int_a^b dx \int_{c(x)}^{d(x)} dy \int_{e(x,y)}^{f(x,y)} dz\,\phi(x,y,z)$$

1.3 Changing Coordinates

Jacobian

It is often convenient to integrate using a different set of coordinates. Change of coordinates involves a scaling factor known as the Jacobian. In two-dimensions for old coordinates x, y and new coordinates u, v we have

$$dA = dx\,dy = J du\,dv$$

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \left\| \begin{array}{cc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{array} \right\|$$

and in 3-dimensions, for old coordinates x, y, z and new coordinates u, v, w we have

$$dV = dx\,dv\,dz = J du\,dv\,dw$$

$$J = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \left\| \begin{array}{ccc} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{array} \right\|$$

Examples

- Cylindrical Polar Coordinates

$$x = r\cos\phi \qquad y = r\sin\theta \qquad z = z$$

$$J = r$$

where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$.

- Spherical Coordinates

$$x = r\sin\theta\cos\phi \qquad y = r\sin\theta\sin\phi \qquad z = r\cos\theta$$

$$J = r^2\sin\theta$$

where $r \in [0, \infty)$, $\phi \in [0, 2\pi)$, $\theta \in [0, \pi]$.

2 Vector Operators

2.1 Grad, Div and Curl

There are some important vector operators related to the operator $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$.

- grad

  The gradient of a scalar field ϕ is the vector field

$$\operatorname{grad}\phi = \vec{\nabla}\phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z}\right)$$

- div

  The divergence of a vector field $\vec{F} = (F_1, F_2, F_3)$ is

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}$$

  If div $\vec{F} = 0$ then we say $\vec{F}$ is solenoidal.

- curl

  The curl of a vector field $\vec{F} = (F_1, F_2, F_3)$ is

$$\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$$

  If curl $\vec{F} = 0$ we say $\vec{F}$ is irrotational.

- Laplacian

  The Laplacian of a scalar field ϕ is the scalar field

$$\Delta\phi = \operatorname{div}\operatorname{grad}\phi = \vec{\nabla}\cdot\vec{\nabla}\phi = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

2.2 Vector Identities

Some important vector identities related to these operators are:

1. $\vec{\nabla}(\phi\psi) = \phi\vec{\nabla}\psi + \psi\vec{\nabla}\phi$
2. $\vec{\nabla}(\phi\vec{F}) = \vec{\nabla}\phi\cdot\vec{F} + \phi\vec{\nabla}\vec{F}$
3. $\vec{\nabla}\times(\phi\vec{F}) = \vec{\nabla}\phi\times\vec{F} + \phi\vec{\nabla}\times\vec{F}$
4. $\vec{\nabla}\cdot(\vec{F}\times\vec{G}) = (\vec{\nabla}\times\vec{F})\cdot\vec{G} - \vec{F}\cdot(\vec{\nabla}\times\vec{G})$
5. $\vec{\nabla}\times(\vec{F}\times\vec{G}) = (\vec{\nabla}\cdot\vec{G})\vec{F} + (\vec{G}\cdot\vec{\nabla})\vec{F} - (\vec{\nabla}\cdot\vec{F})\vec{G} - (\vec{F}\cdot\vec{\nabla})\vec{G}$
6. $\vec{\nabla}(\vec{F}\cdot\vec{G}) = \vec{F}\times(\vec{\nabla}\times\vec{G}) + \vec{G}\times(\vec{\nabla}\times\vec{F}) + (\vec{F}\cdot\vec{\nabla})\vec{G} + (\vec{G}\cdot\vec{\nabla})\vec{F}$
7. $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0$
8. $\vec{\nabla}\times(\vec{\nabla}\phi) = 0$
9. $\vec{\nabla}\cdot\vec{\nabla}\phi = \Delta\phi$

3 Line Integrals and Conservative Fields

3.1 Line Integrals

To integrate a vector field $\vec{F}$ along a curve C we parameterise the curve C in terms of t as $\vec{r}(t)$ and use

$$\int_C \vec{F} \cdot d\vec{l} = \int \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt$$

3.2 Conservative Fields

A vector field $\vec{F}$ is called conservative if there exists a scalar field ϕ such that

$$\vec{F} = \vec{\nabla}\phi$$

A vector field $\vec{F}$ is called path independent if the line integral $\int_C \vec{F} \cdot d\vec{l}$ between any two points is the same for all paths C between those two points.

The following are equivalent: $\vec{F}$ is conservative, $\vec{F}$ is path independent, and $\oint_C \vec{F} \cdot d\vec{l} = 0$.

Proof. i) Conservative $\Rightarrow$ path independence: let $\vec{F} = \vec{\nabla}\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z}\right)$, and consider ϕ restricted to a curve $\vec{r}(t)$, so that on the curve we have $\phi = \phi(\vec{r}(t))$ and

$$\frac{d\phi}{dt} = \frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}$$

Now consider

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{l} &= \int_{t_1}^{t_2} \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}\, dt \\
&= \int_{t_1}^{t_2} \vec{\nabla}\phi \cdot \frac{d\vec{r}}{dt}\, dt \\
&= \int_{t_1}^{t_2} \left(\frac{\partial\phi}{\partial x}\frac{dx}{dt} + \frac{\partial\phi}{\partial y}\frac{dy}{dt} + \frac{\partial\phi}{\partial z}\frac{dz}{dt}\right) dt \\
&= \int_{t_1}^{t_2} \frac{d\phi}{dt} dt \\
&= \phi(t_1) - \phi(t_2)
\end{aligned}$$

by the Fundamental Theorem of Calculus, showing that $\vec{F}$ is path independent.

ii) Path independence $\Rightarrow \oint_C \vec{F} \cdot d\vec{l} = 0$: let C_a and C_b be two curves with the same endpoints P_1 and P_2, then

$$\int_{C_a} \vec{F} \cdot d\vec{l} = \int_{C_b} \vec{F} \cdot d\vec{l}$$

and now consider the closed curve $C = C_a - C_b$, then

$$\oint_C \vec{F} \cdot d\vec{l} = \int_{C_a} \vec{F} \cdot d\vec{l} - \int_{C_b} \vec{F} \cdot d\vec{l} = 0$$

for all closed loops C.

iii) $\oint_C \vec{F} \cdot d\vec{l} = 0 \Rightarrow$ path independence: let $\oint_C \vec{F} \cdot d\vec{l} = 0$ for all closed loops C, and let C_1 and C_2 be two paths between two points a and b. Then we have

$$\oint_C \vec{F} \cdot d\vec{l} = \int_{C_1} \vec{F} \cdot d\vec{l} - \int_{C_2} \vec{F} \cdot d\vec{l} = 0$$
$$\Rightarrow \int_{C_1} \vec{F} \cdot d\vec{l} = \int_{C_2} \vec{F} \cdot d\vec{l}$$

for all C_1 and C_2, so path independence.

iv) Path independence $\Rightarrow$ conservative: let p be some point and let

$$\phi(\vec{x}) = \int_{C(p,\vec{x})} \vec{F} \cdot d\vec{l}$$

where $C(p, \vec{x})$ is any curve from p to $\vec{x} = (x, y, z)$. We will show that $\vec{F} = \nabla\phi$. Componentwise, we have to prove that

$$F_1 = \frac{\partial \phi}{\partial x} = \frac{\partial}{\partial x} \int_{C(p,\vec{x})} \vec{F} \cdot d\vec{l}$$

We choose a path C that goes from p to a point $p' = (x', y, z)$, so that the path from p' to $\vec{x}$ is a straight line. We then have

$$\frac{\partial}{\partial x} \int_{C(p,\vec{x})} \vec{F} \cdot d\vec{l} = \frac{\partial}{\partial x} \int_{C(p,p')} \vec{F} \cdot d\vec{l} + \frac{\partial}{\partial x} \int_{C(p',\vec{x})} \vec{F} \cdot d\vec{l}$$

where the first integral does not depend on x and so is zero when differentiated. We now parameterise $C(p', \vec{x})$ as

$$\vec{r}(t) = t\hat{i} + y\hat{j} + z\hat{k}$$
$$\Rightarrow \frac{d\vec{r}}{dt} = \hat{i}$$

and so we have

$$\begin{aligned} \frac{\partial}{\partial x} \int_{C(p',\vec{x})} \vec{F} \cdot d\vec{l} &= \frac{\partial}{\partial x} \int_{x'}^{x} \vec{F} \cdot \frac{d\vec{r}}{dt} dt \\ &= \frac{\partial}{\partial x} \int_{x'}^{x} \vec{F} \cdot \hat{i} \\ &= \frac{\partial}{\partial x} \int_{x'}^{x} F_1 dt \\ &= F_1 \end{aligned}$$

using the Fundamental Theorem of Calculus. Similarly for the other components. □

4 Surface Integrals

To integrate a vector field $\vec{F}$ over a surface we parameterise the surface as $\vec{r}(u, v)$ and use

$$\iint_S \vec{F} \cdot d\vec{A} = \iint_D du\, dv\, \vec{F}(\vec{r}(u, v)) \cdot \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$$

where D is the domain in $\mathbb{R}^2$ of u, v. Note that the choice $\frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v}$ gives one orientation of the surface; $\frac{\partial \vec{r}}{\partial v} \times \frac{\partial \vec{r}}{\partial u}$ is the other.

5 Integrating Scalars

We can define two- and three-dimensional integrals of scalars by

$$\int_C \phi\, d\vec{l} = \int_C \phi \left| d\vec{l} \right| = \int_{t_1}^{t_2} \phi \left| \frac{d\vec{r}}{dt} \right| dt$$

$$\iint_S \phi\, d\vec{S} = \iint_S \phi \left| d\vec{S} \right| = \iint_D \phi \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dudv$$

6 The Integral Theorems

6.1 Green's Theorem

Let D be a region in the xy plane bounded by a piecewise smooth curve C oriented anti-clockwise. Then if $f(x,y)$ and $g(x,y)$ have continuous first derivatives

$$\iint_D dA \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) = \oint_C (f dx + g dy)$$

Proof. Consider D a simple region (i.e. a region where a double integral can iterated in either order). Then,

$$\begin{aligned}
\iint_D dA \left(\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y} \right) &= \int_c^d dy \int_{a(y)}^{b(y)} dx\, \frac{\partial g}{\partial x} - \int_a^b dx \iint_{c(x)}^{d(x)} dy\, \frac{\partial f}{\partial y} \\
&= \int_c^d dy\, g(x,y) \Big]_{a(y)}^{b(y)} - \int_a^b dx\, f(x,y) \Big]_{c(x)}^{d(x)} \\
&= \int_c^d dy\, g(b(y),y) - \int_c^d dy\, g(a(y),y) - \int_a^b dx\, f(x,d(x)) + \int_a^b dx\, f(x,(c(x)) \\
&= \oint_C dy\, g(x(y),y) + \oint_C dx\, f(x,y(x)) \\
&= \oint_C (f dx + g dy)
\end{aligned}$$

as required. For an arbitrary region, we can divide the region up into many simple regions and sum. □

6.2 Stokes' Theorem

Let S be a piecewise smooth orientable surface with boundary C a piecewise smooth curve oriented so that $\hat{n} \times d\vec{l}$ points into the surface. Let $\vec{F}$ be a continuously differentiable vector field in the neighbourhood of S, then

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{l}$$

Proof. Consider first the simple case $\vec{F} = F_3(x,y,z)\hat{k}$, and a simple region that can be parameterised by x, y, that is $z = h(x,y)$, so that

$$\vec{r} = x\hat{i} + y\hat{j} + h(x,y)\hat{k}$$

is the parameterised surface. We then have

$$\frac{\partial \vec{r}}{\partial x} = \hat{\imath} + \frac{\partial h}{\partial x}\hat{k} \qquad \frac{\partial \vec{r}}{\partial y} = \hat{\jmath} + \frac{\partial h}{\partial y}\hat{k}$$

$$\Rightarrow \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 0 & \frac{\partial h}{\partial x} \\ 0 & 1 & \frac{\partial h}{\partial y} \end{vmatrix} = -\frac{\partial h}{\partial x}\hat{\imath} - \frac{\partial h}{\partial y}\hat{\jmath} + \hat{k}$$

and

$$\operatorname{curl} \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & F_3 \end{vmatrix} = \frac{\partial F_3}{\partial y}\hat{\imath} - \frac{\partial F_3}{\partial x}\hat{\jmath}$$

so

$$\begin{aligned} \operatorname{curl} \vec{F} \cdot \left(\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}\right) &= -\frac{\partial F_3}{\partial y}\frac{\partial h}{\partial x} + \frac{\partial F_3}{\partial x}\frac{\partial h}{\partial y} \\ &= -\frac{\partial}{\partial y}\left(F_3\frac{\partial h}{\partial x}\right) + \frac{\partial}{\partial x}\left(F_3\frac{\partial h}{\partial y}\right) \end{aligned}$$

as

$$\begin{aligned} \frac{\partial}{\partial x}\left(F_3\frac{\partial h}{\partial y}\right) &= \frac{\partial F_3}{\partial x}\frac{\partial h}{\partial y} + \frac{\partial F_3}{\partial h}\frac{\partial h}{\partial x}\frac{\partial h}{\partial y} + F_3\frac{\partial^2 h}{\partial x \partial y} \\ \frac{\partial}{\partial y}\left(F_3\frac{\partial h}{\partial x}\right) &= \frac{\partial F_3}{\partial y}\frac{\partial h}{\partial x} + \frac{\partial F_3}{\partial h}\frac{\partial h}{\partial y}\frac{\partial h}{\partial x} + F_3\frac{\partial^2 h}{\partial y \partial x} \end{aligned}$$

so then

$$\iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = \iint_D dx\, dy \left[\frac{\partial}{\partial x}\left(F_3\frac{\partial h}{\partial y}\right) - \frac{\partial}{\partial y}\left(F_3\frac{\partial h}{\partial x}\right)\right]$$

and we now apply Green's theorem with $f = F_3\frac{\partial h}{\partial x}$ and $g = F_3\frac{\partial h}{\partial y}$, hence

$$\begin{aligned} \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} &= \oint_{\delta D} F_3\frac{\partial h}{\partial x}dx + F_3\frac{\partial h}{\partial y}dy \\ &= \oint_{\delta D} F_3\left(\frac{\partial z}{\partial x}dx + \frac{\partial z}{\partial y}dy\right) \\ &= \oint_{\delta D} F_3\, dz \\ &= \oint_C \vec{F} \cdot d\vec{l} \end{aligned}$$

as required. We can perform similar calculations for $\vec{F} = F_1\hat{\imath}$ and $\vec{F} = F_2\hat{\jmath}$ and sum them to give the general result. For a more general surface S we can split S into a number of simple surfaces and integrate over each of them. □

An application of Stokes' theorem is to show that on a simply connected domain $\operatorname{curl} \vec{F} = 0 \Rightarrow \vec{F}$ conservative. A simply connected domain is a domain such that any smooth curve can be shrunk to a point. Given some curve C we can shrink it to a point and let S be the surface traced by the shrinking, hence any closed curve can be expressed as the boundary of some surface. Then if curl $\vec{F} = 0$,

$$\oint_C \vec{F} \cdot d\vec{l} = \iint_S \operatorname{curl} \vec{F} \cdot d\vec{S} = 0$$

and hence $\vec{F}$ is conservative.

6.3 Gauss' Theorem

Let D be a connected three-dimensional region in $\mathbb{R}^3$ whose boundary is a closed piecewise smooth surface S. Then if $\vec{F}$ is a vector field with continuous first derivatives in a domain containing D

$$\iiint_D \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

Proof. Consider just D simple, and let $\vec{F} = F_3\hat{k}$, so div $\vec{F} = \frac{\partial F_3}{\partial z}$. We now write

$$\iiint_D \operatorname{div} \vec{F}\, dV = \iint_{D_2} \int_{e(x,y)}^{f(x,y)} dz \frac{\partial F_3}{\partial z}$$

where D_2 is the parameter region in the xy plane, so

$$\iiint_D \operatorname{div} \vec{F}\, dV = \iint_{D_2} \Big[F_3(x, y, f(x, y)) - F_3(x, y, e(x, y))\Big]$$

and we parameterise the top surface by

$$\vec{r} = x\hat{i} + y\hat{j} + f(x, y)\hat{k}$$

then

$$\iint_{top} \vec{F} \cdot d\vec{S} = \iint_{D_2} dx\, dy\, \vec{F}(x, y, f(x, y)) \cdot \frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y}$$

and we have

$$\frac{\partial \vec{r}}{\partial x} \times \frac{\partial \vec{r}}{\partial y} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & \frac{\partial f}{\partial x} \\ 0 & 1 & \frac{\partial f}{\partial y} \end{vmatrix} = -\frac{\partial f}{\partial x}\hat{i} - \frac{\partial f}{\partial y}\hat{j} + \hat{k}$$

hence for $\vec{F} = F_3\hat{k}$ we get

$$\iint_{top} \vec{F} \cdot d\vec{S} = \iint_{D_2} dx\, dy\, F_3(x, y, f(x, y))$$

and similarly for the bottom surface,

$$\iint_{bot} \vec{F} \cdot d\vec{S} = -\iint_{D_2} dx\, dy\, F_3(x, y, e(x, y))$$

hence

$$\iiint_D \operatorname{div} \vec{F}\, dV = \iint_S \vec{F} \cdot d\vec{S}$$

and again the proof is similar for other components, and by summing these components works for all $\vec{F}$. As before, we can treat more general domains as a collection of simple domains.

□

7 Vector Potentials

If div $\vec{F} = 0$ then $\vec{F} = \text{curl}\,\vec{A}$ for some $\vec{A}$ called the vector potential. The converse is true only on domains in $\mathbb{R}^3$ with no 3-dimensional obstructions (i.e. domains where any closed surface can be shrunk to a point).

A star-shaped domain is a region D in which there exists a point a such that the line segment between a and any point $x \in D$ lies in D. On such domains if div $\vec{F} = 0$ then we can obtain a vector potential for $\vec{F}$ using the formula

$$\vec{A}(r) = \int_0^1 dt\, \vec{F}(t\vec{r}) \times t\vec{r}$$

The Hodge decomposition of a vector field $\vec{F}$ neither solenoidal nor irrotational on D a simply connected domain with no obstructions to 2-spheres is

$$\vec{F} = \text{curl}\,\vec{A} + \text{grad}\,\phi$$

Part II
Fourier Analysis

8 Fourier Series

8.1 Real Fourier Series

Consider a function $f(x)$ with period l, then the Fourier series expansion of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{2\pi nx}{l}\right)$$

We can find the Fourier coefficients using the following properties of sin and cos:

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \cos\left(\frac{2\pi nx}{l}\right) = \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \sin\left(\frac{2\pi nx}{l}\right) = 0$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \cos\left(\frac{2\pi nx}{l}\right) \sin\left(\frac{2\pi mx}{l}\right) = 0$$

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \cos\left(\frac{2\pi nx}{l}\right) \cos\left(\frac{2\pi mx}{l}\right) = \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \sin\left(\frac{2\pi nx}{l}\right) \sin\left(\frac{2\pi mx}{l}\right) = \frac{1}{2}\delta_{mn}$$

where m, n are positive integers and δ_{mn} is the Kronecker delta,

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases}$$

To find a_0 we integrate both sides over a period

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x) = \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \cos\left(\frac{2\pi nx}{l}\right)}_{=0} + \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \sin\left(\frac{2\pi nx}{l}\right)}_{=0}$$

hence

$$a_0 = \frac{2}{l} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x)$$

To find a_n we multiply across by $\cos\left(\frac{2\pi m x}{l}\right)$ and integrate over a period

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x) \cos\left(\frac{2\pi m x}{l}\right) = \underbrace{\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \frac{a_0}{2} \cos\left(\frac{2\pi m x}{l}\right)}_{=0} + \sum_{n=1}^{\infty} a_n \underbrace{\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \cos\left(\frac{2\pi n x}{l}\right) \cos\left(\frac{2\pi m x}{l}\right)}_{=\frac{1}{2}\delta_{mn}}$$

$$+ \sum_{n=1}^{\infty} b_n \underbrace{\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \sin\left(\frac{2\pi n x}{l}\right) \cos\left(\frac{2\pi m x}{l}\right)}_{=0}$$

hence

$$a_n = \frac{2}{l} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x) \cos\left(\frac{2\pi n x}{l}\right)$$

and similarly

$$b_n = \frac{2}{l} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x) \sin\left(\frac{2\pi n x}{l}\right)$$

8.2 Complex Fourier Series

The complex Fourier series of a function $f(x)$ with period l is

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{l}\right)$$

where the complex Fourier coefficients are given by

$$c_n = \frac{1}{l} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x) \exp\left(-\frac{2\pi i n x}{l}\right)$$

which comes from a similar method to before, using

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \exp\left(\frac{2\pi i x}{l}(n-m)\right) = l\delta_{mn}$$

One feature of the complex Fourier series is that if $f(x)$ is real then $\overline{c_n} = c_{-n}$, where the overline denotes the complex conjugate.

8.3 Dirichlet's and Parseval's Theorems

Dirichlet's Theorem: If $f(x)$ is periodic with a finite number of minima and maxima in one period, and with a finite number of discontinuities in one period, and

$$\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, |f(x)|^2 < \infty$$

then the Fourier series for f is convergent, and converges to $f(x)$ for all points where f is continuous. For points a where $f(x)$ is discontinuous it converges to

$$\frac{1}{2}\left[\lim_{x\to a_+} f(x) + \lim_{x\to a_-} f(x)\right]$$

i.e. it extrapolates across the discontinuity.

Parseval's Theorem: The L^2 norm of $f(x)$ is

$$\frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, |f(x)|^2 = \frac{1}{4}a_0^2 + \frac{1}{2}\sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Proof. (complex case) We have that

$$\begin{aligned}
\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, |f(x)|^2 &= \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x)\overline{f(x)} \\
&= \int_{-\frac{l}{2}}^{\frac{l}{2}} dx \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{l}\right) \sum_{n=-\infty}^{\infty} \overline{c_m} \exp\left(-\frac{2\pi i m x}{l}\right) \\
&= \sum_{n,m} c_n \overline{c_m} \int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, \underbrace{\exp\left(\frac{2\pi i x}{l}(n-m)\right)}_{=\delta_{mn}} \\
&= \sum_{n,m} c_n \overline{c_m} \delta_{mn} l \\
&= l \sum_{n=-\infty}^{\infty} |c_n|^2
\end{aligned}$$

□

9 Fourier Integrals

Consider the complex Fourier series,

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi i n x}{l}\right) \qquad c_n = \frac{1}{l}\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x) \exp\left(-\frac{2\pi i n x}{l}\right)$$

$$\Rightarrow f(x) = \sum_{n=-\infty}^{\infty} \frac{2\pi}{l}\left[\frac{1}{2\pi}\int_{-\frac{l}{2}}^{\frac{l}{2}} dx\, f(x) \exp\left(-\frac{2\pi i n x}{l}\right)\right] \exp\left(\frac{2\pi i n x}{l}\right)$$

and as l gets large we have $\delta k = \frac{2\pi}{l}$, and can write $k = \frac{2\pi}{l}n$, and argue that in the limit $l \to \infty$ the sum over n becomes an integral over k, hence

$$f(x) = \int_{-\infty}^{\infty} dk \left[\frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x) \exp(-ikx)\right] \exp(ikx)$$

the Fourier integral of f. We may write this as

$$f(x) = \int_{-\infty}^{\infty} dk\, \widetilde{f(k)} \exp(ikx)$$

where

$$\widetilde{f(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx\, f(x)\exp(-ikx)$$

is known as the Fourier transform of $f(x)$. The Fourier integral formula holds if $f(x)$ is L^1, that is, if it satisfies

$$\int_{-\infty}^{\infty} dx\, |f(x)| < \infty$$

10 The Dirac Delta Function

10.1 Definition and Properties

The Dirac delta function (which is not, strictly speaking, a function) may be defined by

$$\delta(x) = \begin{cases} 0 & x \neq 0 \\ \infty & x = 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x)dx = 1$$

with the characteristic property

$$\int_{-\infty}^{\infty} \delta(x)f(x)dx = f(0)$$

We then have

$$\int_{-\infty}^{\infty} \delta(x-a)f(x)dx = f(a)$$

established using the substitution $y = x - a$.

We can consider the derivative of the delta function by integrating by parts

$$\begin{aligned}\int_{-\infty}^{\infty} \delta'(x)f(x)dx &= \delta(x)f(x)\Big]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \delta(x)f'(x)dx \\ &= -\int_{-\infty}^{\infty} \delta(x)f'(x)dx \\ &= -f'(0)\end{aligned}$$

We consider $\delta(ax)$ by letting $y = ax \Rightarrow dy = adx$ so

$$\int_{-\infty}^{\infty} \delta(ax)f(x)dx = \int_{-\infty}^{\infty} \frac{1}{a}\delta(y)f\left(\frac{y}{a}\right) = \frac{1}{a}f(0)$$

if $a > 0$, and

$$\int_{-\infty}^{\infty} \delta(ax)f(x)dx = \int_{\infty}^{-\infty} \frac{1}{a}\delta(y)f\left(\frac{y}{a}\right) = \int_{-\infty}^{\infty} -\frac{1}{a}\delta(y)f\left(\frac{y}{a}\right) = -\frac{1}{a}f(0)$$

if $a < 0$, so then in general

$$\int_{-\infty}^{\infty} \delta(ax)f(x)dx = \frac{1}{|a|}f(0)$$

Now let $h(x)$ be a smooth function, and consider

$$\int_{-\infty}^{\infty} \delta(h(x))f(x)dx$$

If $h(x)$ has no zeros this is identically zero. Suppose $h(x)$ has one zero, $h(x_1) = 0$ and suppose $h'(x_1) > 0$. Then we can write

$$\int_{-\infty}^{\infty} \delta(h(x)) f(x) dx = \int_{a}^{b} \delta(h(x)) f(x) dx$$

where $h'(x) > 0$ on (a, b), with $x_1 \in (a, b)$. This means we can invert $h(x)$ on the interval; let $y = h(x)$ then $x = h^{-1}(y)$, and also

$$dy = h'(x) dx \Rightarrow dx = \frac{dy}{h'(h^{-1}(y))}$$

giving

$$\int_{a}^{b} \delta(h(x)) f(x) dx = \int_{h(a)}^{h(b)} \delta(y) f(h^{-1}(y)) \frac{dy}{h'(h^{-1}(y))}$$

and $y = 0$ for $x = x_1$, so this integrates to

$$\frac{f(h^{-1}(0))}{h'(h^{-1}(0))} = \frac{f(x_1)}{h'(x_1)}$$

If $h' < 0$ then everything is the same except one of the limits of integration will change giving a minus, so then

$$\int_{-\infty}^{\infty} \delta(h(x)) f(x) dx = \frac{f(x_1)}{|h'(x_1)|}$$

If h has multiple zeros then we can split the integral up into multiple intervals to get

$$\int_{-\infty}^{\infty} \delta(h(x)) f(x) dx \quad = \sum_{x_i : h(x_i) = 0} \frac{f(x_i)}{|h'(x_i)|}$$

so

$$\delta(h(x)) = \sum_{x_i : h(x_i) = 0} \frac{\delta(x - x_i)}{|h'(x_i)|}$$

10.2 Delta Function and Fourier Integrals

Consider

$$\delta(x) = \int_{-\infty}^{\infty} dk \, \widetilde{\delta(k)} \exp(ikx)$$

where

$$\widetilde{\delta(k)} = \frac{1}{2\pi} \int_{-\infty}^{\infty} dx \, \delta(x) \exp(-ikx) = \frac{1}{2\pi}$$

$$\Rightarrow \delta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \exp(ikx)$$

giving us the orthogonality relation

$$\delta(x - x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \, \exp\left[ik(x - x')\right]$$

We also have that the Fourier transform of the constant function $f(x) = 1$ is

$$\widetilde{f(k)} = \widetilde{1(k)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \exp(-ikx) = \delta(k)$$

Parseval's Theorem for Fourier Integrals: also called Plancherel's formula,

$$\int_{-\infty}^{\infty} dx\, |f(x)|^2 = 2\pi \int_{-\infty}^{\infty} dk\, |\widetilde{f(k)}|^2$$

Proof.

$$\begin{aligned}
\int_{-\infty}^{\infty} dx\, |f(x)|^2 &= \int_{-\infty}^{\infty} dx\, f(x)\overline{f(x)} \\
&= \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx'\, f(x)\overline{f(x')}\delta(x-x') \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dk\, f(x)\overline{f(x)} \exp\left[ik(x-x')\right] \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk \left(\int_{-\infty}^{\infty} dx\, f(x)\exp(ikx)\right)\left(\int_{-\infty}^{\infty} dx'\, \overline{f(x')}\exp(-ikx')\right) \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, 2\pi\widetilde{f(k)}\overline{2\pi\widetilde{f(k)}} \\
&= 2\pi\int_{-\infty}^{\infty} dk\, |\widetilde{f(k)}|^2
\end{aligned}$$

as required. □

Part III
Ordinary Differential Equations

11 First Order ODEs

These are equations of the form

$$y' + p(x)y = f(x)$$

with general solution

$$y = y_p + Cy_h$$

where y_p is a particular solution to the inhomogeneous equation, and y_h is the solution to the homogeneous version. We can solve this equation by multiplying across by an integrating factor λ such that $\lambda' = p\lambda$, as then we have

$$\lambda y' + p\lambda y = \lambda f$$

$$\Rightarrow \lambda y' + \lambda' y = \lambda$$

$$\Rightarrow (\lambda y)' = \lambda f$$

which we can solve by integrating. It follows that we must have

$$\lambda = \exp(px)$$

if p constant, or in general,

$$\lambda = \exp\left(\int_a^x p(z)dz\right)$$

where a is an arbitrary constant.

12 Second Order ODEs

These are equations of the form

$$a(x)y'' + b(x)y' + c(x)y = f(x)$$

with general solution

$$y = C_1 y_1 + C_2 y_2 + y_p$$

where y_1 and y_2 are solutions of the homogeneous equation, and y_p is the particular solution to the inhomogeneous version.

12.1 Homogeneous Constant Coefficients

In the homogeneous constant coefficient case

$$ay'' + by' + cy = 0$$

we guess solution $y = e^{\lambda x}$ which gives

$$a\lambda^2 + b\lambda + c = 0$$

known as the auxiliary equation, with roots λ_1 and λ_2. The general solution is then

$$y = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$$

In the case that $\lambda_1 = \lambda_2 = \lambda$ then the general solution is

$$y = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$$

We can solve the inhomogeneous case easily in the case that $f(x) = Fe^{rx}$, by guessing $y = Ce^{rx}$ which gives

$$ar^2C + brC + cC = F \Rightarrow C = \frac{F}{ar^2 + br + c}$$

and the general solution is

$$y = C_1 e^{\lambda_1 x} + C_2 x e^{\lambda_2 x} + Ce^{rx}$$

In the case that r coincides with one of the roots of the auxiliary equation, then we instead guess $y = Cxe^{rx}$ (and in the exceptional cases that both roots are equal and equal to r, $y = Cx^2e^{rx}$).

12.2 Solving Using Fourier Analysis

We can analyse periodic driving forces using Fourier analysis by writing $f(x)$ as a Fourier series, hence

$$ay'' + by' + cy = \sum_{n=-\infty}^{\infty} c_n \exp\left(\frac{2\pi inx}{l}\right)$$

and then solve

$$ay_n'' + by_n' + cy_n = c_n \exp\left(\frac{2\pi inx}{l}\right)$$

giving general solution

$$y = \sum_{n=-\infty}^{\infty} y_n$$

Similarly, if $f(x)$ is non-periodic but decays at $\pm\infty$ we can write it as a Fourier integral

$$f(x) = \int_{-\infty}^{\infty} dk \widetilde{f(k)} e^{ikx}$$

and then solve

$$ay_k'' + ky_n' + cy_k = \widetilde{f(k)} e^{ikx}$$

and integrate over the solutions

$$y = \int_{-\infty}^{\infty} dk\, y_k$$

12.3 Cauchy-Euler Equations

These are equations of the form

$$\alpha x^2 y'' + \beta x y' + \gamma y = 0$$

which can be solved using the substitution

$$x = e^z \Rightarrow z = \log x$$

so that

$$y' = \frac{d}{dx} y = \frac{dz}{dx}\frac{dy}{dz} = \frac{1}{x}\frac{dy}{dz}$$

$$y'' = \frac{d}{dx} y' = \frac{d}{dx}\frac{1}{x}\frac{dy}{dz} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x}\frac{d}{dx}\frac{dy}{dz} = -\frac{1}{x^2}\frac{dy}{dz} + \frac{1}{x^2}\frac{d^2y}{dz^2}$$

The equation becomes

$$\alpha\left(-\frac{dy}{dz} + \frac{d^2y}{dz^2}\right) + \beta\frac{dy}{dz} + \gamma y = 0$$

or

$$\alpha\frac{d^2y}{dz^2} + (\beta - \alpha)\frac{dy}{dz} + \gamma y = 0$$

hence we have auxiliary equation

$$\alpha\lambda^2 + (\beta - \alpha)\lambda + \gamma\lambda = 0$$

and solution

$$y = C_1 e^{\lambda_1 z} + C_2 e^{\lambda_2 z} = C_1 x^{\lambda_1} + C_2 x^{\lambda_2}$$

(A more direct approach would be to substitute $y = x^\lambda$ in the original equation, though in the case of repeated roots you need to know to multiply the second solution by $\log x$.)