Theorems

Linear Operators

Theorem 1 (Hamilton-Cayley) Let $M \xrightarrow{T} M$ be a linear operator on a vector space M, u_i a basis for M, so $Tu_j = \alpha^i_j u_i$ with matrix $A = (\alpha^i_j)$. Then the characteristic polynomial of T is $p = \det(A - XI)$ and

$$p(T) = 0$$

Theorem 2 The eigenvalues of T are the zeros of the minimal polynomial.

Theorem 3 If $p, f, g \in K[x]$, K a field and p irreducible then p divides $fg \Rightarrow p$ divides f or p divides g.

Theorem 4 (Unique Factorisation) Let $f \in K[x]$, degree $f > 0$, then

$$f = \alpha p_1 \dots p_k$$

where $\alpha \in K$ and $p_1 \dots p_k$ are irreducible and monic, uniquely up to reordering of factors.

Theorem 5 (Primary Decomposition) Let $M \xrightarrow{T} M$ a linear operator which satisfies a polynomial equation with only linear factors

$$(T - \lambda_1 \mathbb{I})^{r_1} \dots (T - \lambda_k \mathbb{I})^{r_k} = 0$$

with λ_i distinct scalars, r_i positive integers, then M is the direct sum of the generalised eigenspaces,

$$M = \ker(T - \lambda_1 \mathbb{I})^{r_1} \oplus \dots \oplus (T - \lambda_k \mathbb{I})^{r_k}$$

and $x = x_1 + \dots x_k$ uniquely.

Theorem 6 If $M \xrightarrow{T} M$ diagonalisable and N T-invariant then T_N is diagonalisable.

Theorem 7 Let S, T be commuting linear operators on M, then each eigenspace of S is T-invariant and vice versa, and if both S and T diagonalisable then they are simultaneously diagonalisable.

Linear Forms

Theorem 8 Let $u_1 \dots u_n$ be a basis for M, then $u^1 \dots u^n$ are a basis for the dual space M^*, called the basis dual to $u_1 \dots u_n$.

Theorem 9 A system $f_1 = 0, \dots, f_m = 0$ of homogeneous linear equations of rank r on an n-dimensional vector space has an $(n - r)$-dimensional solution space.

Scalar Products

Theorem 10 Let M be a Euclidean or Hilbert space, then
i) $||\alpha x|| = |\alpha|\,||x||$,
ii) if $x \neq 0$ then $\frac{x}{||x||}$ has norm 1,
iii) $|(x\,|\,y)| \leq ||x||\,||y||$ (Cauchy-Schwarz inequality),
iv) $||x+y|| \leq ||x||\,||y||$ (triangle inequality).

Theorem 11 Let $(\cdot\,|\,\cdot)$ be a scalar product on a vector space M, and N a finite dimensional vector subspace such that $(\cdot\,|\,\cdot)_N$ non-degenerate, then

$$M = N \oplus N^{\perp}$$

Theorem 12 Let $(\cdot\,|\,\cdot)$ be a symmetric or hermitian scalar product on a finite dimensional space M. Then M has a basis u_i of mutually orthogonal vectors, i.e. $(u_i\,|\,u_j) = 0$ if $i \neq j$, i.e. the matrix of the scalar product is diagonal.

Theorem 13 (Sylvester's Theorem) Let $u^1 \dots u^n$ and $w^1 \dots w^n$ be linear coordinates on a real or complex vector space M and let

$$\begin{aligned} F &= \left|u^1\right|^2 + \cdots + \left|u^r\right|^2 - \left|u^{r+1}\right|^2 - \cdots - \left|u^{r+s}\right|^2 + 0\left|u^{r+s+1}\right|^2 + \cdots + 0\left|u^n\right|^2 \\ &= \left|w^1\right|^2 + \cdots + \left|w^t\right|^2 - \left|w^{t+1}\right|^2 - \cdots - \left|w^{t+k}\right|^2 + 0\left|w^{u+k+1}\right|^2 + \cdots + 0\left|w^n\right|^2 \end{aligned}$$

then $r = t$ and $s = k$.

Adjoints

Theorem 14 Let $M \xrightarrow{T} M$ have matrix $A = (\alpha^i_j)$ with respect to an orthonormal basis, then T^* has matrix A^t (Euclidean) or $\overline{A}^t$ (Hilbert).

Theorem 15 Let $M \xrightarrow{T} M$ be self-adjoint and M be a Hilbert space then all the eigenvalues of T are real numbers.

Theorem 16 If A an $n \times n$ Hermitian matrix then all the roots of its characteristic polynomial are real.

Theorem 17 Let N be invariant under T, then $N^{\perp}$ invariant under T^*.

Theorem 18 (Spectral Theorem) Let $M \xrightarrow{T} M$ be either a self-adjoint operator on a finite dimensional Euclidean space M or a normal operator on a finite dimensional Hilbert space M, then M has an orthonormal basis of eigenvectors of T, thus $M = M_1 \oplus \cdots \oplus M_k$, a direct sum of mutually orthogonal eigenspaces.

Theorem 19 (Heisenberg Uncertainty Relation) Let P, Q be linear operators satisfying commutation relation $PQ - QP = \alpha\mathbb{I}$, $\alpha \in \mathbb{C}$, then

$$(\Delta P)(\Delta Q) \leq \frac{1}{2}|\alpha|$$

Tensors

Theorem 20 Let u_i be a basis for M then e.g. $u^i \otimes u_j \otimes u^k$ is a basis for $M^* \otimes M \otimes M^*$.

Theorem 21 Contraction is well-defined.

Theorem 22 $D(Ax_1 \dots Ax_n) = \det A D(x_1 \dots x_n)$ for all $x_i \in K^n$.

Theorem 23 If $A, B \in K^{n\times n}$ then $\det AB = \det A \det B$.

Theorem 24 $A \in K^{n\times n}$ invertible $\Leftrightarrow \det A \neq 0$.

Theorem 25 (Cramer's Rule) If $A \in K^{n\times n}$ invertible and $Ax = b$ then

$$x_i = \frac{D(a_1 \dots b \dots a_n)}{\det A} \quad b \text{ in } i^{th} \text{ slot}$$

Theorem 26 (Inverse Formula) The (i, j)-entry of A^{-1} is given by

$$\frac{D(a_1 \dots e_j \dots a_n)}{\det A} \quad e_j \text{ in } i^{th} \text{ slot}$$

Theorem 27 Let $T \in \mathcal{T}^r M$, then $\sum_{\phi\in S_r} \varepsilon^\phi \phi \cdot T$ is skew-symmetric.

Theorem 28 Let $\dim M = n$ and u_i a basis for M, then
i) $M^{(r)} = \{0\}, M_{(r)} = \{0\}$ if $r > n$,
ii) $\{u^{i_1} \wedge \dots \wedge u^{i_r}\}_{i_1<\dots<i_r}$ a basis for $M^{(r)}$, $\{u_{i_1} \wedge \dots \wedge u_{i_r}\}_{i_1<\dots<i_r}$ a basis for $M_{(r)}$, for each $0 \leq r \leq n$.

Theorem 29 The skew-symmetriser satisfies

$$\mathcal{A}\left[(\mathcal{A}S) \otimes T\right] = \mathcal{A}\left[S \otimes T\right] = \mathcal{A}\left[S \otimes (\mathcal{A}T)\right]$$

and

$$\mathcal{A}(S \otimes T) = (-1)^{st}\mathcal{A}(T \otimes S)$$

Theorem 30 The wedge product is bilinear, associative, super-commutative and satisfies

$$R_1 \wedge \dots \wedge R_k = \frac{(r_1 + \dots + r_k)!}{r_1! \dots r_k!}\mathcal{A}\left(R_1 \otimes \dots \otimes R_k\right)$$

Push-forward and Pull-back

Theorem 31 Let $M \xrightarrow{T} M$ be a linear map of finite dimensional K-vector spaces then for each integer $r \geq 1$ the push-forward

$$\underbrace{M \otimes \dots \otimes M}_{r} \xrightarrow{T_*} \underbrace{N \otimes \dots \otimes N}_{r}$$

is a covariant functor from K-vect fd to K-vect fd, and the pull-back

$$\underbrace{M^* \otimes \dots \otimes M^*}_{r} \xleftarrow{T^*} \underbrace{N^* \otimes \dots \otimes N^*}_{r}$$

is a contravariant functor.

Theorem 32 The push-forward and pull-back preserve tensor products, commute with permutations and preserve wedge products.

Orientation

Theorem 33 The volume form $u^1 \wedge \cdots \wedge u^n$ is independent of choice of standard basis $u_1 \ldots u_n$, and if $(\cdot \,|\, \cdot)$ has components $g_{ij} = (w_i \,|\, w_j)$ with respect to a positively oriented basis $w_1 \ldots w_n$ then

$$\text{vol} = \sqrt{|\det g_{ij}|}\, w^1 \wedge \cdots \wedge w^n$$

Theorem 34 Let $u_1 \ldots u_n$ be a standard basis for M, then

$$*u^1 \wedge \cdots \wedge u^r = s_{r+1} \ldots s_n u^{r+1} \wedge \cdots \wedge u^n$$

where $s_i = (u_i \,|\, u_j) = \pm 1$.

Continuity

Theorem 35 $B_X(a, r)$ is open in X.

Theorem 36 Let $M \supset X \xrightarrow{f} Y \subset N$, then f is continuous at $a \Leftrightarrow$ for each V open in Y such that $f(a) \in V$ there exists W open in X such that $fW \subset V$.

Theorem 37 Let $X \xrightarrow{f} Y$, X, Y topological spaces, then f is continuous $\Leftrightarrow$ V open in $Y \Rightarrow f^{-1}V$ open in X.

Theorem 38 Let

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[dr, "gf"'] & Y \arrow[d, "g"] \\ & Z \end{tikzcd}$$

then f, g continuous $\Rightarrow gf$ continuous.

Differentiability

Theorem 39 Let $M \supset V \xrightarrow{f} W \subset N$ be differentiable at $a \in V$. Then the derivative $f'(a)$ is uniquely determined by the formula

$$\begin{aligned} f'(a)h &= \lim_{t \to 0} \frac{f(a+th) - f(a)}{t} \\ &= \frac{d}{dt} f(a+th)\Big|_{t=0} \\ &= \text{the directional derivative of } f \text{ at } a \text{ along } h \end{aligned}$$

Theorem 40 Let $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}^m$ be differentiable, V open, where $f(x) = (f^1(x), \ldots, f^m(x))$, $f^i(x) = f^i(x_1, \ldots, x_n)$. Then the derivative

$$\mathbb{R}^n \xrightarrow{f'(a)} \mathbb{R}^m$$

is the $m \times n$ matrix

$$f'(a) = \left(\frac{\partial f^i(a)}{\partial x^j} \right) \quad i = 1 \ldots m \ , \ j = 1 \ldots n$$

Theorem 41 (Chain Rule For Functions on Finite Dimensional Real or Complex Vector Space) Let $U \xrightarrow{g} V \xrightarrow{f} W$ and $U \xrightarrow{f \cdot g} W$ where U, V, W are open subsets of finite dimensional real or complex vector spaces. Let g be differentiable at a, f differentiable at $g(a)$, then $f \cdot g$ is differentiable at a and

$$\Big(f \cdot g \Big)' = f'\Big(g(a) \Big) g'(a)$$

Theorem 42 Let $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}$, V open, then f is $C^1 \Leftrightarrow \frac{\partial f}{\partial x^i}$ exists and is continuous for all i.

Theorem 43 If $\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}$, V open, is C^2, then

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}$$

Theorem 44 (Mean Value Theorem for Functions on Finite Dimensional Normed Spaces) Let $M \supset V \xrightarrow{f} N$ be C^1. Let $x, y \in V$ such that

$$[x, y] = \{ tx + (1-t)y \,|\, 0 \leq t \leq 1 \} \subset V$$

Let

$$\Big|\Big| f' \left[tx + (1-t)y \right] \Big|\Big| \leq k \quad \forall 0 \leq t \leq 1$$

then

$$\Big|\Big| f(x) - f(y) \Big|\Big| \leq k \Big|\Big| x - y \Big|\Big|$$

Theorem 45 (Inverse Function Theorem) Let $M \supset V \xrightarrow{f} N$ be a C^r function on open V, with M, N finite dimensional real or complex vector spaces. Let $a \in V$ at which

$$M \xrightarrow{f'(a)} N$$

is invertible, then there exists an open neighbourhood W of a such that

$$W \xrightarrow{f} f(W)$$

is a C^r diffeomorphism onto open $f(W)$ in N.

Manifolds

Theorem 46 (Implicit Function Theorem) Let $f = (f^1 \ldots f^l)$ be C^r real-valued functions on an open set V in $\mathbb{R}^n$, so

$$\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}^l$$

$$\mathbb{R}^n \xrightarrow{f'(x)} \mathbb{R}^l \quad (l \times n \text{ matrix})$$

and let

$$X = \{x \in V | f(x) = 0\}$$

be the space of solutions of the l equations, $f^1 = 0, \dots, f^l = 0$. Let $a \in X$ be a point at which rank $f'(a) = l$ with (say) the first l columns of $f'(a)$ being linearly independent. Then there exists an open neighbourhood U of a in X such that

$$x^{l+1} \dots x^n$$

are coordinates on X with domain U, and $x^1 \dots x^l$ are C^r functions of $x^{l+1} \dots x^n$ on U.

Thus, if $X = \{x \in V | f(x) = 0, \text{ rank } f'(x) = l\}$, then X is an $(n-l)$-dimensional C^r manifold.

Theorem 47 Let X be a smooth n-dimensional manifold and let $a \in X$, then T_aX is a real n-dimensional vector space and if $y = (y^i)$ coordinates at a then

$$\frac{\partial}{\partial y^1}_a \cdots \frac{\partial}{\partial y^n}_a$$

a basis for T_aX.

Theorem 48 Let $X \xrightarrow{\phi} Y$ be smooth, and $f \in C^\infty(Y)$, then

$$\phi^* df = d\phi^* f$$

ie the pull-back commutes with differentials, and the following diagram is commutative:

$$\begin{tikzcd}
\phi^* f \; C^\infty(X) \arrow[d, "d"] & C^\infty(Y) \; f \arrow[l, "\phi^*"'] \arrow[d, "d"] \\
\phi^* df = d\phi^* f \; \Omega^1(X) & \Omega^1(Y) \; df \arrow[l, "\phi^*"']
\end{tikzcd}$$

Theorem 49 (Chain Rule for Maps of Manifolds) Let

$$\begin{tikzcd}
X \arrow[rr, "\phi"] \arrow[dr, "\psi \cdot \phi"'] & & Y \arrow[dl, "\psi"] \\
& Z &
\end{tikzcd}$$

be a commutative diagram of smooth maps of manifolds, then

$$\begin{tikzcd}
TX \arrow[rr, "\phi_*"] \arrow[dr, "(\psi \cdot \phi)_*"'] & & TY \arrow[dl, "\psi_*"] \\
& TZ &
\end{tikzcd}$$

is a commutative diagram, ie $(\psi \cdot \phi)_* = \psi_* \cdot \phi_*$, or $(\psi \cdot \phi)'(x) = \psi'(\phi(x))\phi'(x)$