Manifolds

Coordinate System Let V be an open set of a topological space X. An n-dimensional coordinate system on X with domain V is a homeomorphism y of V onto an open set $y(V)$ of $\mathbb{R}^n$.

Manifold A topological space X is called an n-dimensional manifold if for each $a \in X$ there exists an n-dimensional coordinate system with domain a neighbourhood of a.

Function Let $y = (y^1 \dots y^n)$ be coordinates with domain V. Let $V \xrightarrow{f} \mathbb{R}$, then there exists a unique F, domain $y(V)$, such that

$$f(x) = F(y^1(x) \dots y^n(x))$$

for all $x \in V$.

C^r-compatible Let $y = (y^i)$ with domain V and $z = (z^i)$ with domain W be coordinate systems on a manifold X. We say y and z are C^r-compatible if y^i is a C^r function of $z^1 \dots z^n$ and z^i is a C^r function of $y^1 \dots y^n$, on $V \cap W$.

C^r manifold A topological space X is called a C^r manifold if a collection of mutually C^r-compatible coordinate systems are given whose domains cover all of X. Such a collection is called an atlas on X. Similarly for C^∞ or smooth manifold.

Implicit Function Theorem Let $f = (f^1 \dots f^l)$ be C^r real-valued functions on an open set V in $\mathbb{R}^n$, so

$$\mathbb{R}^n \supset V \xrightarrow{f} \mathbb{R}^l$$

$$\mathbb{R}^n \xrightarrow{f'(x)} \mathbb{R}^l \quad (l \times n \text{ matrix})$$

and let

$$X = \{x \in V | f(x) = 0\}$$

be the space of solutions of the l equations, $f^1 = 0, \dots, f^l = 0$. Let $a \in X$ be a point at which rank $f'(a) = l$ with (say) the first l columns of $f'(a)$ being linearly independent. Then there exists an open neighbourhood U of a in X such that

$$x^{l+1} \dots x^n$$

are coordinates on X with domain U, and $x^1 \ldots x^l$ are C^r functions of $x^{l+1} \ldots x^n$ on U.

Thus, if $X = \{x \in V | f(x) = 0, \operatorname{rank} f'(x) = l\}$, then X is an $(n-l)$-dimensional C^r manifold.

Coordinates at a Let $a \in X$ a smooth manifold. Coordinates $y = (y^i)$ are called coordinates at a if their domain is a neighbourhood of a.

Smooth at a A real-valued function f on a smooth manifold X is called smooth at a if its domain is a neighbourhood of a and there exist coordinates y^i at a such that $f = F(y^1(a), \ldots, y^n(a))$ on a neighbourhood of a with F C^∞.

Partial Derivative If f smooth at a, y^i coordinates at a, with $f = F(y^1, \ldots, y^n)$ on a neighbourhood of a then we write

$$\frac{\partial f}{\partial y^j}(a) = \frac{\partial F}{\partial x^j}(y^1(a), \ldots, y^n(a))$$

called the partial derivative of f with respect to the j^{th} coordinate with respect to the coordinates y^i.

Smooth Path A parametrised path α in X, domain U open in $\mathbb{R}$

$$\begin{aligned} \mathbb{R} \supset U &\xrightarrow{\alpha} X \\ t &\mapsto \alpha(t) \end{aligned}$$

is called smooth if $y^i(\alpha(t))$ is a C^∞ function of t (when defined) for any coordinates (y^i) on X.

Tangent Vector For each parameter $t \in U$ define an operator $\dot{\alpha}(t)$ acting on functions f smooth at a by

$$\dot{\alpha}(t) f = \frac{d}{dt} f(\alpha(t))$$

= the rate of change of f along α at parameter t. $\dot{\alpha}(t)$ is called a tangent vector to X at the point $\alpha(t)$.

Tangent Space The set of all tangent vectors to X at a is called the tangent space to X at a, denoted $T_a X$.

Basis for Tangent Space Let X be a smooth n-dimensional manifold and let $a \in X$, then $T_a X$ is a real n-dimensional vector space and if $y = (y^i)$ coordinates at a then

$$\frac{\partial}{\partial y^1_a} \cdots \frac{\partial}{\partial y^n_a}$$

a basis for $T_a X$.

Tangent Bundle If X a smooth manifold we write $TX = \bigcup_{x \in X} T_x X$ the set of all tangent vectors to X, called the tangent bundle of X.

Velocity Vector $\dot{\alpha}(t) \in T_{\alpha(t)} X$ and is called the velocity vector of $\alpha(t)$ at parameter t, ie is the rate of change of the parametrised path with respect to t.

Differential of f If f smooth at $a \in X$ a smooth manifold then the differential of f at a, denoted df_a is the linear form on $T_a X$ given by

$$\langle df_a, v \rangle = vf = \dot{\alpha}(t) f = \frac{d}{dt} f(\alpha(t))$$

for $v \in T_a X$, $v = \dot{\alpha}(t)$ say. Thus $df_a \in T_a^* X$, the dual of the tangent space, and measures the rate of change of f at a.

Scalar Field Let X be a smooth manifold and V open in X. A smooth valued function f, domain V is called a scalar field, and we denote by $C^\infty(V)$ the set of all scalar fields with domain V.

Smooth Map A map $X \xrightarrow{\phi} Y$ of smooth manifolds is called smooth if
i) V open in $Y \Rightarrow \phi^{-1} V$ open in X,
ii) if $f \in C^\infty(V)$ then $\phi^* f = f \cdot \phi \in C^\infty(\phi^{-1}(V))$. $\phi^* f$ is called the pull-back of f under ϕ.

Differential 1-form Let X be a smooth manifold and V open in X. A differential 1-form ω is a function on V such that $x \mapsto \omega_x \in T_x^* X$.

Differential (map) We denote by $\Omega^1(V)$ the set of differential 1-forms with domain V. Then for each $f \in C^\infty(V)$ we have $df \in \Omega^1(V)$, so $C^\infty(V) \xrightarrow{d} \Omega^1(V)$ a linear map called the differential.

Push-forward If $X \xrightarrow{\phi} Y$ smooth we define the push-forward $TX \xrightarrow{\phi_*} TY$ by

$$\begin{aligned}
[\phi_* v] f &= \frac{d}{dt} f\Big(\phi(\alpha(t))\Big) \\
&= \frac{d}{dt} (\phi^* f)(\alpha(t)) \\
&= \dot{\alpha}(t)[\phi^* f] \\
&= v[\phi^* f]
\end{aligned}$$

for all f smooth at $\phi(x)$, $x \in X$, $v = \dot{\alpha}(t) \in T_x X$.

Pull-back If $X \xrightarrow{\phi} Y$ a smooth map, ω a differential 1-form on Y with domain V open, we define $\phi^*\omega$ to be the differential 1-form on X with domain $\phi^{-1}V$, given by

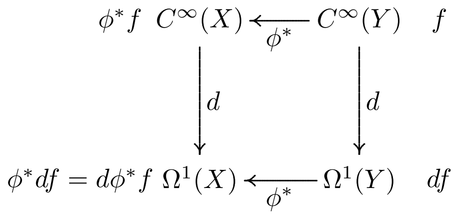
$$\langle (\phi^*\omega)_x, v \rangle = \langle \omega_{\phi(x)}, \phi_* v \rangle$$

for each $v \in T_x X, x \in X$. $\phi^*\omega$ is called the pull-back of ω under ϕ.

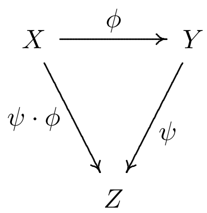
Pull-back Commutes with Differentials Let $X \xrightarrow{\phi} Y$ be smooth, and $f \in C^\infty(Y)$, then

$$\phi^* df = d\phi^* f$$

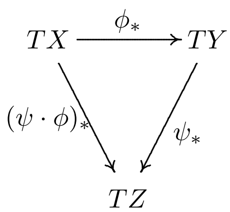
ie the pull-back commutes with differentials, and the following diagram is commutative:



Chain Rule for Maps of Manifolds Let



be a commutative diagram of smooth maps of manifolds, then



is a commutative diagram, ie $(\psi \cdot \phi)_* = \psi_* \cdot \phi_*$, or $(\psi \cdot \phi)'(x) = \psi'(\phi(x))\phi'(x)$