Super Happy Fun 141 Revision Notes

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These notes cover (very badly) some of the lecture material from MP Fry's 141 course from 2006-2007. Of extraordinarily dubious helpfulness for the current 141 course.

Contents

1 Momentum and Gravity

- **Momentum:** The *momentum* $\vec{P}$ of a body of mass m moving with speed $\vec{v}$ is:

$$\vec{P} = m\vec{v}$$

- **Force:** The change in total momentum of a body or system equal to the sum of external forces $\vec{F}_{ext}$ on the body or system:

$$\frac{d\vec{P}_{tot}}{dt} = \sum \vec{F}_{ext}$$

- **Gravity:** The *gravitational attraction* between two bodies of mass M_1, M_2 is:

$$\vec{F} = -GM_1M_2\,\frac{\vec{r}}{r^3}$$

or,

$$\vec{F} = -\frac{GM_1M_2}{r^2}\hat{r}$$

where $\vec{r}$ is the separation of the masses and $r = |\vec{r}|$.

- **Mass in sphere:** For a mass m inside a sphere, we have:

$$\vec{F} = -\frac{GmM(r)}{r^2}\hat{r}$$

where

$$M(r) = \frac{4\pi r^3 \rho}{3}$$

giving

$$\vec{F} = -\frac{4\pi\rho Gmr}{3}\hat{r}$$

Note that the density ρ may vary with radius.

2 Non-inertial Reference Frames

- **Inertial and non-inertial, pseudo-forces:** An *inertial (non-accelerating) reference frame* is one in which the equation $\vec{F} = m\vec{a}$ is valid. In a non-inertial (accelerating) frame it becomes necessary to introduce a *pseudo-force* such that $\vec{F} = m\vec{a}$ holds in that frame. The pseudo-force on a particle of mass m in a non-inertial frame accelerating with acceleration $\vec{a}$ is:

$$\vec{F}_{pseudo} = -m\vec{a}$$

- **Rotating frames:** The essential relationship between a frame rotating with angular velocity $\vec{\omega}$ and an inertial frame is given by the expression:

$$\left(\frac{d}{dt}\right)_{inertial} = \left(\frac{d}{dt}\right)_{rotating} + \vec{\omega}\times$$

which holds for any vector. Here the $\times$ denotes the vector cross product.

- **Rotating and accelerating frames:** For a frame that is both rotating with angular velocity $\vec{\omega}$ and accelerating away from the origin of the inertial frame with acceleration $\ddot{\vec{R}}$, then the relationship between the acceleration of a body as viewed in the inertial frame, $\vec{a}_I$, and the accleration $\vec{a}$, velocity $\vec{v}$ and position $\vec{r}$ of the body in the non-inertial frame is:

$$\vec{a}_I = \ddot{\vec{R}} + \vec{a} + \dot{\vec{\omega}} \times \vec{r} + 2\vec{\omega} \times \vec{v} + \vec{\omega} \times (\vec{\omega} \times \vec{r})$$

We also have

$$\vec{a}_I = \frac{\vec{F}}{m}$$

where $\vec{F}$ is the sum of the true forces on m, giving the final expression for the acceleration $\vec{a}$ of a body in a rotating and inertial frame:

$$m\vec{a} = \vec{F} - m\ddot{\vec{R}} - m\dot{\vec{\omega}} \times \vec{r} - 2m\vec{\omega} \times \vec{v} - m\vec{\omega} \times (\vec{\omega} \times \vec{r})$$

Here, $\vec{F}$ = true force on m, $-m\ddot{\vec{R}}$ = pseudo-force due to acceleration of frame, $-2m\vec{\omega} \times \vec{v}$ = Coriolis force, $-m\vec{\omega} \times (\vec{\omega} \times \vec{r})$ = centripetal force.

- **Earth's surface:** For the case of a body close to the Earth's surface, $\vec{F} = -m\vec{g}$, $\ddot{\vec{R}} = \vec{\omega} \times (\vec{\omega} \times \vec{R})$, $\dot{\vec{\omega}} = 0$ and the centripetal force is negligibly small. This gives the following expression for the accleration $\vec{a}$ of a body close to the Earth's surface:

$$\vec{a} = -\vec{g} - \vec{\omega} \times (\vec{\omega} \times \vec{R}) - 2\vec{\omega} \times \vec{v}$$

3 Charged Particles In Electric And Magnetic Fields

- **Coulomb's law:** Coulomb's Law for the force between two charged particles of charge q_1, q_2 is:

$$\vec{F} = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}\hat{r}$$

where ϵ_0 is the permittivity of free space, and r is the separation of the two charges. Note that the signs of the charges determine whether the force is attractive or repulsive.

- **Electric and magnetic forces:** The force on a particle of charge q in an electric field $\vec{E}$ is given by:

$$\vec{F} = q\vec{E}$$

The force on a particle of charge q moving with velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by:

$$\vec{F} = q\vec{v} \times \vec{B}$$

In solving problems involving these forces, determine the x, y, z components of the force and hence of the acceleration of the particle. Then solve, partially by guessing what $x(t)$, $y(t)$ and $z(t)$ solutions will look like.

4 Energy

- **Kinetic energy:** The *kinetic energy* of a body of mass m, velocity $\vec{v}$ is denoted by

$$T = \frac{1}{2}mv^2$$

and the change in kinetic energy is given by:

$$T_2 - T_1 = \frac{1}{2}mv^2(t_2) - \frac{1}{2}mv^2(t_1) = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\,dt$$

This is also equal to the work W done on a body by a force. If $\vec{F}$ is only a function of the position $\vec{r}$ of the body, then:

$$T_2 - T_1 = \int_1^2 \vec{F} \cdot d\vec{r}$$

- **Conservative force:** A *conservative force* is one such that

$$\int_1^2 \vec{F} \cdot d\vec{r} = \text{path independent}$$

i.e, the change in kinetic energy is the same no matter what path is taken to move the body from point 1 to point 2.

- **Potential energy:** For a conservative force there is then a scalar function V known as the *potential energy*, given by:

$$\int_1^2 \vec{F} \cdot d\vec{r} = V(\vec{r}_1) - V(\vec{r}_2)$$

and so

$$T_2 - T_1 = \int_1^2 \vec{F} \cdot d\vec{r} = V(\vec{r}_1) - V(\vec{r}_2)$$

and

$$T_1 + V_1 = T_2 + V_2$$

so

$$T + V = E = \text{total energy} = \text{constant}$$

for a conversative force.

In this case, we also have the following relationship between force and potential energy:

$$\vec{F}(\vec{r}) = -\vec{\nabla} V(\vec{r})$$

where

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$$

known as the *gradient*.

If V is centrally symmetric, or F a central force, then:

$$\vec{F}(r) = -\frac{dV(r)}{dr}\hat{r}$$

The condition for a conservative force is:

$$\vec{\nabla} \times \vec{F} = 0$$

Gravity and the electro-magnetic force (Coulomb's Law) are both conservative.

- **Non-conservative Forces:** If a *non-conservative force* is present, the total energy is not constant, and the rate of change of energy is given by:

$$\frac{d\vec{E}}{dt} = \vec{F}_{non-cons} \cdot \vec{v}$$

- **Equilibrium:** A particle is in *equilibrium* when $\frac{dV}{dr} = 0$. It is in stable equilibrium when $\frac{dV}{dr} = 0$ and $\frac{d^2V}{dr^2} > 0$. (Although the notation here uses $V(r)$, this of course holds for $V(x, y, z)$,$V(r, \theta)$ and so on.)

  A particle is *bounded* when $E < 0$. It is just bounded when $E = 0$, and in this case when applied to particles in orbit, the velocity of the particle is known as the escape velocity.

5 The Harmonic Oscillator

- **General solution:** A *harmonic oscillator* is a particle subject to a linear restoring force:

$$\vec{F} = -kx\hat{i}$$

$$\Rightarrow \vec{F} = m\ddot{\vec{x}} = -kx\hat{i}$$

$$\Rightarrow \ddot{x} + \frac{k}{m}x = 0$$

  where x is the displacement of the particle from equilibrium, and k is the spring constant.

  The *general solution* (or homogeneous solution) to this equation is:

$$x(t) = A\sin\omega t + B\cos\omega t$$

  where A and B are constants fixed by the inital conditions, and $\omega = \sqrt{\frac{k}{m}}$.

  The potential energy of a harmonic oscillator is:

$$V = \frac{1}{2}kx^2$$

- **Forced oscillator:** A *forced oscillator* is a harmonic oscillator driven by a time varying external force, $\vec{F}(t)$. The equation of motion is:

$$m\ddot{x} + kx = F(t)$$

We already know the general solution to the homogeneous equation $m\ddot{x}_h + kx_h = 0$ to be $x_h = A\sin\omega t + B\cos\omega t$. As the restoring force is linear, it follows that the most general solution to a problem of this sort is:

$$x(t) = x_h + x_p$$

where x_p is known as the *particular solution*; it is the solution of:

$$m\ddot{x}_p + kx_p = F(t)$$

In most cases, $F(t)$ will be of the form $F(t) = F_0 \sin\omega_0 t$ (or cos as the case may be), then the particular solution will be $x_p = C\sin\omega_0 t$ where C is an arbitrary constant.

The general solution is then:

$$x(t) = A\sin\omega t + B\cos\omega t + C\sin\omega_0 t$$

Solving for the constants, say for intial conditions $x(o) = \dot{x}(0) = 0$, will lead to a solution of the form:

$$x(t) = \frac{\frac{F_0}{m}}{\omega^2 - \omega_0^2}\left(\sin\omega t - \frac{\omega_0}{\omega}\sin\omega_0 t\right)$$

When $\omega_0 = \omega$, *resonance* occurs.

- **Harmonic oscillator with friction:** The equation of motion is of the form

$$m\ddot{x} = -kx - \beta\dot{x}$$

where β is the coefficient of friction.

To solve this, let $x = e^{at}$, a an arbitrary constant. This leads to a quadratic in a with solutions of the form $a = -\frac{\beta}{2m} \pm i\omega$, with $i = \sqrt{-1}$, $\omega = \frac{\sqrt{4mk-\beta^2}}{2m}$. Combining the possible solutions should give you a solution of the form:

$$x(t) = e^{\frac{-\beta t}{2m}}(A\cos\omega t + B\sin\omega t)$$

It is also possible to have harmonic oscillators with friction and driving forces. In this case, the solution $x(t)$ above becomes the homogeneous solution, and there is a particular solution x_p to the equation $m\ddot{x}_p = -kx_p - \beta\dot{x}_p + F(t)$.

6 Centre of Mass

- **Centre of mass of 2 particles:** The *centre of mass* of two bodies of masses m_1, m_2 is defined by:

$$\vec{R} = \frac{m_1\vec{r_1} + m_2\vec{r_2}}{m_1 + m_2}$$

In the absence of external forces, the centre of mass moves with constant velocity, as all internal forces between particles cancel out.

$$(m_1 + m_2)\ddot{\vec{R}} = \sum F_{ext}$$

- **Centre of mass of n particles:** The centre of mass of n particles is given by:

$$\vec{R} = \frac{\sum_{i=1}^{n} m_i\vec{r}_i}{\sum_{i=1}^{n} m_i}$$

- **Centre of mass frame:** It is often useful to consider the motion of masses in relation to their centre of mass frame. For instance, in the centre of mass frame, the total momentum is always zero.
- **Energy in centre of mass frame:** We can also split the total energy of a two-particle system (interacting with a conservative force) up into the motion of the centre of mass and motion about the centre of mass (plus potential energy).

  Then, the energy due to the motion of the centre of mass is given by:

  $$E_{cm} = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}} \cdot \dot{\vec{R}}$$

  and the energy due to rotation about the centre of mass is:

  $$E_{rot} = \frac{1}{2}\frac{m_1 m_2}{m_1 + m_2}\dot{\vec{r}} \cdot \dot{\vec{r}} + V(\vec{r})$$

  where $\vec{r}$ is the separation vector of the two masses, and $V(\vec{r})$ is the potential energy of the system.
- **Reduced mass:** We also define the *reduced mass* of two masses to be:

  $$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

  and then the total energy of a two-body system may be written:

  $$E_{tot} = \frac{1}{2}(m_1 + m_2)\dot{\vec{R}} \cdot \dot{\vec{R}} + \frac{1}{2}\mu\dot{\vec{r}} \cdot \dot{\vec{r}} + V(\vec{r})$$

7 Angular Momentum

- **Angular momentum:** The *angular momentum* $\vec{J}$ of a mass m about a point O is given by:

  $$\vec{J} = \vec{r} \times \vec{p}$$

  or

  $$\vec{J} = m\vec{r} \times \vec{v}$$
- **Torque:** The *torque* about the point 0 is given by:

  $$\vec{N} = \vec{r} \times \vec{F}$$

  The torque is equal to the rate of change of angular momentum, hence:

  $$\dot{\vec{J}} = \vec{r} \times \vec{F}$$

  The direction of $\vec{J}$ and $\dot{\vec{J}}$ are given by the right hand rule. In the absence of external torques, angular momentum is conserved.
- **Moments of inertia:** The *moment of inertia* of a body about an axis in the body is given by:

  $$I = \sum m_i r_{\perp}^2$$

  that is, the sum of: each particle of mass composing the body multiplied by the perpendicular distance to the axis squared.

  Moments of inertia are related to angular momentum by the expression:

  $$\vec{J} = I\vec{\omega}$$

  where ω is the angular velocity of the body as it rotates about that axis. However, $\vec{J}$ and $\vec{\omega}$ need not be parallel.

- **Parallel axis theorem:** The *parallel axis theorem* states that if I_{cm} is the moment of inertia through the centre of mass of a body, and I is the moment of inertia about a parallel axis a distance R away, then:
$$I = I_{cm} + MR^2$$
with M the mass of the body.

- **Kinetic Energy Of A Rigid Body:** The rotational kinetic energy of a body is given by:
$$T_{rot} = \frac{1}{2}\vec{\omega} \cdot \vec{J} = \frac{1}{2}I\omega^2$$
The total kinetic energy is then:
$$T_{tot} = T_{trans} + T_{rot}$$
$$\Rightarrow T_{tot} = \frac{1}{2}M\vec{v} \cdot \vec{v} + \frac{1}{2}I\omega^2$$
where $\vec{v}$ is the translational velocity of the centre of mass.
Note that when a body rolls without slipping, then:
$$v = \omega R$$

8 Orbits (Two Body Central Force Problem)

- **Useful relationships:** Letting $\vec{r}$ be the separation of two bodies, then their angular momentum about their centre of mass is:
$$\vec{J}_{cm} = \mu\vec{r} \times \dot{\vec{r}}$$
If in polar co-ordinates then
$$J_{cm} = \mu r^2\dot{\theta}$$
The energy due to motion about the centre of mass can be rewritten as:
$$E_{rot} = \frac{1}{2}\mu\dot{r}^2 + \frac{J_{cm}^2}{2\mu r^2} + V(r)$$
Knowing these is useful for problems involving orbits.