# Matrix Theory

a journal club introduction

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I review the old idea that M-theory (in Minkowski spacetime, or compactified on a lightlike direction) is described by Matrix Theory, namely the low energy theory on D0 branes aka super Yang-Mills in no spatial dimensions, where the physical degrees of freedom are N by N matrices. I sketch (with ever-increasing sketchiness) the Matrix Theory conjecture and some of its successes and puzzles. I suggest the topic is still topical by mentioning recent work on Lorentz invariance of Matrix Theory and on non-Lorentz invariance of its duality web.

## Contents



*... but, I assure you, one of the reasons for doing science, especially the kind of science I do (elementary particle physics, high energy theory) is that it makes your head feel funny, Goddamned strange.*

*– Sidney Coleman on the similarities between sciences fiction and fact*

*Unfortunately, no one can be told what the Matrix is. You have to see it for yourself.*

*– Morpheus*

## 1 Introduction

**Motivation** These notes were prepared to give a journal club on an "old topic nobody talks about anymore": Matrix Theory. Matrix Theory was, or is, an approach to making sense of M-theory. M-theory was introduced as a unification of the five superstring theories, with the eponymous letter allegedly hypothetically standing for 'Membrane' or 'Magical' or 'Mystery' or 'Mother'. Following the *BFSS Matrix Theory conjecture* [1], and a stronger follow-up conjecture by BFSS' second S [2], 'Matrix' became an alternative possibility: the claim being that M-theory could be described using a simple quantum mechanical system of matrices.

### The BFSS Matrix Theory conjecture [1]

M-theory in flat spacetime is described by the *Matrix Theory action*:

$$S = \int d\tau \mathrm{Tr}\left(\tfrac{1}{2R}\dot{X}^i\dot{X}_i + \tfrac{1}{4}\tfrac{R}{4\pi^2}[X^i,X^j][X_i,X_j]\right) + \text{fermions}\,, \quad (1.1)$$

where $X^i$, $i = 1,\ldots,9$, are $N \times N$ matrices, in the limit $N \to \infty$, $R \to \infty$

### The Susskind Matrix Theory conjecture [2]

M-theory in flat spacetime *compactified on a compact null circle with $N$ units of momentum in the null direction* is described by the *Matrix Theory action* (1.1) for finite $N$.

These conjectures inspired a flurry of activity between 1996 and 2000/2001, as can be seen from their citation counts, see figure 1 (which shows that the BFSS conjecture remains part of the global 'background' of string theory/M-theory research) and figure 2 (which shows perhaps more accurately the drop-off in activity on Matrix Theory as a first principles definition of M-theory). However, if you squint you can see a clear uptick in recent citations in figure 2, which I conjecture to be statistically significant. This uptick includes recent work on Lorentz invariance of Matrix Theory, and the non-Lorentz invariance of its dual incarnations, which I'll mention (very) briefly below.

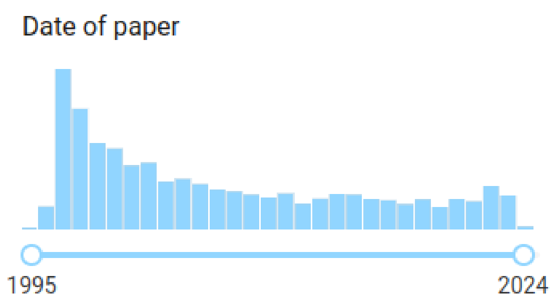


Figure 1: References to the BFSS Matrix Theory conjecture of [1]: large $N$ Matrix Theory describes M-theory in Minkwoski spacetime

**The literature** Matrix Theory was subjected to a number of reviews during its inflationary phase. These include ones by Bigatti and Susskind [3], Bilal [4] and Banks [5]. Aspects related to U-duality are treated in detail in the review by Obers and Pioline [6]. The most comprehensive overview of the state of the art at freeze-out is found in the review by Taylor [7] (the third installment in a trilogy of overlapping reviews, starting with [8] and [9]).

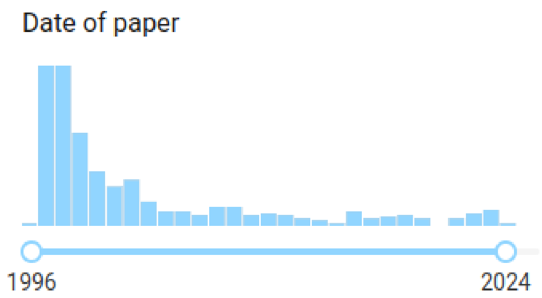


Figure 2: References to the Susskind Matrix Theory conjecture of [2]: finite $N$ Matrix Theory describes M-theory on a compact null circle with $N$ units of momentum

## 2 M-theory and String Theory

### 2.1 M-theory on a circle

**M-theory to type IIA** M-theory compactified on a circle gives rise to weakly coupled type IIA string theory, in the limit where the circle shrinks to zero size [10]. Let's take this circle to be parametrised by a coordinate $x^{10} \sim x^{10} + 2\pi$ and to have radius $R_{10}$. The only parameter in uncompactified M-theory is the eleven-dimensional Planck length, $\ell_p$. Once we compactify, we can combine this with the circle radius to define the ten-dimensional string length, $\ell_s$, and string coupling, $g_s$. The eleven-dimensional and ten-dimensional quantities are related as:

$$R_{10} = \ell_s g_s \,, \quad \ell_p = g_s^{1/3} \ell_s \,, \tag{2.1}$$

$$g_s = (R_{10}/\ell_p)^{3/2} \,, \quad \ell_s = \ell_p^{3/2} R_{10}^{-1/2} \,. \tag{2.2}$$

**Kaluza-Klein reduction of massless fields** The massless bosonic fields of M-theory are the eleven-dimensional metric and a three-form gauge field, $A_{(3)}$. These can be Kaluza-Klein reduced to obtain the massless bosonic fields of ten-dimensional type IIA string theory. The Kaluza-Klein metric can be written simply as

$$\mathrm{d}s_{11}^2 = R_{10}^2 (\mathrm{d}x^{10} + C_{(1)})^2 + \mathrm{d}s_{10}^2 \,, \tag{2.3}$$

or

$$\frac{\mathrm{d}s_{11}^2}{\ell_p^2} = e^{4\Phi/3} (\mathrm{d}x^{10} + C_{(1)})^2 + e^{-2\Phi/3} \frac{\mathrm{d}s_{10}^2}{\ell_s^2} \,. \tag{2.4}$$

The metric $\mathrm{d}s_{10}^2$ gives the ten-dimensional string frame metric, the Kaluza-Klein vector $C_{(1)}$ is the RR one-form, and $\Phi$ is the dilaton. The string coupling is related to the dilaton by $g_s \equiv e^{\Phi}$ (or more precisely to the asymptotic value of the dilaton). Meanwhile, the three-form reduces as:

$$A_{(3)} = C_{(3)} + B_{(2)} \wedge \mathrm{d}x^{10} \,, \tag{2.5}$$

giving the RR three-form $C^{(3)}$ and the NSNS two-form $B^{(2)}$. Note that the convention here is to take all $p$-forms to be dimensionless.

### 2.2 D0 branes

**D0 branes from Kaluza-Klein excitations** In the Kaluza-Klein reduction, we restrict to the zero modes on the circle. However, the tower of Kaluza-Klein excitations (of the massless fields in eleven dimensions) carrying non-zero momentum on the eleventh dimension also have a string theory interpretation. These are identified with D0 branes. Let $p_{\hat\mu}$, $\hat\mu = 0, 1, \ldots, 10$, denote the eleven-dimensional momentum. The momentum in the circular direction is quantised as $p_{10} = N/R$, and the eleven-dimensional massless condition $p_{\hat\mu}p^{\hat\mu} = 0$ implies that the ten-dimensional (string frame) mass-squared of a Kaluza-Klein excitation is:

$$M^2 \equiv -p_\mu p^\mu = (\tfrac{N}{R})^2 = (\tfrac{N}{\ell_s g_s})^2 \,. \tag{2.6}$$

The tension of a single D0 brane is known to be $T_{D0} = \frac{1}{\ell_s g_s}$, matching the above formula.

**Bosonic D0 brane action from dimensional reduction** A massless state (also called a momentum mode) in eleven-dimensions has the following worldline action:

$$S = \tfrac{1}{2\ell_p^2} \int \mathrm{d}\tau \lambda \, \hat g_{\hat\mu\hat\nu} \dot X^{\hat\mu} \dot X^{\hat\nu} \,, \tag{2.7}$$

where $\hat g_{\hat\mu\hat\nu}$ denotes the eleven-dimensional metric, and $\lambda$ is an auxiliary worldline field (essentially the einbein/metric on the worldline). The factor of $\ell_p^{-2}$ is inserted on dimensional grounds. For the Kaluza-Klein form of the metric of equation (2.3), this action is:

$$S = \tfrac{1}{2\ell_p^2} \int \mathrm{d}\tau \lambda \left( R^2 (\dot X^{10} + C_\mu \dot X^\mu)^2 + g_{\mu\nu} \dot X^\mu \dot X^\nu \right) \,. \tag{2.8}$$

Let's assume the background fields ($R, C_\mu$ and $g_{\mu\nu}$) do not depend on the coordinate $x^{10}$. We can dimensionally reduce this action using the following steps. Pass to the Hamiltonian form of the action solely for the $X^{10}$ direction, defining $\mathcal{H} = \dot X^{10} P_{10} - \mathcal{L}$, where $\mathcal{L}$ denotes the Lagrangian and $P_{10} = \frac{\lambda}{\ell_p^2} R^2 (\dot X^{10} + C_\mu \dot X^\mu)$. Using $\mathcal{L} = \dot X^{10} P_{10} - \mathcal{H}$, we have an equivalent action:

$$S = \int \mathrm{d}\tau \left( \dot X^{10} P_{10} + \tfrac{1}{2} \left( -\tfrac{\ell_p^2}{\lambda R^2} P_{10}^2 + \tfrac{\lambda}{\ell_p^2} g_{\mu\nu} \dot X^\mu \dot X^\nu \right) + P_{10} C_\mu \dot X^\mu \right) \,, \tag{2.9}$$

related to (2.8) by integrating out $P_{10}$. Instead of doing that, integrate by parts on the first term, so that $X^{10}$ plays the role of a Lagrange multiplier enforcing that $P_{10}$ is constant. Assuming this has been imposed, we can integrate out $\lambda$ to obtain:

$$\begin{aligned} S &= |P_{10}| \left( -\tfrac{1}{R} \int \mathrm{d}\tau \sqrt{-g_{\mu\nu} \dot X^\mu \dot X^\nu} + \mathrm{sgn}(P_{10}) \int \mathrm{d}\tau \, C_\mu \dot X^\mu \right) \\ &= |P_{10}| \left( -\tfrac{1}{\ell_s g_s} \int \mathrm{d}\tau \sqrt{-g_{\mu\nu} \dot X^\mu \dot X^\nu} + \mathrm{sgn}(P_{10}) \int \mathrm{d}\tau \, C_\mu \dot X^\mu \right) \,. \end{aligned} \tag{2.10}$$

where I assumed that $R$ was constant and used the basic relationship (2.1). For $P_{10} = 1$ this gives the action for a single D0 brane, of tension $\frac{1}{\ell_s g_s}$, coupling electrically to the one-form $C_\mu$. (Normally one writes a factor of the tension in front of this coupling, which can be done here on switching to conventions where $C_\mu$ or $X^\mu$ are dimensionful rather than dimensionless.)

**Bosonic non-abelian D0 brane action** We are more interested however in the description of multiple D0 branes. In this case, there are extra degrees of freedom resulting from open strings stretching between the D0 branes. For $N$ D0 branes, the worldline scalar fields corresponding to the spacetime coordinates transverse to the D0 brane are promoted to $N \times N$ matrices. Let us denote these by $X^i$, $i = 1, \ldots, 9$. These live in the adjoint of $u(N)$, so are hermitian $N \times N$ matrices which transform under local $U(N)$ gauge transformations. The worldvolume theory then also includes a $U(N)$ gauge field $A$ with covariant derivative $D_\tau X^i = \dot X^i +$

$i[A, X^i]$. The low energy theory, in flat spacetime, is then $U(N)$ super-Yang-Mills.

We can however give an action which generalises (2.10). This action follows from arguments using T-duality covariance [11]. While a single D0 brane only couples to the metric and the RR one-form, this action includes couplings to the NSNS $B$-field and to *all* RR $p$-forms. There are various subtleties arising from the non-abelian nature of the coordinates, which I will not discuss. The action is:

$$\begin{aligned} S = &- T_{D0} \int \mathrm{d}\tau \, \mathsf{STr} \left( e^{-\phi} \sqrt{\det Q^i{}_j} \sqrt{-P[g_{00}] - P[E_{0i}](Q^{-1} - \delta)^i{}_k E^{kj} P[E_{j0}]} \right) \\ &+ T_{D0} \int \mathsf{STr} \left( P[e^{i\lambda^{-1} \iota_X \iota_X} (\sum_n C_{(n)} e^B)] \right) , \end{aligned} \tag{2.11}$$

with

$$Q^i{}_j = \delta^i_j + i\lambda^{-1}[X^i, X^k] E_{kj} \,, \quad \lambda \equiv 2\pi \ell_s^2 \,, \quad E_{\mu\nu} = g_{\mu\nu} + B_{\mu\nu} \,. \tag{2.12}$$

This has a lot going on. Let me unpack. The index $i$ in (2.11) denotes a target space 9-dimensional index. The notation $P[\dots]$ denotes the (non-abelian) pullback to the worldvolume (worldline) assuming a static gauge choice to identify the worldsheet and target space time coordinates, $\tau \equiv x^0$, so that:

$$P[g_{00}] \equiv g_{00} + 2 g_{i0} D_\tau X^i + g_{ij} D_\tau X^i D_\tau X^j \,, \quad P[E_{0i}] \equiv E_{0i} + D_\tau X^j E_{ji} \,. \tag{2.13}$$

The notation $\iota_X$ denotes the interior product, $(\iota_X C_{(p)})_{\mu_1 \dots \mu_{p-1}} = X^j C_{j \mu_1 \dots \mu_{p-1}}$ Relative to (2.10) I have extracted a factor of $T_{D0}$ from the RR fields. Finally, the notation 'STr' denotes a symmetrised trace prescription. This won't be relevant to us when we take the relevant low energy or Matrix Theory limit.

It's great this action exists, but it's mostly too much information for us. Let's specialise to flat spacetime, with no gauge fields, where we can write[1]

$$S = - T_{D0} \int \mathrm{d}\tau \, \mathsf{STr} \left( \sqrt{\det(\delta^i_j + i\lambda^{-1}[X^i, X_j])} \sqrt{1 - \delta_{ij} D_\tau X^i D_\tau X^j} \right) , \tag{2.14}$$

and expand using $\det(I + A) = 1 + \mathrm{tr} A + \frac{1}{2}((\mathrm{tr} A)^2 - \mathrm{tr} A^2)$ to get the leading order interactions:

$$S = T_{D0} \int \mathrm{d}\tau \mathsf{Tr} (\tfrac{1}{2} D_\tau X^i D_\tau X_i + \tfrac{1}{4} \lambda^{-2} [X^i, X^j][X_i, X_j]) \,. \tag{2.15}$$

We dropped a constant term, and used the fact that the trace here is of a product of two matrices and hence automatically symmetric. What we get is the bosonic part of super-Yang-Mills in $0+1$ dimensions. There is of course no field strength for the gauge field in this case. In practice we often gauge fix $A = 0$.

**Conventions for the D0 brane action** This action appears in the literature with various normalisations. Let's explicitly restore the factors of $\ell_s$ and $g_s$:

$$S = \tfrac{1}{g_s \ell_s} \int \mathrm{d}\tau \mathsf{Tr} (\tfrac{1}{2} D_\tau X^i D_\tau X_i + \tfrac{1}{4} \tfrac{1}{4\pi^2 \ell_s^4} [X^i, X^j][X_i, X_j]) \,. \tag{2.16}$$

The coefficient of the commutator squared term is often made nicer by picking string units where $2\pi \ell_s^2 = 1$.

We can alternatively rewrite in terms of M-theory variables $R = \ell_s g_s$ and $\ell_p = \ell_s g_s^{1/3}$:

$$S = \tfrac{1}{R} \int \mathrm{d}\tau \mathsf{Tr} (\tfrac{1}{2} D_\tau X^i D_\tau X_i + \tfrac{1}{4} \tfrac{R^2}{4\pi^2 \ell_p^6} [X^i, X^j][X_i, X_j]) \,. \tag{2.17}$$

[1] Exercise: figure out how I made the dimensions consistent.

Here one often uses Planck units, $\ell_p = 1$. This gives the action in the form used in (1.1).

Finally, we can let $X^i = \lambda \Phi^i$. Then we obtain

$$S = T_{D0}\lambda^2 \int \mathrm{d}\tau \mathsf{Tr}(\tfrac{1}{2}D_\tau \Phi^i D_\tau \Phi_i + \tfrac{1}{4}[\Phi^i, \Phi^j][\Phi_i, \Phi_j]) \tag{2.18}$$

This can be viewed more precisely as the bosonic part of super-Yang-Mills in one-dimension. The Yang-Mills coupling is

$$g^2_{\mathsf{YM}} = T^{-1}_{D0}\lambda^{-2} = \frac{1}{4\pi^2}\frac{g_s}{\ell_s^3}\,. \tag{2.19}$$

**Supersymmetric low energy D0 action** The super-Yang-Mills action extends (2.16) with the addition of a sixteen-component spinor of $\mathrm{SO}(9)$, denoted by $\theta$, each component of which is again an $N \times N$ hermitian matrix transforming in the adjoint of $U(N)$. The action is:

$$S = \tfrac{1}{g_s \ell_s}\int \mathrm{d}\tau \mathsf{Tr}(\tfrac{1}{2}D_\tau X^i D_\tau X_i + \tfrac{1}{4}\tfrac{1}{4\pi^2\ell_s^4}[X^i, X^j][X_i, X_j] + \tfrac{1}{2\pi\ell_s^2} i\theta^T D_\tau \theta - \tfrac{1}{(2\pi\ell_s^2)^2}\theta^T\gamma^i[X_i, \theta])\,, \tag{2.20}$$

where $\gamma^i$ are gamma matrices realising the $\mathrm{SO}(9)$ Clifford algebra.

## 3 Two derivations of Matrix Theory

The goal of these derivations is to find a description of M-theory compactified on a lightlike direction. This means we are in Minkowski spacetime with lightcone coordinates $x^\pm$ and we identify $x^- \sim x^- + 2\pi R$. We want to show that this compactification can be described by the same action that appears in the low energy limit on D0 branes. The lightlike compactification is known as the Discrete LightCone Quantisation (DLCQ): the idea is to use $x^+$ as the time coordinate for quantisation (which would be usual lightcone quantisation) and then to compactify $x^-$ (which then has discrete momenta $p_-$, quantised in units of $1/R$). Given the DLCQ, one takes the limit $N, R \to \infty$, with $p_-$ fixed, to decompactify and return to Minkowski spacetime (Lorentz boosts can change the value of $R$ so it is not sufficient to just make it large). This provides the link from the Susskind Matrix Theory conjecture – which is what the below derivations establish – and the BFSS one. I avoid using the terminology 'DLCQ' below and instead refer to this situation as a lightlike or null compactification.

**Basic relativistic kinematics** The very first thing we should do[2] is remind ourself of some basic facts about relativistic kinematics. Say we are working with Minkowski spacetime with coordinates $X^\mu = (X^0, X^{10}, X^i)$ and momenta $P^\mu = (E, P^{10}, P^i)$. The metric is $\eta_{\mu\nu} = (-1, 1, \delta_{ij})$. Physical states have $P^\mu P_\mu = -M^2$ and $E > 0$. We therefore have

$$E - (P^{10})^2 - P^i P^i = M^2 \Rightarrow E - (P^{10})^2 \geq 0 \Rightarrow E = P^0 \geq |P^{10}|\,. \tag{3.1}$$

I now define lightcone coordinates

$$X^\pm = \tfrac{1}{\sqrt{2}}(X^0 \pm X^{10})\,, \tag{3.2}$$

such that $\eta_{+-} = -1$, $\eta_{++} = \eta_{--} = 0$. For the momentum, it follows that

$$-2P^+P^- + P^iP^i = -M^2 \Rightarrow P^- = \frac{1}{2P^+}(P^iP^i + M^2) \tag{3.3}$$

[2]This was actually – due to incompetence – the last section of these notes to be written.

while

$$P^+ = \tfrac{1}{\sqrt{2}}(P^0 + P^{10}) \geq \tfrac{1}{\sqrt{2}}(|P^{10}| + P^{10}) \geq 0\,. \tag{3.4}$$

In lightcone quantisation, $X^+$ is treated as the time coordinate. We take $P^-$ to be the Hamiltonian. We see that the momenta $P^+$ are positive, which is consistent by the mass-shell condition with positive energy. When we compactify the $X^-$ direction, this means that $P^+ = N/R$ with $N$ a positive integer.

### 3.1 Treating the null compactification as a boosted spatial compactification

This is Seiberg's derivation [12]. It uses three descriptions of the null compactification:

- Description 1: M-theory on a spatial circle
- Description 2: M-theory on an almost lightlike circle with an exact lightlike limit
- Description $\tilde{1}$: M-theory on a spatial circle with rescaled Planck length

Description 2 is obtained from description 1 by applying a boost transformation. States of interest in some fixed energy range in description 2 are mapped to states in description 1 in an energy range that vanishes in the lightlike limit. Description $\tilde{1}$ is obtained from description 1 by rescaling the Planck length such that the this energy range in description 1 is finite and fixed. This leads to a type IIA description via Matrix Theory.

I'll use coordinates $x'$ for description 1, coordinates $\bar{x}$ for description 2, and – because it is my favourite – reserve clean coordinates $x$ for description 3 which will be used in the next subsection.

**Description 1** This is M-theory on a spatial circle of radius $R_s$. Let's denote the coordinates by $(x'^0, x'^{10}, x^i)$, $i = 1, \ldots, 9$, with

$$x'^{10} \sim x'^{10} + 2\pi R_s\,, \tag{3.5}$$

and metric

$$\mathrm{d}s^2 = -(\mathrm{d}x'^0)^2 + (\mathrm{d}x'^{10})^2 + \mathrm{d}x^i\mathrm{d}x^i\,. \tag{3.6}$$

**Boosting from description 1 to description 2** We apply a boost transformation to description 1 to obtain description 2. This is M-theory on an almost lightlike circle. The boost defines new coordinates

$$\bar{x}^0 = \tfrac{1}{\sqrt{1-\beta^2}}(x'^0 - \beta x'^{10})\,, \quad \bar{x}^{10} = \tfrac{1}{\sqrt{1-\beta^2}}(x'^{10} - \beta x'^0)\,, \tag{3.7}$$

both of which are periodically identified as

$$\bar{x}^0 \sim \bar{x}^0 - 2\pi R_s \tfrac{\beta}{\sqrt{1-\beta^2}}\,, \quad \bar{x}^{10} \sim \bar{x}^{10} + 2\pi R_s \tfrac{1}{\sqrt{1-\beta^2}}\,. \tag{3.8}$$

Now switch to lightcone coordinates:

$$\bar{x}^\pm = \tfrac{1}{\sqrt{2}}(\bar{x}^0 \pm \bar{x}^{10})\,, \tag{3.9}$$

such that the metric is

$$\mathrm{d}s^2 = -2\mathrm{d}\bar{x}^+\mathrm{d}\bar{x}^- + \mathrm{d}x^i\mathrm{d}x^i\,. \tag{3.10}$$

The lightcone coordinates are identified as:

$$\bar{x}^+ \sim \bar{x}^+ + 2\pi R_s \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{1-\beta}{1+\beta}}\,, \quad \bar{x}^- \sim \bar{x}^- - 2\pi R_s \tfrac{1}{\sqrt{2}}\sqrt{\tfrac{1+\beta}{1-\beta}}\,. \tag{3.11}$$

**Parametrising the boost** Let's consider the limit of an infinite boost, for which $\beta \to 1$. Parametrise this by $\beta = 1 - \omega^{-2}$ with $\omega \to \infty$. (One convenient (but by no means unique) choice of boost parameter could be

$$\beta = \frac{1 - \frac{1}{2\omega^2}}{1 + \frac{1}{2\omega^2}}\,, \tag{3.12}$$

but I won't explicitly use this below.) In this limit,

$$\bar{x}^+ \sim \bar{x}^+ + O(1/\omega)\,, \quad \bar{x}^- \sim \bar{x}^- - 2\pi R_s \omega + O(1/\omega)\,. \tag{3.13}$$

If we define

$$R = R_s \omega \tag{3.14}$$

to be fixed in the limit, then we get the identifications

$$\bar{x}^+ \sim \bar{x}^+\,, \quad \bar{x}^- \sim \bar{x}^- - 2\pi R\,. \tag{3.15}$$

We then take the limit $\omega \to \infty$, with $R$ fixed, to be our definition of the null compactification. In the original unboosted description 1, this implies that $R_s \to 0$.

**Relating energy and momentum** The energy and momentum of description 1 are related to that of description 2 by:

$$p'_0 = \tfrac{1}{\sqrt{2}}\left(\sqrt{\frac{1-\beta}{1+\beta}}\bar{p}_+ + \sqrt{\frac{1+\beta}{1-\beta}}\bar{p}_-\right)\,, \quad p'_{10} = \tfrac{1}{\sqrt{2}}\left(\sqrt{\frac{1-\beta}{1+\beta}}\bar{p}_+ - \sqrt{\frac{1+\beta}{1-\beta}}\bar{p}_-\right)\,. \tag{3.16}$$

In the infinite boost limit, we get

$$p'_0 \approx \tfrac{1}{2}\tfrac{1}{\omega}\bar{p}_+ + \omega(1 - \frac{1}{4\omega^2})\bar{p}_-\,, \quad p'_{10} \approx \tfrac{1}{2}\tfrac{1}{\omega}\bar{p}_+ - \omega(1 - \frac{1}{4\omega^2})\bar{p}_-\,. \tag{3.17}$$

Hence for $\omega \to \infty$ we have:

$$p'_{10} \approx -\omega \bar{p}_-\,, \quad p'_0 + p'_{10} \approx \tfrac{1}{\omega}\bar{p}_+\,. \tag{3.18}$$

Now, it follows from the brief discussion of relativistic kinematics above that $\bar{p}_- = -\bar{p}^+ = -N/R$, with $N > 0$. As $\omega = R/R_s$, states with $N$ units of momentum on the null circle map in the limit to states with $N$ units of momentum on the spatial circle:

$$p'_{10} = \frac{N}{R_s}\,. \tag{3.19}$$

Now let's consider what happens to the energy. The lightcone energy is $\bar{p}^- = -\bar{p}_+$. We also have $p'_0 = -E'$. Write $E' = \Delta E' + \frac{N}{R_s}$ to separate the contribution to the energy from the rest mass of the momentum modes on the circle. Then we find

$$\Delta E' \approx \tfrac{1}{\omega}\bar{p}^- = \tfrac{R_s}{R}\bar{p}^-\,. \tag{3.20}$$

So if we are interested in describing the physics in the lightlike compactified theory at lightfront energy $\bar{p}^-$, say of $O(1)$, we are concentrating on an energy range $\Delta E'$ which in fact goes to zero in the limit.

**From description 2 to string theory in description 1** The above arguments indicate that the lightlike compactification on a null circle of radius $R$ of M-theory (description 2, in the limit) can be understood as an infinite boost limit of a spatial compactification, on a circle of radius $R_s = R/\omega$, which goes to zero in the limit. This sounds like good news, because in this limit M-theory reduces to type IIA sting theory, something we understand much better! States with $N$ units of momentum on the null circle get mapped to states with $N$ units of momentum on the spatial circle, i.e. D0 branes, and states of finite energy in the lightlike compactification get mapped to states with (vanishing) energy $\Delta E'$ in the spatial compactification. We also have that

$$g_s = (R_s/\ell_p)^{3/2} \to 0\,, \quad \ell_s = \ell_p^{3/2} R_s^{-1/2} \to \infty\,. \tag{3.21}$$

This is string theory at weak coupling but with the string mass $M_s = 1/\ell_s$ going to zero. This is not actually a very tractable limit of the theory (e.g. all the string oscillator modes are becoming massless). But according to our energy mapping, we're only interested in states in the energy range $\Delta E' \sim \omega^{-1}\bar{p}^-$. Note that $M_s \sim \omega^{-1/2}$ so $\Delta E'/M_s \sim \omega^{-1/2} \to 0$, that is the energy range we care about goes to zero faster than the string mass does. So we can try to 'zoom in' on this energy range.

**From description 1 to description $\tilde{1}$** Instead of describing physics at small energies and fixed Planck scale, we can fix the energy and let the Planck scale become large in the limit. This amounts to 'transposing the problem to an auxiliary M-theory' [13].

The argument for choosing the Planck scale rescaling goes something like the following. Firstly, let's consider the energy $\bar{p}_+$ in the lightlike compactification. Suppose we perform a boost transformation here, with $\beta = \tanh\gamma$. The lightcone coordinates transform as $\bar{x}^\pm \to e^{\mp\gamma}\bar{x}^\pm$, so the radius of the lightlike circle transforms as $R \to e^\gamma R$. Meanwhile the momenta transform as $\bar{p}_\pm \to e^{\pm\gamma}p_\pm$. So $\bar{p}_+$ and $R$ transform the same way. We conclude that $\bar{p}_+ \sim R/\ell_p^2$, which matches what you would expect from dimensional grounds. It follows from (3.20) that $\Delta E' \sim R_s/\ell_p^2$.

Now transpose to an auxiliary M-theory with Planck length $\tilde{\ell}_p$. The states we're interested in now have energy $\Delta\tilde{E} \sim R_s/\tilde{\ell}_p^2$. The simplest thing to do now is to identify this with the energies $\bar{p}_+$ in the lightlike compactification. Thus we make the identification:

$$\frac{R_s}{\tilde{\ell}_p^2} = \frac{R}{\ell_p^2} \Rightarrow \tilde{\ell}_p^2 = \frac{1}{\omega}\ell_p^2\,. \tag{3.22}$$

We also keep transverse lengths fixed. This amounts to requiring the transverse coordinates $\tilde{x}^i$ in descripton $\tilde{1}$ to obey $\tilde{x}^i/\tilde{\ell}_p = x^i/\ell_p$, or $\tilde{x}^i = \omega^{-1/2}x^i$.

**From description 2 to string theory in description $\tilde{1}$** Now we combine all the above steps. M-theory with Planck length $\ell_p$ compactified on a lightlike circle of radius $R$, with $N$ units of lightlike momentum, is equivalent, by an infinite boost and a transposition to the auxiliary description $\tilde{1}$, to the $\omega \to \infty$ limit of M-theory with Planck length $\tilde{\ell}_p$ on a spatial circle of radius $R_s$, with $N$ units of spatial momentum, with

$$R_s = \omega^{-1}R\,, \quad \tilde{\ell}_p = \omega^{-1/2}\ell_p \tag{3.23}$$

with $R$ and $\ell_p$ held fixed. We can reduce this to type IIA string theory. As we have $N$ units of momentum on the circle, we are in the sector of type IIA with $N$ D0 branes. The string coupling and string length are :

$$\tilde{g}_s = (R_s/\tilde{\ell}_p)^{3/2} = \omega^{-3/4}g_s\,, \quad \tilde{\ell}_s = \tilde{\ell}_p^{3/2}R_s^{-1/2} = \omega^{-1/4}\ell_s\,. \tag{3.24}$$

where $g_s$ and $\ell_s$ are fixed, with $R = g_s l_s$. Now we have both $\tilde{g}_s \to 0$ and $\tilde{\ell}_s \to 0$, so we're at weak coupling and the string mass scale becomes infinite. This is a limit we can work with. Note that the combination $\tilde{g}_s/\tilde{\ell}_s^3$ is fixed: this is the Yang-Mills coupling of the low energy theory on D0 branes.

**From description $\tilde{1}$ to Matrix Theory** We can now write down the non-abelian D0 action in the type IIA obtained from description $\tilde{1}$, and apply the above scaling rules:

$$\begin{aligned} S &= -\frac{1}{\tilde{\ell}_s \tilde{g}_s} \int \mathrm{d}\tau \, \mathsf{STr} \left( \sqrt{\det(\delta^i_j + i\tilde{\lambda}^{-1}[\tilde{X}^i, \tilde{X}_j])} \sqrt{1 - D_\tau \tilde{X}^i D_\tau \tilde{X}_i} \right) \\ &= -\frac{\omega}{\ell_s g_s} \int \mathrm{d}\tau \, \mathsf{STr} \left( \sqrt{\det(\delta^i_j + i\omega^{-1/2}\lambda^{-1}[X^i, X_j])} \sqrt{1 - \omega^{-1} D_\tau X^i D_\tau X^i} \right) \\ &= -\frac{1}{R} \int \mathrm{d}\tau \, \mathsf{Tr} \left( \omega - \tfrac{1}{2} D_\tau X^i D_\tau X^i - \tfrac{1}{4}\lambda^{-2}[X^i, X^j][X_i, X_j] + O(1/\omega) \right) . \end{aligned} \tag{3.25}$$

The leading term gives a constant piece involving the D0 energy $N\omega/R = N/R_s$, which we ignore: this is the same term that we subtracted in (3.20) when comparing to the energy of the lightlike compactification. This is exactly like discarding the formally divergent constant rest mass of a relativistic point particle in the non-relativistic limit. In the alternative way of taking this limit which I will present in the next subsection, this naively divergent term will cancel against a contribution from the RR one-form. The finite term gives the Matrix Theory Lagrangian:

$$S = \frac{1}{R} \int \mathrm{d}\tau \, \mathsf{Tr} \left( \tfrac{1}{2} D_\tau X^i D_\tau X^i + \tfrac{1}{4}\lambda^{-2}[X^i, X^j][X_i, X_j] \right) . \tag{3.26}$$

You might wonder about the justification that this is the relevant description of all physics in this limit. This amounts to observing that the effect of $g_s \to 0$ and $l_s \to 0$ is to decouple all string theory modes except those of massless open strings stretched between the D0 branes. This means we are guaranteed to get the low energy super-Yang-Mills description of the D0 system.

## 3.2 Treating the null compactification as a limit

Here's an alternative derivation, which avoids mucking about with the Planck length. It uses again three descriptions:

- Description 1: M-theory on a spatial circle
- Description 2: M-theory on an almost lightlike circle with an exact lightlike limit
- Description 3: M-theory on a spatial circle with an exact lightlike limit

Description 1 and 2 are the same as before. Description 3 is obtained from description 2 by a simple change of coordinates.

**From description 2 to description 3** After boosting from description 1, we have periodically identified lightcone coordinates

$$\bar{x}^+ \sim \bar{x}^+ + 2\pi R_s \tfrac{1}{\sqrt{2}} \sqrt{\tfrac{1-\beta}{1+\beta}} \,, \quad \bar{x}^- \sim \bar{x}^- - 2\pi R_s \tfrac{1}{\sqrt{2}} \sqrt{\tfrac{1+\beta}{1-\beta}} \,. \tag{3.27}$$

and metric

$$\mathrm{d}s^2 = -2\mathrm{d}\bar{x}^+ \mathrm{d}\bar{x}^- + \mathrm{d}x^i \mathrm{d}x^i \,. \tag{3.28}$$

Now we define new coordinates

$$x^- = \bar{x}^- \quad x^+ = \bar{x}^+ + \tfrac{1-\beta}{1+\beta} \bar{x}^- \,, \tag{3.29}$$

such that

$$x^- \sim x^- - 2\pi R_s \tfrac{1}{\sqrt{2}} \sqrt{\tfrac{1+\beta}{1-\beta}} \,, \quad x^+ \sim x^+ \,, \tag{3.30}$$

and the metric is

$$\mathrm{d}s^2 = -2\mathrm{d}x^+\mathrm{d}x^- + 2\tfrac{1-\beta}{1+\beta}(\mathrm{d}x^-)^2 + \mathrm{d}x^i\mathrm{d}x^i\,. \tag{3.31}$$

For $\beta < 1$, we can treat $x^+$ as the time direction and $x^-$ as a compact spatial coordinate. For $\beta \to 1$, $x^-$ becomes null. Let's let $R = R_s\frac{1}{\sqrt{2}}\sqrt{\frac{1+\beta}{1-\beta}}$ and assume this is fixed for $\beta \to 1$. Rescaling $x^-$we can rewrite this metric as:

$$\mathrm{d}s^2 = 2\tfrac{1-\beta}{1+\beta}R^2(\mathrm{d}x^- - \tfrac{1}{2R}\tfrac{1+\beta}{1-\beta}\mathrm{d}x^+)^2 - \tfrac{1}{2R^2}\tfrac{1+\beta}{1-\beta}(\mathrm{d}x^+)^2 + \mathrm{d}x^i\mathrm{d}x^i\,, \tag{3.32}$$

where now $x^- \sim x^- - 2\pi$. The radius squared of the $x^-$ circle is $2\frac{1-\beta}{1+\beta}R^2$ and this goes to zero for $\beta \to 1$, meaning that this limit takes us to type IIA string theory.

**From description 3 to type IIA string theory** We therefore reduce on the $x^-$ direction using (2.3). We have

$$R_- = \ell_s g_s = \sqrt{2}\sqrt{\tfrac{1-\beta}{1+\beta}}R\,,\quad C_{(1)} = -\tfrac{1}{2R}\tfrac{1+\beta}{1-\beta}\mathrm{d}x^+\,,\quad \mathrm{d}s^2_{10} = -\tfrac{1}{2R^2}\tfrac{1+\beta}{1-\beta}(\mathrm{d}x^+)^2 + \mathrm{d}x^i\mathrm{d}x^i\,, \tag{3.33}$$

along with $\ell_p = g_s^{1/3}\ell_s$. Solving for the string theory parameters, and writing $\beta \approx 1 - \frac{1}{\omega^2}$ again, we have

$$g_s = (\sqrt{2}\sqrt{\tfrac{1-\beta}{1+\beta}}R/\ell_p)^{3/2} \approx \omega^{-3/2}\hat{g}_s\,,\quad \ell_s = \ell_p^{3/2}(\sqrt{2}\sqrt{\tfrac{1-\beta}{1+\beta}})^{-1/2}R^{-1/2} \approx \omega^{1/2}\hat{\ell}_s\,, \tag{3.34}$$

where $\hat{g}_s \equiv (R/\ell_p)^{3/2}$ and $\hat{\ell}_s \equiv \ell_p^{3/2}R^{-1/2}$ are fixed in the limit. We similarly have

$$C_{(1)} \approx -\tfrac{\omega^2}{R}\mathrm{d}x^+\,,\quad \mathrm{d}s^2_{10} \approx -\omega^2(\mathrm{d}x^+)^2 + \mathrm{d}x^i\mathrm{d}x^i\,. \tag{3.35}$$

These expressions define the IIA description of the limit leading to the null compactification of M-theory. Relative to the previous description $\tilde{1}$, the limit involves not only the string length and coupling, but also the background metric and one-form field.

Here are some technical observations:

- the limit appears singular at the level of the background fields
- the limit for the metric has the same form as a non-relativistic limit, with $\omega$ playing the role of the speed of light. This generalises to curved backgrounds and other branes, leading to a duality web of 'non-Lorentzian decoupling limits' [14]
- if we use the boost (3.12), the expressions (3.34) and (3.35) are true for finite $\omega$ (i.e. replacing the $\approx$ with an $=$)
- technically we again have $g_s \to 0$ and $\ell_s \to \infty$, which we said was a complicated limit previously. The string length becoming infinite is compensated by the scaling of the metric itself. We could instead work with a string length $\tilde{\ell}_s = \omega^{-1/2}\hat{\ell}_s$ which goes to zero in the limit, if we change the metric to $\mathrm{d}\tilde{s}^2 = -(\mathrm{d}x^+)^2 + \omega^{-2}\mathrm{d}x^i\mathrm{d}x^i$, such that

$$\frac{1}{\ell_s^2}\mathrm{d}s^2_{10} = \frac{1}{\tilde{\ell}_s^2}\mathrm{d}\tilde{s}^2_{10}\,. \tag{3.36}$$

  The scaling of the transverse coordinates $x^i$ shows that essentially this is recovering the previous description $\tilde{1}$.

**And back to Matrix Theory again** Despite the singular nature of the limit in spacetime, it leads to a finite D0 worldvolume action – which is that of Matrix Theory. Let's relabel $x^+ \equiv x^0$ and start again with the non-abelian action[3], noting that $g_{00} = -\omega^2$ and we have to include the one-form coupling, so that:

$$\begin{aligned} S &= -\tfrac{1}{\ell_s g_s}\int \mathrm{d}\tau\, \mathrm{STr}\left(\sqrt{\det(\delta^i_j + i\lambda^{-1}[X^i,X_j])}\sqrt{\omega^2 - D_\tau X^i D_\tau X_i}\right) - \int \mathrm{d}\tau \mathrm{STr}(P[C_{(1)}]) \\ &= -\tfrac{\omega^2}{\hat\ell_s \hat g_s}\int \mathrm{d}\tau\, \mathrm{STr}\left(\sqrt{\det(\delta^i_j + i\omega^{-1}\hat\lambda^{-1}[X^i,X_j])}\sqrt{1 - \omega^{-2} D_\tau X^i D_\tau X^i}\right) + \int \mathrm{d}\tau \omega^2 \tfrac{N}{R} \\ &= \tfrac{1}{R}\int \mathrm{d}\tau\, \mathrm{Tr}\left(\tfrac12 D_\tau X^i D_\tau X^i + \tfrac14 \hat\lambda^{-2}[X^i,X^j][X_i,X_j] + O(1/\omega)\right) . \end{aligned} \tag{3.37}$$

where now the divergent terms cancel.

# 4 Matrix Theory successes and puzzles

Let me write the Matrix Theory Lagrangian one more time, using convenient units:

$$S = \tfrac{1}{R}\int \mathrm{d}\tau\, \mathrm{Tr}\left(\tfrac12 \dot X^i \dot X^i + \tfrac14 [X^i,X^j][X_i,X_j] + \left(i\theta^T\dot\theta - \theta^T\gamma^i[X_i,\theta]\right)\right) . \tag{4.1}$$

The momenta and Hamiltonian are:

$$P_i = \tfrac{1}{R}\dot X_i\,, \quad H = \tfrac{1}{2R}\dot X^2 - \tfrac{1}{4R}[X^i,X^j][X_i,X_j]\,. \tag{4.2}$$

The classical equations of motion are:

$$\ddot X^i + [[X^i,X^j],X^j] = 0\,. \tag{4.3}$$

We also have the Gauss constraint $[X^i,\dot X^i] = 0$ from the equation of motion of the Yang-Mills gauge field, which we've fixed to zero.

I'll now discuss – briefly – some of the M-theoretic physics captured by this Lagrangian. I follow most closely the review [7], to which I refer you for further details and references.

## 4.1 Gravitons and branes

**Gravitons from classical solutions** Matrix Theory describes (super)gravitons in 11-dimensions. Classically, we can find solutions of the equations of motion (4.3) consisting of commuting matrices:

$$X^i = \begin{pmatrix} x^i_1 + v^i_1\tau & 0 & \dots & \\ 0 & x^i_2 + v^i_2\tau & \dots & \\ & & \ddots & \\ & & & x^i_N + v^i_N\tau \end{pmatrix} . \tag{4.4}$$

From (4.2), the $a^{\text{th}}$ entry on the diagonal has momenta $(P_a)_i = \frac{1}{R}v^i_a$ and energy $E_a = \frac{v_a^2}{2R}$, such that $E_a = \frac{\sum_i p^i_a p^i_a}{2p^+}$, which is the expected energy of a massless state in lightcone coordinates with $p^+ = 1/R$, compare (3.3). We interpret this solution as describing $N$ gravitons in 11-dimensions, each with lightcone momentum $p^+ = 1/R$.

We describe a *single* graviton with $p^+ = N/R$ if all $x^i_a$ are equal and all $v^i_a$ are equal.

[3]Referring to the abelian action (2.10), we realise we need to now insert a minus sign in the coupling to the RR one-form, because the lightcone momentum $p_- = -N/R$ is negative.

**Supergravitons from quantisation** The $N=1$ Matrix Theory is

$$S=\frac{1}{R}\int \mathrm{d}\tau\,\left(\tfrac{1}{2}\dot{X}^i\dot{X}^i+i\theta^T\dot{\theta}\right)\,. \tag{4.5}$$

The interesting part of the quantum theory comes from the 16-component fermion. Writing $\theta^\alpha$ we have the usual anticommutation relations $\{\theta^\alpha,\theta^\beta\}=\delta^{\alpha\beta}$. From these we can define eight creation and eight annihilation operators, $\theta^\alpha_\pm=\frac{1}{\sqrt{2}}(\theta^\alpha\pm i\theta^{\alpha+8})$ obeying $\{\theta^\alpha_+,\theta^\beta_-\}=\delta^{\alpha\beta}$, $(\theta^\alpha_\pm)^2=0$. Then we can build a Fock space from a vacuum $|0\rangle$ with states $|0\rangle,\theta^\alpha_+|0\rangle,\theta^\alpha_+\theta^\beta_+|0\rangle$ and so on. This gives 256 states, and corresponds to the physical polarisations of the massless states in eleven-dimensions: the graviton (44), three-form (84) and gravitino (128).

For $N>1$, we can write $X^i=x^i1+\tilde{X}^i$ where $x^i=\frac{1}{N}X^i$ and $\mathrm{tr}\tilde{X}^i=0$. This corresponds to decomposing $U(N)=U(1)\times SU(N)$. The $U(1)$ part, corresponding to $x^i$, describes the centre of mass of the system. Making a similar decomposition for the fermions, we find that the Matrix Theory action splits into a $U(1)$ part given by (4.5) and a $SU(N)$ part given by the original Lagrangian restricted to traceless matrices. Hence we get the 256 supergraviton states from the $U(1)$ part. The $SU(N)$ theory is then believed to have a single zero energy bound state.

**Membranes from Matrix Theory and Matrix Theory from Membranes** Matrix Theory can also describe the membrane of M-theory. In fact, Matrix Theory can be *derived* as a matrix regularisation of the light-cone quantisation of the supermembrane – this was in fact done in the 1980s [15]. After lightcone quantisation, the bosonic degrees of freedom on the membrane worldvolume are 9 scalars $X^i(\tau,\sigma^a)$ which depend on the worldvolume time coordinate $\tau$ and two spatial coordinates $\sigma^a$. The functional dependence on $\sigma^a$ can be regularised by treating the coordinates as $N\times N$ matrices (i.e. with row and column indices playing the role of discretised coordinates), with the large $N$ limit returning to the functional description. For instance, the membrane dynamics involves the 'Poisson bracket' $\{X^i,X^j\}=\epsilon^{ab}\partial_aX^i\partial_bX^j$, $a=1,2$, defined using the two spatial coordinates of the membrane worldvolume, and the matrix regularisation transforms this into a matrix commutator. The details depend on the topology of the membrane.

A simple example is the spherical membrane. We can choose three of the coordinate matrices $X^i$ to be given by $X^A=\frac{2r}{N}J^A$, $A=1,2,3$, where $J^A$ form the $N$-dimensional representation of SU$(2)$, $[J^A,J^B]=i\epsilon^{ABC}J_C$. Recall the quadratic Casimir for the spin-$j$ representation is $C_2(j)=j(j+1)$ and we have $N=2j+1$ hence $C_2(N)=\frac{1}{4}(N^2-1)$. Then $\sum_A(X^A)^2=\frac{4r^2}{N^2}C_2(N)\mathbb{I}=r^2(1-N^{-2})\mathbb{I}$ and for large $N$ we recover the defining equation for a sphere of radius $r$. This is an example of a 'fuzzy' or 'non-commutative' geometry, something which naturally appears when studying the description of multiple D-branes. Arbitrary functions on the sphere can be expanded in spherical harmonics: these are symmetric traceless polynomials (of degree $l$) in the embedding coordinates. The matrix approximation (for $N<l$) to these polynomials just uses the matrices $X^A$ for these embedding coordinates.

## 4.2 Interactions

**Multiple objects from multiple blocks** A very important observation about Matrix Theory is that it can describe multi-graviton (or multi-brane, or multi-whatever) states simply by viewing these as corresponding to block diagonal matrix configurations, for example:

$$X^i=\begin{pmatrix}X^i_1 & 0\\ 0 & X^i_2\end{pmatrix}\,, \tag{4.6}$$

where here $X_1$ and $X_2$ are $N_1 \times N_1$ and $N_2 \times N_2$ matrices, respectively, and $N_1 + N_2 = N$. Indeed we already made use of this when describing classical graviton solutions, above. We can define a notion of 'separation' between two such blocks via

$$r^i = \tfrac{1}{N_1}\text{tr}X_1^i - \tfrac{1}{N_2}\text{tr}X_2^i\,, \quad r \equiv \sqrt{r^i r^i}\,. \tag{4.7}$$

Classically such a block diagonal decomposition leads to a set of distinct non-interacting subsectors, each described by its own $U(N_i)$ Matrix Theory Lagrangian.

**Graviton scattering** We can use Matrix Theory to study interactions between M-theoretic objects viewed as subblocks (as in (4.6)) of the matrices $X^i$. The classic example is two-graviton scattering. Here we suppose that initially (at $\tau = -\infty$) we have a configuration of the form:

$$X^i = \begin{pmatrix} X_1^i & 0 \\ 0 & X_2^i \end{pmatrix} \tag{4.8}$$

describing two gravitons, one with $N_1$ units of lightcone momentum, and the other with $N_2$ units of lightcone momentum, and a final configuration (at $\tau = +\infty$) of the form:

$$X^i = \begin{pmatrix} X_3^i & 0 \\ 0 & X_4^i \end{pmatrix} \tag{4.9}$$

describing two gravitons, one with $N_3$ units of lightcone momentum, and the other with $N_4$ units of lightcone momentum. Evidently $N_1 + N_2 = N_3 + N_4 = N$. The interactions between these gravitons leads to some complicated intermediate state (which will not be block diagonal).

If we consider quantum mechanical fluctuations about a classical background of the form (4.6), the off-diagonal fluctuations are of course not zero. However, they will generically be massive, with masses that grow like $r^2$, with $r$ the separation (4.7). For some graviton interaction, such as the two-to-two scattering above, we can then compute the amplitude perturbatively by assuming these are well-separated gravitons, and integrating out these very massive modes. We do this at fixed $N_1, N_2, N_3, N_4$ and $N$. Then at the end to compare to M-theory in uncompactified Minkowski spacetime we are supposed to take $N_1, \ldots, N_4$ and $N$ to infinity (along with $R$).

This procedure was shown in a number of cases to successfully reproduce classical 11-dimensional supergravity scattering amplitudes. Now, the Matrix Theory conjecture(s) make no reference to supergravity, so a priori one should not necessarily expect any agreement. Indeed, the perturbative Matrix Theory calculation outlined in the previous paragraph is not what you should do if you really want to take the large $N$ limit to make contact with M-theory. Instead of fixing $N$ and assuming $r$, the separation between the Matrix Theory objects, is large, you should fix $r$ and then take $N$ to be large. The issue with this is that for large $N$ we can not really describe each block as its own independent configuration (the size of the wavefunction for each block grows with $N$).

When doing computations at finite $N$, one is dealing (according to our derivations of Matrix Theory) with M-theory compactified on a lightlike circle of radius $R$, or equivalently a spatial circle of radius $R_s$ with $R_s \to 0$. We should trust 11-dimensional SUGRA when all radii are large (i.e. at large $g_s$). On the other hand, the 10-dimensional D0 brane description is valid for small radius (i.e. small $g_s$). Put differently, the D0 brane system describes states with 11-dimensional momenta $N/R_s$ which are large compared to the Planck scale. This is not the supergravity regime, where all momenta should be small compared to the Planck scale.

Generically, for calculations that are finite $N$ in practice, one does not expect to be able to compare with classical supergravity. It is further expected that to make the connection with classical supergravity work, subtleties in the large $N$ limit will appear and need to be dealt with.

Despite these expectations, a lot of effort was expended into computing various Matrix Theory amplitudes and trying to compare them with supergravity quantities, including multi-graviton interactions. The remarkable thing was that some quantities agree precisely! This was explained using supersymmetry: some particular calculations (at one- and two-loops in particular) are in fact protected in Matrix Theory by supersymmetric non-renormalisation theorems. This explains why Matrix Theory can compute supergravity quantities in the first place, despite the fact one is not doing the calculation in the regime where supergravity applies: anything that matches with supergravity is a protected quantity. It's still non-trivial that Matrix Theory produces the correct supergravity answer for these cases (and if it didn't, of course that would signal the Matrix Theory conjecture was just plain wrong).

The best-studied computation of this form involved two-to-two graviton scattering with no longitudinal momentum transfer, i.e. $N_1 = N_3$, $N_2 = N_4$. Let's confirm some of the expectations outlined above. We can effectively do this calculation in the $N = 2$ theory. We expand the matrices $X^i$ in terms of a classical background plus a fluctuation, $Y$:

$$X^i = \begin{pmatrix} x_1^i & 0 \\ 0 & x_2^i \end{pmatrix} + Y^i = (x_c^i + y_c^i)\mathbb{I}_2 + \tfrac{1}{2} r^i \sigma_3 + y_\alpha^i \sigma^\alpha \tag{4.10}$$

where $x_c^i = \frac{1}{2}(x_1^i + x_2^i)$ is the centre of mass position of the classical configuration, and $r^i = \frac{1}{2}(x_1^i - x_2^i)$ is the separation. We conveniently parametrised $Y$ in terms of the Pauli sigma matrices, such that:

$$Y^i = \begin{pmatrix} y_c^i + y_3^i & y_1^i - iy_2^i \\ y_1^i + iy_2^i & y_c^i - y_3^i \end{pmatrix} \tag{4.11}$$

To quadratic order in the fluctuations, the Matrix Theory action is:

$$S = S_{\text{cl}} + \tfrac{1}{R}\int \mathrm{d}\tau \left( \tfrac{1}{2}\mathrm{tr}(\dot{Y}^i \dot{Y}^i) - r^2(y_1^2 + y_2^2) - (r\cdot y_1)^2 - (r \cdot y_2)^2 \right) , \tag{4.12}$$

where $S_{\text{cl}}$ denotes the classical action evaluated on the background $X^i$. We see indeed that the off-diagonal parts of the fluctuations have mass squareds of order $r^2$.

The classic Matrix Theory calculation is then to compute the quantum effective action:

$$e^{iS_{\text{eff}}[x]} \sim \int \mathcal{D}y \, e^{iS_{\text{cl}}[x] + iS_{\text{fluct}}[y]} . \tag{4.13}$$

In practice this is not done using (4.12) but by reintroducing the gauge field (which was gauge fixed to zero above) and working in background field gauge. The result is an effective potential for the interaction of two gravitons. The one- and two-loop effective potential can then be shown to match the expected result from classical supergravity.

Let me sketch this matching in slightly more detail. In this case, one can treat the effective potential as an expansion in powers of the relative velocity $v$, and inverse powers of the separation $r$. On dimensional grounds one can identify which terms could arise in classical supergravity (by virtue of involving integral powers of the 11-dimensional Newton's constant) and which could not. Then one checks the coefficient of these terms and finds, at least for certain leading terms, that they match. It might be expected that higher-loop calculations in Matrix Theory could further correct these coefficients, which would invalidate this agreement.

However, it has been shown that this is not possible due to supersymmetry non-renormalisation theorems.

In this example, there are also other terms (in the $r$, $v$ expansion) appearing from the Matrix Theory calculation which could be viewed as Matrix Theory predictions for quantum corrections to classical supergravity. However the status of these is less clear (for instance, attempts to reproduce the effects of the known curvature corrections to 11-dimensional supergravity from Matrix Theory were not successful).

**Lorentz invariance** An immediate outstanding issue since the formulation of the link between Matrix Theory and M-theory in 11-dimensional Minkowski spacetime was the question of how – or whether – 11-dimensional Lorentz invariance appears. Recently a number of papers have argued that 11-dimensional Lorentz invariance can be shown (in the relevant large $N$ limit) using soft theorems [16–19]. Soft theorems are statements about the behaviour of amplitudes in limits where an external momentum corresponding to a massless particle (e.g. a graviton or photon) vanishes, and are closely connected to (asymptotic) symmetries. It's amusing(?) to note that soft theorems date to the 1960s, the Matrix Theory conjecture to the 1990s, and now they have been combined in the 2020s.

### 4.3 Duality

Finally, let's consider the Matrix Theory description of M-theory on a $p$-dimensional torus with radii $R_a$, $a = 1, \dots, p$. According to Seiberg's derivation, when we pass to description $\tilde{1}$, we have to introduce radii $\tilde{R}_a$ such that $\tilde{R}_a/\tilde{\ell}_p = R_a/\ell_p$, i.e. so that $\tilde{R}_a = \omega^{-1/2} R_a$. Taking $\omega \to \infty$ means that the torus shrinks to zero size. In the string theory description this tells us we should T-dualise on the torus directions. We reach the same conclusion in description 3, noting that there $(R_a/\ell_s) \to 0$.

This means the D0 branes of Matrix Theory become D$p$ branes, and in place of Matrix Theory we have SYM in $1+p$ dimensions. This picture works perfectly for $p = 0, 1, 2, 3$, for which the string coupling is going to zero or a constant in the limit. Note that the $p = 1$ case is known as Matrix String Theory [20], and there's also a $p = -1$ case (which follows formally from a timelike T-duality) known as the IKKT Matrix Model [21]. For $p > 3$, the string coupling diverges meaning that we need to deal with strong coupling. For $p = 4$ this means lifting SYM on D$4$ branes to a theory living on M5 branes. For $p = 5$ this means S-dualising SYM on D$5$ branes to a theory living on NS5 branes. For $p = 6$ the D6 should lift to the Kaluza-Klein monopole in M-theory, and it's much less clear whether the Matrix Theory description works, or is useful, in this case. More details about (U-)duality and Matrix Theory are reviewed nicely in [5, 6].

In my second derivation of Matrix Theory, I ended up taking a limit of the metric and RR-one form given by (3.35). I noted this was like a non-relativistic limit, as the time coordinate was scaling like $\omega^2$, with $\omega \to \infty$ and the spatial coordinates were fixed. When one T-dualises, what you get from this are generalised non-relativistic limits where all 'longitudinal' coordinates, including time and $p$ spatial coordinates (associated with a $p$-brane worldvolume), scale like $\omega^2$ relative to the 'transverse' spatial coordinates. Taking the S-dual of the $p = 1$ case gives a non-relativistic limit associated with a fundamental string (rather than a D1 brane). This is known as 'non-relativistic string theory' [22, 23]: it has a non-relativistic target space but a relativistic worldsheet. There is also a 'non-relativistic M-theory' which is U-dual to the null compactification leading to Matrix Theory, and which uplifts both type IIA non-relativistic string theory and the $p = 2$ version of Matrix Theory on D2 branes. This corner of M-theory has been recently revived with a view to making sense of quantum gravity in its non-relativistic corner. One can also argue that these provide, via duality, another 'definition' of M-theory/string on a null circle. See [14] for an updated roadmap of the 'non-Lorentzian duality web'.

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