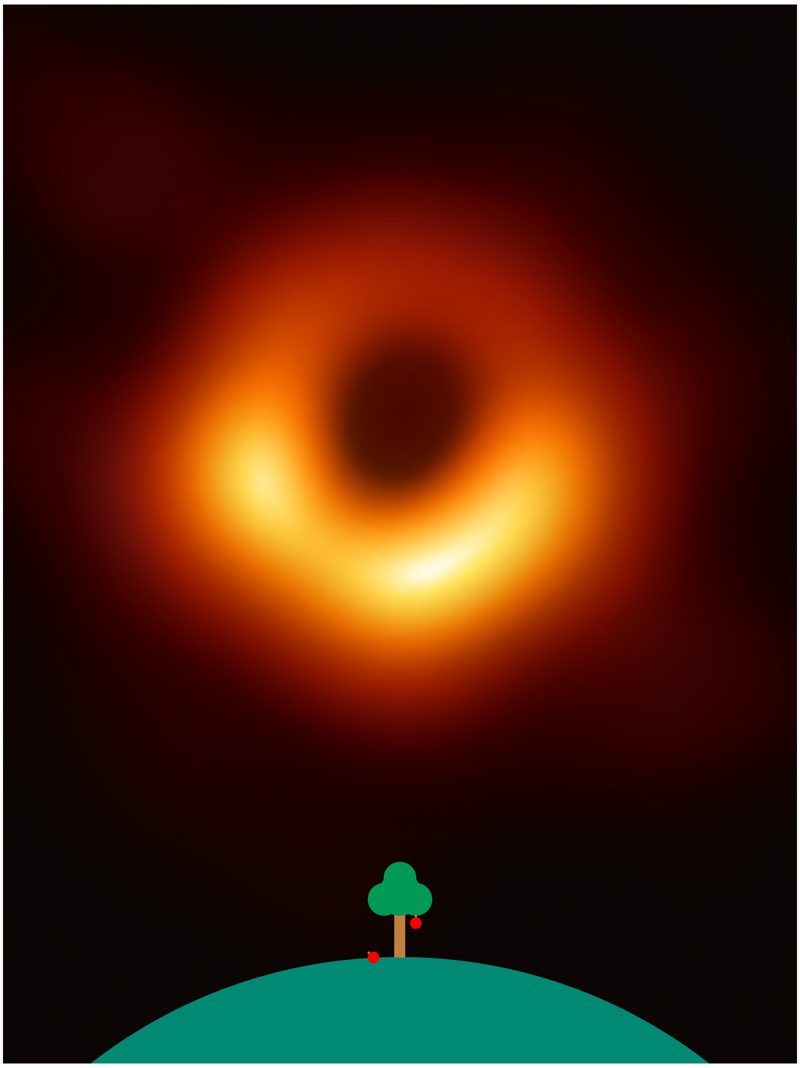
General Relativity

Chris Blair

Notes for an MSc course at the Vrije Universiteit Brussel

Last updated May 2, 2022



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Prologue

This is a course on gravity, from the black death to black holes.

In 1665, a plague outbreak forced the University of Cambridge to close, sending Isaac Newton back to his family home (and apple trees). There, he began to develop a theory of gravity, explaining why moons and fruit fall the same way, and leading to the famous expression

$$\vec{F} = -\frac{Gm_1 m_2}{r^2}\frac{\vec{r}}{r}, \tag{0.1}$$

detailing the gravitational force between two objects of mass m_1 and m_2 separated by the vector $\vec{r}$ of length $r \equiv |\vec{r}|$. This is an attractive, long-ranged force between any two massive bodies in the universe, which is relevant on extremely large length-scales. However, on the scale of individual particles, gravity is "weak". For instance, the gravitational force between a proton and an electron is 39 orders of magnitude less than the electrical force expressed by Coulomb's law, $F_c = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r^2}$, which is the other classic inverse square force law in physics.

Newton's Law of Universal Gravitation is not universal, and the whole Newtonian system of the world is an approximation to relativistic physics. At high velocities, excluding gravity, the theory of Special Relativity applies. The absolute time of human intuition and Newtonian dynamics breaks down, with the introduction of *spacetime* as the setting for physics. Special relativistic spacetime is characterised by the invariance of the speed of light, c, in all inertial (non-accelerating) frames. Energy and mass are related via Einstein's catchphrase equation $E = mc^2$. More accurately, this equation is a consequence of the relativistic relationship between energy and momentum, and so applies also to *massless* particles, such as the photon.

The special relativistic spacetime is a fixed structure, criss-crossed by the coordinate axes of privileged observers in inertial frames, whose worldviews are related by Lorentz transformations. General relativity elevates this spacetime from a background in which physics happens to a dynamical geometry which evolves in time and interacts with the matter it contains. Gravity is reinterpreted in terms of the curvature of the geometry, which is related dynamically to the energy and momentum content of the matter.

The Einstein field equations, which are the explicit mathematical statement of the relation between curvature and matter, are not quite as catchy as $E = mc^2$. However, we can extract from a *solution* of these equations a particularly striking expression, involving (as it should) not only m and c but also G, the gravitational constant. This is the expression for the Schwarzschild radius: $r_s = 2Gm/c^2$, which defines a *black hole.* The idea here is the following. As Einstein gravity interacts with *all* energy and momentum, it impacts the motion of both massive and massless particles: light bends in gravitational fields. If enough mass is gathered in a small enough spatial volume, the impact on light will be severe. If a total mass m is contained within a spherical region of radius r_s, then this region forms a black hole, defined by the property that no light (and hence no other matter) can propagate from $r < r_s$ to $r = \infty$.

Black holes can be viewed as the simplest things in the universe. No matter what matter went into their formation, they are in a sense formed of pure geometry, characterised solely by their mass M, as well as perhaps by their electric charge and angular momentum. Simultaneously, for a variety of reasons both classical and quantum, they are the most complicated things in physics.

1.4 billion years ago, a pair of black holes collided and merged. These black holes were 36 and 29 times the mass of the Sun, and combined into a single black hole 62 times the mass of the Sun. The remaining 3 solar masses was transformed into energy (by $E = mc^2$) and radiated away in the form of *gravitational waves*: localised perturbations in the structure of spacetime itself, travelling outwards from the merger at the speed of light. At the instant of collision, to an observer capable of "seeing" in both the electromagnetic and gravitational spectrum, this was the single most powerful source of energy in the entire visible universe, far exceeding the mundane electromagnetic output of all the stars in the sky.

On Earth, 1.4 billion years ago, there were no such observers. The lifeless super-continent Columbia was in the process of breaking up, the ocean home to single-celled organisms. As the gravitational waves travelled outwards through the vacuum, on Earth, continents wandered, seas and mountains appeared, disappeared, reappeared, and an incomprehensible number of lifeforms lived brief (but meaningful) existences. With the timeless inevitability of the laws of physics in vacuum, the gravitational waves of this long-ago merger finally reached and passed through Earth, on the 14th September, 2015, just after one set of the great-great-descendants of those single celled organisms had turned on a new scientific sense.

The idea of the LIGO gravitational wave detector is to split light from a laser into two separate beams, which then travel in orthogonal directions down "arms" of length L, reflect off a mirror before being recombined in a detector (see figure 1). If the distances travelled by the two beams are the same, then they will still be in sync when they recombine. If the distances travelled are not the same, then they will be out of sync, and the difference in distance can be inferred from the resulting interference observed in the light received.

A gravitational wave will briefly cause the lengths of each arm to fluctuate, with one arm first stretching while the other compresses, then vice versa. The amount of fluctuation is minuscule: perhaps $10^{-21}L$. The arms used by LIGO are about 4 km long, meaning that the length changes detected are of the order of 10^{-18}m.[1] The radius of the *proton* is 10^{-15}m. Think about that the next time you read that general relativity is the theory of large scales and quantum mechanics the theory of small scales.

The fluctuations $\delta L/L$ (known as the *strain*) detected from that first gravitational wave signal are shown in figure 2. As the gravitational waves propagate through the apparatus, $\delta L/L$ begins to oscillate, with increasing frequency, before the oscillations suddenly damp and die away.

This reflects the pair of black holes beginning to spiral closer together (in the "inspiral" phase of the merger), gaining angular velocity while radiating away energy. This angular velocity is directly related to the frequency of the oscillations measured by the detector, and it peaks at the instant of the merger, after which the signal drops off (in the "ringdown" phase).

LIGO is now observing gravitational wave signals (involving both black holes and neutron stars) at the rate of about one a week. Future detectors may have sufficiently high sensitivity to upgrade this to the order of several hundred *a day*. These signals link us to events that happened at immense distances from Earth, involving objects whose existence and fundamental properties we still do not fully understand. In order to begin to describe gravitational waves, black holes, and indeed the whole history of the universe, we need to study General Relativity.[2]

[1] In fact, to be able to observe this sort of change, the light in each arm is reflected multiple times between the original mirror and a second one, to allow the difference in path length travelled to build up.

[2] Many courses on General Relativity begin by motivating the subject with a discussion of GPS satellites, the time-keeping of which requires knowledge of the effects of curved spacetime described by the theory. For this author, there is more of a scientific thrill to detecting one billion year old black hole collisions than there is to ordering an Uber.

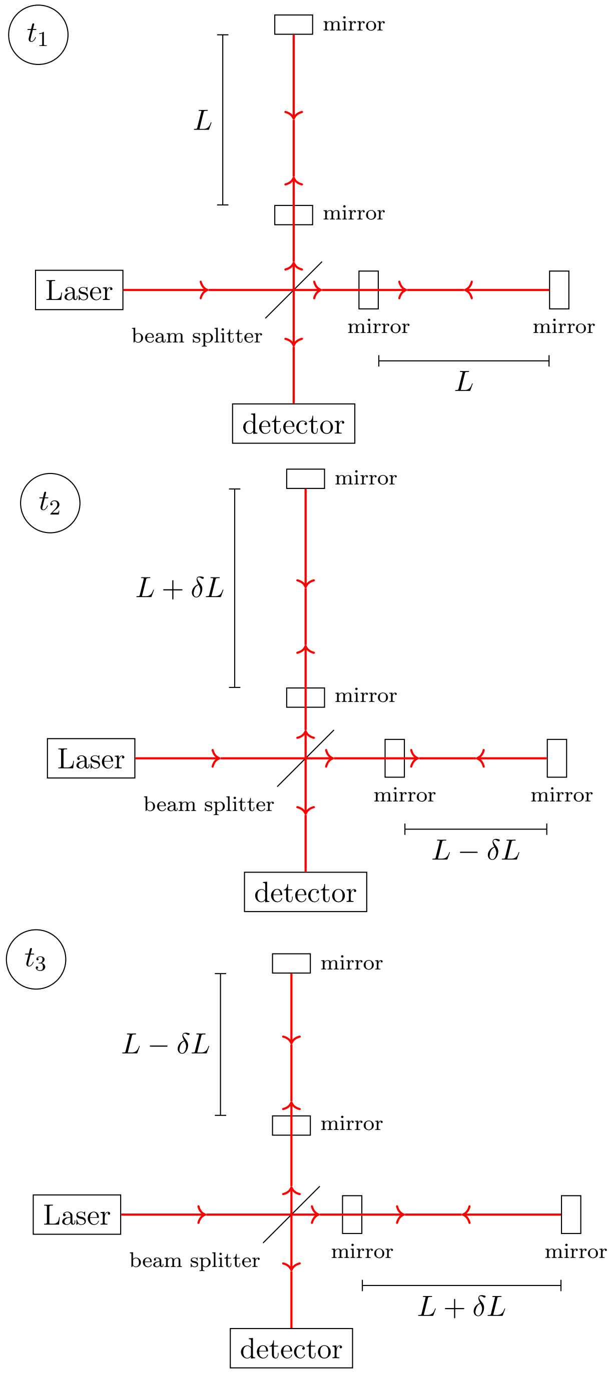


Figure 1: Gravitational wave detector. A passing gravitational wave causes the lengths of the paths taken by the split laser beam to fluctuate, for instance in the order shown with $t_1 < t_2 < t_3$.

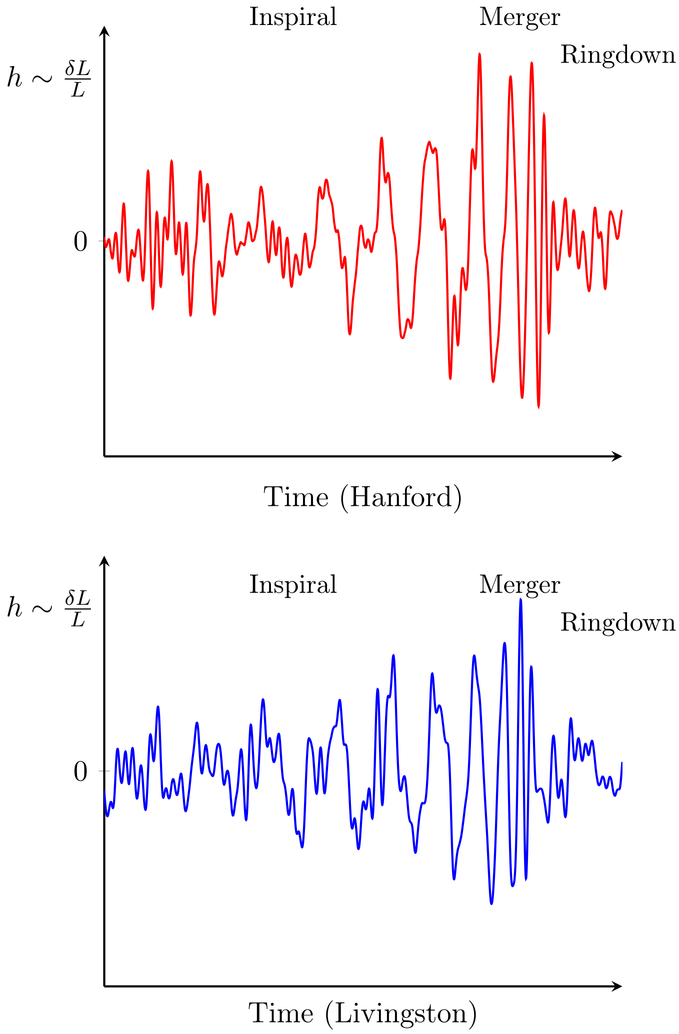


Figure 2: A chirping comes across the sky: the first gravitational wave detection. The *strain* is $\delta L/L$, where L is the length of the arm that the laser light travels, and δL is the fluctuation (positive and negative) due to the passage of the gravitational wave. Hanford and Livingston are the locations of LIGO's two detectors. The frequency increases as the merging black holes draw closer together ("inspiral") and is directly related to the angular velocity of their orbits as they rotate around each other, before cutting off at the merger. The data here for the graphs is taken from `https://www.gw-openscience.org/events/GW150914/`, and the original detection paper is: B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], "Observation of Gravitational Waves from a Binary Black Hole Merger," Phys. Rev. Lett. **116**, no. 6, 061102 (2016), arXiv:1602.03837.

About these notes

These notes accompany a one-semester, Masters level course (24 hours worth of lectures). The overall outline is close to the previous version given by Ben Craps, and I thank Ben for sharing his handwritten lecture notes. I also thank the students of the 2019/20 to 2021/22 courses for their questions during the lectures and constructive feedback, and also for their keen eye for typos in these notes.

The precise contents of these notes are based, mainly, on the following sources:

- *Spacetime and Geometry* by Sean Carroll
- Lecture notes by Harvey Reall and Ulrich Sperhake for Part II and Part III of the Mathematical Tripos, University of Cambridge, available online at:
  - `http://www.damtp.cam.ac.uk/user/hsr1000/teaching.html`
  - `http://www.damtp.cam.ac.uk/user/us248/Lectures/lectures.html`
- *Gravity* by James Hartle
- *Geometry, Topology and Physics* by Mikio Nakahara
- "The basics of gravitational wave theory," by Éanna Flanagan and Scott Hughes, New J. Phys. **7** (2005) 204, arXiv:gr-qc/0501041
- "The basic physics of the binary black hole merger GW150914," by B. P. Abbott *et al.* [LIGO Scientific and Virgo Collaborations], Annalen Phys. **529** (2017) no.1-2, 1600209, arXiv:1608.01940

Many other sources are available: each one constituting a specific chart with which to explore the space of General Relativity. Although we do not recommend building up a complete atlas, the use of one set of coordinates to understand the subject is not recommended!

I would appreciate being informed of typos and errors at christopher.blair@vub.be.

1 Spacetime

1.1 Newtonian space and time

In Newtonian physics, time and space form a four-dimensional continuum whose individual points we call events. An observer using Cartesian spatial coordinates associates to each event a set of four numbers, (t, x, y, z).

There is a clear distinction between the temporal coordinate t and the spatial coordinates (x, y, z). For any two events p and $\tilde{p}$, with $p = (t, x, y, z)$, $\tilde{p} = (\tilde{t}, \tilde{x}, \tilde{y}, \tilde{z})$, we can order them such that p occurs before $\tilde{p}$ $(t < \tilde{t})$, p occurs after $\tilde{p}$ $(t > \tilde{t})$, or the events are simultaneous $(t = \tilde{t})$. Furthermore, these distinctions are *absolute*: all observers agree on the time interval Δt between any two events. Thus the time interval is an *invariant* of the Newtonian world.

The other invariant is the spatial separation between simultaneous events, given by the Cartesian expression:

$$(\Delta s)^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \,. \tag{1.1}$$

This is the same in all choices of Cartesian coordinates: it is clearly invariant under a shift of the origin, $\vec{x} \mapsto \vec{x}' = \vec{x} + \vec{a}$, and also under rotations, $\vec{x} \mapsto \vec{x}' = A\vec{x}$, where A is an orthogonal matrix, obeying $A^T A = I$, such that

$$(\Delta s')^2 = \Delta\vec{x}'^T \Delta\vec{x}' = \Delta\vec{x}^T A^T A \Delta\vec{x} = \Delta\vec{x}^T \Delta\vec{x} = (\Delta s)^2 \,. \tag{1.2}$$

In index notation, let's write $x^i = (x, y, z)$, with i a three-dimensional spatial index. Then equivalently we have $x^i \mapsto x'^i = A^i{}_j x^j$ and we require $\delta_{ij} = A^k{}_i A^l{}_j \delta_{kl}$, where

$$\delta_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{1.3}$$

The symmetric matrix δ_{ij} is a first example of a "metric": it measures the norms of vectors in $\mathbb{R}^3$, or equivalently the distances between points, with $(\Delta s)^2 = \delta_{ij} \Delta x^i \Delta x^j$.

1.2 Special relativistic spacetime

The postulates of special relativity

In special relativity, we again have a four-dimensional continuum of events. However, there is no longer an absolute separation between time and space, and there is no well-defined notion of whether two events are simultaneous. We call this continuum spacetime. Our description of physics in spacetime is governed by the postulates of special relativity.

Postulates of special relativity

1. The laws of physics are the same in all inertial (non-accelerating) frames.
2. The speed of light in vacuum is a constant, c, in all inertial frames.

There is therefore a preferred class of observers: inertial (non-accelerating) observers, each of which labels events in spacetime by a set of coordinates (t, x, y, z) associated to their frame. In practical terms, we view (x, y, z) as a Cartesian coordinate system, which could be constructed using a series of rigid rods extended through space. These are accompanied by synchronised clocks placed at every point in space. Then measurements can be taken.

The invariant of the spacetime of special relativity is the *spacetime interval*:

$$(\Delta s)^2 = -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \,. \tag{1.4}$$

It is convenient to introduce an index notation. We use Greek indices, $\mu, \nu, \rho, \sigma, \ldots$ as labels for four-dimensional components. Thus our coordinates are $x^\mu = (ct, x, y, z)$. We also write $x^\mu = (x^0, x^i)$, with $i = 1, 2, 3$ labelling the spatial components as before, and $x^0 = ct$. Notice that the speed of light gives x^0 the dimensions of length.

We rewrite the spacetime interval in terms of the *Minkowski metric*:

$$\eta_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{1.5}$$

so that

$$(\Delta s)^2 = \eta_{\mu\nu} \Delta x^\mu \Delta x^\nu \,. \tag{1.6}$$

The inverse metric is denoted by $\eta^{\mu\nu}$.

Null, timelike and spacelike trajectories

We can distinguish between three classes of spacetime intervals.

- $(\Delta s)^2 = 0$, then the interval is *null.* Massless particles, or light rays, travel on null trajectories.
- $(\Delta s)^2 > 0$, then the interval is spacelike.
- $(\Delta s)^2 < 0$, then the interval is timelike. Massive particles (and hence observers) travel on timelike trajectories.

If two events p and q are timelike separated, then it is possible for an observer starting at p to travel between them. On the other hand, if they are spacelike separated, then it is impossible for an observer starting at p to travel to q.

The *lightcone* of a point p consists of all points q such that the spacetime interval between p and q is null. Thus it is the set of all points which can be reached by a light ray passing through p.

This causal structure is shown in figure 3.

A special role is played by straight paths (whether timelike, null, or spacelike): they represent geodesics of Minkowski spacetime (paths minimising, or maximising, the "distance" between two points). The *proper time* $\Delta\tau$ experienced by an observer travelling on a straight path is defined by

$$(c\Delta\tau)^2 = -(\Delta s)^2 \,. \tag{1.7}$$

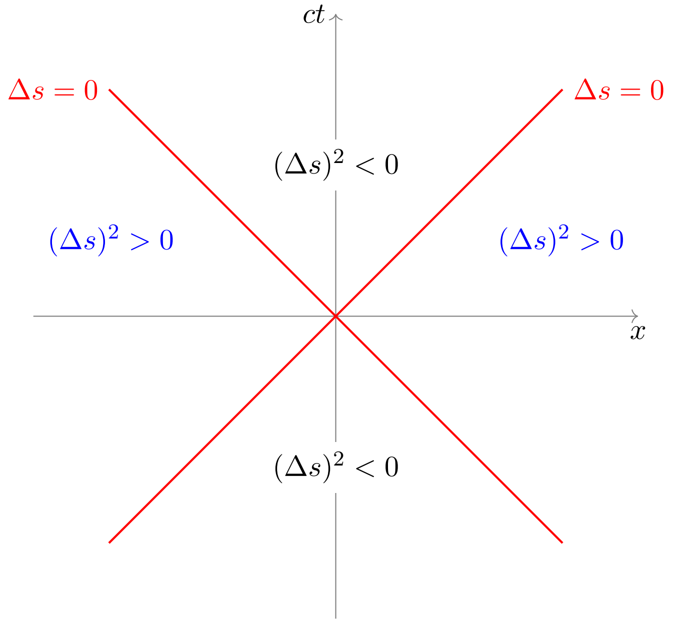


Figure 3: Causal structure of Minkowski spacetime

More generally, an arbitrary path can be viewed as a curve $x^\mu(\lambda)$ in spacetime, parametrised by some parameter $\lambda \in \mathbb{R}$. We can measure distances along such a curve by integrating the infinitesimal line element:

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu \,, \tag{1.8}$$

leading to the following general expressions for *proper distance* Δs (on a spacelike curve for which $ds^2 > 0$ always):

$$\Delta s = \int d\lambda \sqrt{\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \,, \tag{1.9}$$

and *proper time* $\Delta\tau$ (on a timelike curve for which $ds^2 < 0$ always):

$$c\Delta\tau = \int d\lambda \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}} \,. \tag{1.10}$$

(For a path between p and q such that $x^\mu(\lambda_0)$ are the coordinates of p and $x^\mu(\lambda_1)$ are the coordinates of q, the range of integration is from λ_0 to λ_1.)

This means that time experienced is observer dependent. Consider two observers, Alice and Bob (who may or may not be twins). Suppose they take two separate journeys through spacetime, depicted on the diagram 4. We work with coordinates such that the point p corresponds to $(0,0,0,0)$, where Alice and Bob start with synchronised clocks. Alice does nothing and eventually ends up at the point p', corresponding to $(ct_f, 0, 0, 0)$. Meanwhile, Bob shoots off with velocity v in the x-direction, before accelerating and decelerating about the turnaround point q and returning to Alice with velocity $-v$ in the x-direction, arriving back at p' to meet Alice.

According to the inertial coordinates we have chosen, Alice's worldline is given by $x^\mu(\lambda) = (c\lambda, 0, 0, 0)$, with $\lambda \in [0, t_f]$. The elapsed proper time for Alice is

$$\tau_A = \int_0^{t_F} d\lambda \,. \tag{1.11}$$

The precise details of Bob's worldline are in fact unimportant. We know it is of the form

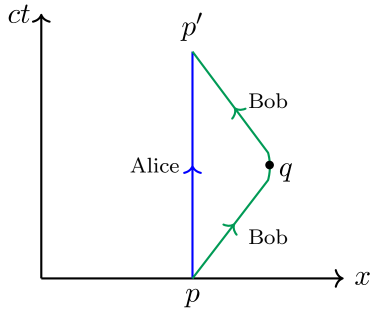


Figure 4: Alice and Bob, a spacetime odyssey.

$x^\mu(\lambda) = (c\lambda, x^i(\lambda))$. The proper time that elapses for Bob is

$$\tau_B = \int_0^{t_F} d\lambda \sqrt{1 - \frac{v^2(\lambda)}{c^2}}\,, \quad v^2 \equiv \delta_{ij} \frac{dx^i}{d\lambda}\frac{dx^j}{d\lambda}\,. \tag{1.12}$$

The factor $\sqrt{1-\frac{v^2(\lambda)}{c^2}}$ is less than one because $v(\lambda)$ is not identically zero for his worldline, and the Euclidean norm means that $v^2 \geq 0$. As a result, no matter what path is taken by Bob, it must be the case that the integral of τ_B comes out less than the integral for τ_A. Thus:

$$\tau_B < \tau_A\,, \tag{1.13}$$

and Bob returns to find that Alice has aged more. (This result holds regardless of the precise details of the decelerating/accelerating turnaround phase. We would need to work out the details of this to repeat the calculation from Bob's rest-frame. This rest-frame, however, is obviously not inertial. If it was then we would naively find the infamous twin paradox, namely that both Bob and Alice would by relativity calculate that the other had aged less.)

In fact, this thought experiment has been carried out in reality, for instance by synchronising atomic clocks and sending one off in a plane, leading to the expected result.

Transformations between inertial frames

The transformation from one inertial frame, with coordinates x^μ, to another, with coordinates x'^μ, is achieved by a Poincaré transformation. These transformations include spacetime translations by a constant four-vector a^μ,

$$x^\mu \mapsto x'^\mu = x^\mu + a^\mu\,, \tag{1.14}$$

and Lorentz transformations (spacetime "rotations"):

$$x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu\,, \tag{1.15}$$

where to preserve the spacetime interval we require

$$\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}\,. \tag{1.16}$$

The set of all matrices Λ obeying (1.16) forms the Lorentz group $O(1,3)$. This is the generalisation of the orthogonal group of spatial rotations, $O(3)$, to spacetime transformations. By taking the determinant of both sides, we see that $\det\Lambda = \pm 1$. We can restrict to the "proper orthochronous" Lorentz group $SO(1,3)^{\uparrow}$ consisting of those elements of $O(1,3)$ with $\det\Lambda = +1$ and $\Lambda^0{}_0$ positive. This is the group of transformations preserving the direction of time: it is also the component of $O(1,3)$ which can be continuously connected to the identity matrix.

The combination of Lorentz transformations (1.15) and spacetime translations (1.14) forms what is known as the Poincaré group. As the symmetry group of Minkowski spacetime, it is of fundamental importance in special relativistic theories, especially quantum field theory. We will not study it in detail here, but you should certainly do so elsewhere!

To gain some understanding of the transformations that are allowed, we can write down for example the Lorentz transformation generating rotations in the xy plane:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \theta \in [0, 2\pi) , \tag{1.17}$$

and that generating what you might have previously thought of as a genuine Lorentz transformation, namely a so-called "boost" acting on t and x:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \phi \in (-\infty, \infty) . \tag{1.18}$$

This leads to

$$\begin{aligned} ct' &= ct\cosh\phi - x\sinh\phi , \\ x' &= -ct\sinh\phi + x\cosh\phi . \end{aligned} \tag{1.19}$$

The point $x' = 0$ corresponds to the worldline defined by $x = ct\tanh\phi$ in the original frame: thus relative to this frame it is moving with velocity $v = c\tanh\phi$. Rewriting in terms of v, the transformation (1.19) takes the form usually written in elementary treatments of relativity:

$$\begin{aligned} t' &= \gamma(t - vx/c^2) , \\ x' &= \gamma(x - vt) , \end{aligned} \tag{1.20}$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$.

Exercise 1.1 (Lorentz transformations)

Show that $(\Lambda^0{}_0)^2 \geq 1$.

Verify that the transformations (1.17) and (1.18) indeed obey the defining condition (1.16), and that the expressions (1.19) and (1.20) are equivalent.

1.3 Physics, vectors and tensors

Physics in inertial frames

The laws of physics are the same in all inertial frames, which are related to each other by Lorentz transformations and spacetime translations. What form do these laws take?

Let's begin with the laws governing the motion of particles. Consider a curve $x^\mu(\lambda)$ in Minkowski spacetime, parametrised by some parameter $\lambda \in \mathbb{R}$. The rate of change of the curve with respect to λ is $\frac{dx^\mu}{d\lambda}$, and at each particular value of λ gives the *tangent vector* to the curve at that point. What can we do with this tangent vector? We can use the Minkowski metric to measure its norm: at any given point on the curve, the tangent vector will be timelike, spacelike, or null, and the curve is said to be timelike, spacelike or null (for a given range of the parameter λ) according to whether the tangent vector $\frac{dx^\mu}{d\lambda}$ is timelike, spacelike, or null.

Massive particles travel on timelike curves. Let's use proper time τ as the parameter for such a curve. Then the tangent vector is the four-velocity:

$$U^\mu = \frac{dx^\mu(\tau)}{d\tau}\,, \tag{1.21}$$

and as $c^2 d\tau^2 = -\eta_{\mu\nu} dx^\mu dx^\nu$ we have

$$\eta_{\mu\nu} U^\mu U^\nu = -c^2\,. \tag{1.22}$$

In the rest frame of the particle, we have $U^\mu = (c, 0, 0, 0)$. If m is the rest mass of the particle, then we can also define the momentum four-vector

$$p^\mu = mU^\mu\,, \tag{1.23}$$

which obeys $\eta_{\mu\nu} p^\mu p^\nu = -m^2 c^2$. This equation effectively defines the rest mass in a Lorentz invariant manner: up to a factor of $-c^2$, it is given by the norm of the four-momentum. The time component is $p^0 = E/c$, where E corresponds to the energy of the particle: in general, $E = c\sqrt{m^2c^2 + \vec{p}^{\,2}}$. (You can see why only the $\vec{p} = 0$ version of this equation caught on.)

The equation governing the motion of free massive particles is:

$$\frac{d}{d\tau} p^\mu(\tau) = m \frac{d^2}{d\tau^2} x^\mu(\tau) = 0\,. \tag{1.24}$$

Now, under a Lorentz transformation, $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$, the tangent vector transforms in the same way:

$$U^\mu = \frac{dx^\mu}{d\tau} \mapsto U'^\mu = \frac{dx'^\mu}{d\tau} = \Lambda^\mu{}_\nu \frac{dx^\nu}{d\tau} = \Lambda^\mu{}_\nu U^\nu\,, \tag{1.25}$$

hence also $p^\mu \mapsto p'^\mu = \Lambda^\mu{}_\nu p^\nu$. Suppose we have a particle that is not free, but responding to some force. The requirement that the laws of physics remain invariant under Lorentz transformations means that this force must also be a four-vector, f^μ, such that

$$\frac{d}{d\tau} p^\mu(\tau) = m \frac{d^2}{d\tau^2} x^\mu(\tau) = f^\mu\,, \tag{1.26}$$

and under Lorentz transformations $f^\mu \mapsto f'^\mu = \Lambda^\mu{}_\nu f^\nu$.

Indeed, all other physical quantities that enter into the description of relativistic physics in inertial frames must have a well-defined transformation under Lorentz transformations: and all physical laws must be written in a way that is *covariant* under such transformations. This will ensure that the form of these laws is indeed the same in all inertial frames.

Vectors and tensors in special relativity

Fundamentally, this means that all physical quantities that we will use have a well-defined *geometric* definition. We will make this very precise in section 3. However, let's set out the basic definitions now in a slightly simpler fashion.

In special relativity, we can define a vector to be a quantity X^μ transforming under Lorentz transformations "the same way" as the coordinates, that is:

$$X^\mu \mapsto X'^\mu = \Lambda^\mu{}_\nu X^\nu \,. \tag{1.27}$$

Examples of vectors are provided as above by the tangent vectors to curves, such as particle worldlines, through spacetime. Indeed the general *definition* of a vector on a more general spacetime is in terms of tangents and curves.

As well as vectors, we have covectors, which carry a *lower* index. Denoting a covector by ω_μ, we specify its transformation under Lorentz transformations to be:

$$\omega_\mu \mapsto \omega'_\mu = (\Lambda^{-1})^\nu{}_\mu \omega_\nu \,. \tag{1.28}$$

Covectors are dual to vectors, in grossly simplified terms you can think of them as "row vectors" while vectors with upper indices are "column vectors". This in particular means that given a vector and a covector we can form a *scalar* quantity $\omega_\mu X^\mu$ which is invariant under Lorentz transformations: $\omega_\mu X^\mu = \omega'_\mu X'^\mu$.

More generally, we have tensors carrying r upper indices and s lower indices (which we can refer to as an (r, s) tensor), transforming under Lorentz transformations as

$$T^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} \mapsto T'^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} = \Lambda^{\mu_1}{}_{\rho_1}\dots\Lambda^{\mu_r}{}_{\rho_r}(\Lambda^{-1})^{\sigma_1}{}_{\nu_1}\dots(\Lambda^{-1})^{\sigma_s}{}_{\nu_s}T^{\rho_1\dots\rho_r}{}_{\sigma_1\dots\sigma_s} \,. \tag{1.29}$$

The most important tensor in relativity (special or general) is the metric. The metric is a (0,2) tensor field which is *symmetric* and *non-degenerate*. Hence we can write it in components (in special relativity) as $\eta_{\mu\nu} = \eta_{\nu\mu}$, and there exists an inverse $\eta^{\mu\nu}$ such that $\eta^{\mu\rho}\eta_{\rho\nu} = \delta^\mu_\nu$. The Minkowski metric of special relativity is of course invariant under Lorentz transformations, but not invariant under more general coordinate transformations, while the Kronecker delta δ^μ_ν is a $(1, 1)$ tensor and invariant under all coordinate transformations. The metric gives a map taking two vectors, X, Y and giving a scalar $\eta(X, Y) = \eta_{\mu\nu}X^\mu Y^\nu$. Thus it defines an inner product on vectors, which we can use to measure norms, amongst other geometrically useful quantities.

Another important practical consequence of the metric's existence is that we can use it to identify vectors and covectors: if X^μ is a vector then $\eta_{\mu\nu}X^\nu$ is a covector, and if ω_μ is a covector then $\eta^{\mu\nu}\omega_\nu$ is a vector. To reduce the number of indices in expressions, we actually use the metric to raise and lower, meaning we write $X_\mu \equiv \eta_{\mu\nu}X^\nu$ and $\omega^\mu \equiv \eta^{\mu\nu}\omega_\nu$. In special relativity, the metric is constant, and diagonal, so the only thing to be concerned about here is the fact that the minus sign means $X_0 = -X^0$, $X_i = X^i$ $(i = 1, 2, 3)$.

Here we are automatically using the fact that one can sum over, or *contract*, repeated upper and lower indices, whether on different tensors, or on the same tensor. In the latter case, we obtain what is known as a *contraction* of a tensor. For instance, given an (r,s) tensor, we can generically produce rs individual contractions which are themselves $(r-1,s-1)$ tensors, given by:

$$(T')^{\mu_1\dots\mu_{r-1}}{}_{\nu_1\dots\nu_{s-1}} = T^{\mu_1\dots\mu_{i-1}\mu\mu_i\dots\mu_{r-1}}{}_{\nu_1\dots\nu_{j-1}\mu\nu_j\dots\nu_{s-1}}\,, \tag{1.30}$$

where here we contract the i^{th} upper index with the j^{th} lower index. Mechanically, the process is just so: one takes the tensor components and *contracts* one upper index with one lower index by labelling each with the same dummy index instructing us to sum over them. The result gives the components of a new tensor.

We will also have cause to impose symmetry properties on tensors, as already apparent from the existence of the metric. We define symmetrisation by

$$T^{(\mu_1\dots\mu_n)} = \frac{1}{n!}\left(T^{\mu_1\dots\mu_n} + \text{permutations of } 1,\dots,n\right)\,, \tag{1.31}$$

and antisymmetrisation by

$$T^{[\mu_1\dots\mu_n]} = \frac{1}{n!}\left(T^{\mu_1\dots\mu_n} + \text{signed permutations of } 1,\dots,n\right)\,, \tag{1.32}$$

where the sign is $+1$ for even permutations and -1 for odd permutations. For example,

$$\begin{aligned} T^{(\mu\nu)} &= \frac{1}{2}(T^{\mu\nu} + T^{\nu\mu})\,, \\ T^{(\mu\nu\rho)} &= \frac{1}{6}(T^{\mu\nu\rho} + T^{\nu\rho\mu} + T^{\rho\mu\nu} + T^{\mu\rho\nu} + T^{\nu\mu\rho} + T^{\rho\nu\mu})\,, \end{aligned} \tag{1.33}$$

$$\begin{aligned} T^{[\mu\nu]} &= \frac{1}{2}(T^{\mu\nu} - T^{\nu\mu})\,, \\ T^{[\mu\nu\rho]} &= \frac{1}{6}(T^{\mu\nu\rho} + T^{\nu\rho\mu} + T^{\rho\mu\nu} - T^{\mu\rho\nu} - T^{\nu\mu\rho} - T^{\rho\nu\mu})\,. \end{aligned} \tag{1.34}$$

Symmetry properties can be useful when contracting:

$$\begin{aligned} &T^{(\mu_1\dots\mu_n)}T'_{[\mu_1\dots\mu_n]} = 0\,, \\ &T^{(\mu_1\dots\mu_n)}T'_{\mu_1\dots\mu_n} = T^{(\mu_1\dots\mu_n)}T'_{(\mu_1\dots\mu_n)}\,, \quad T^{[\mu_1\dots\mu_n]}T'_{\mu_1\dots\mu_n} = T^{[\mu_1\dots\mu_n]}T'_{[\mu_1\dots\mu_n]}\,. \end{aligned} \tag{1.35}$$

If we have a tensor such that $S_{\mu_1\dots\mu_n} = S_{(\mu_1\dots\mu_n)}$ we say that it is symmetric on the indices $\mu_1\dots\mu_n$. If a tensor such that $A_{\mu_1\dots\mu_n} = A_{[\mu_1\dots\mu_n]}$ we say it is antisymmetric on the indices $\mu_1\dots\mu_n$.

Physical fields in special relativity (and more generally) are tensors of various types. The simplest are scalar fields, $\phi(x)$, which are just functions on spacetime. More interestingly, the electromagnetic field strength $F^{\mu\nu}$ is an antisymmetric tensor, $F^{\nu\mu} = -F^{\mu\nu}$. Its components give the electric field E^i and magnetic field B_k, with $F^{0i} = \frac{1}{c}E^i$ and $F^{ij} = \epsilon^{ijk}B_k$. Maxwell's equations in the presence of a source $J^\mu = (c\rho, J^i)$ are:

$$\partial_\mu F^{\nu\mu} = J^\nu\,, \quad \partial_{[\mu}F_{\nu\rho]} = 0\,. \tag{1.36}$$

We can solve the latter equation by writing $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$, where A_μ is the electomagnetic gauge potential. Notice here we are raising and lowering indices using the Minkowski metric without a second thought. In the presence of an electromagnetic field, the relativistic generalisation of the Lorentz force law for the motion of a particle is of the form (1.26) with $f^\mu = qU^\nu F^\mu{}_\nu$, where q is the charge of the particle. This can be checked to reduce to $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$ for small velocities.

Note that in defining the components of the B-field, we used the alternating symbol ϵ^{ijk} which is totally antisymmetric. The four-dimensional covariant version of this is the Levi-Civita symbol $\epsilon_{\mu\nu\rho\sigma}$ defined such that $\epsilon_{0123} = +1$. (This is not a true tensor as under certain Lorentz transformations it changes sign: more generally it transforms under general coordinate transformations up to a Jacobian factor. This means that more precisely it is something known as a tensor density, or a pseudo-tensor.)

Any physical system will have an energy-momentum tensor, $T_{\mu\nu}$, which is symmetric, and conserved, $\partial_\mu T^{\mu\nu} = 0$. We can view $T^{\mu\nu}$ as the conserved quantity resulting from invariance under spacetime translations. Intuitively, it describes the flux of four-momentum across surfaces of constant x^μ. The energy-momentum tensor will play an important role in general relativity, and we will describe it in more detail later in the course.

Vectors and tensor in mathematics

Before bringing gravity into the picture, let's have a first look at how one should really think about vectors and tensors mathematically. Physicists tend to refer to, if not think of, the objects $X^\mu, \omega_\mu, \ldots$ as vectors, covectors and tensors in their own right. This is not mathematically accurate. In fact, X^μ and ω_μ are not the vector and covector themselves, but instead are the *components* of a particular vector and covector in a certain basis. The Lorentz transformations, which we naively used above to "define" tensors, are then particular examples of a change of basis.

The other mathematical distinction that needs to be made concerns how and where exactly a vector is defined. Any vector must be an element of a vector space. Hence geometrically we have to associate a vector space to every point of spacetime. Then the objects $X^\mu(x)$ which we naively call 'vectors' are really the components of so-called vector fields, which at each x in spacetime give a vector which is an element of a particular vector space at x.

This is called the tangent space at x. The name is due to the fact that we can intrinsically define all vectors in the tangent space as being tangent vectors to curves through the point x. An intuitive basis for the tangent space at x is provided by the *coordinate basis*. We can define curves which correspond to moving away from the point x along the coordinate axes:

$$
\begin{aligned}
x_{(0)}(\lambda) &= (x^0 + \lambda, x^1, x^2, x^3)\,, \\
x_{(1)}(\lambda) &= (x^0, x^1 + \lambda, x^2, x^3)\,, \\
x_{(2)}(\lambda) &= (x^0, x^1, x^2 + \lambda, x^3)\,, \\
x_{(3)}(\lambda) &= (x^0, x^1, x^2, x^3 + \lambda)\,.
\end{aligned}
\tag{1.37}
$$

The bracketed indices label the curves themselves.

For each of these curves we can define a tangent vector which we denote here by

$$
e_{(\mu)} \equiv \frac{dx_{(\mu)}}{d\lambda}\,, \tag{1.38}
$$

such that $e_{(0)} = (1,0,0,0)$, $e_{(1)} = (0,1,0,0)$ and so on. This is perhaps a complicated way to define the usual unit vectors which are the standard basis for the vector space $\mathbb{R}^4$. Note the brackets around the μ on $e_{(\mu)}$. This notation is used to label the vectors themselves, so (μ) should not be thought of as the component index itself. It is common to write these coordinate basis unit vectors as

$$e_{(\mu)} = \frac{\partial}{\partial x^\mu} \equiv \partial_\mu \,. \tag{1.39}$$

The reason for this notation is because tangent vectors at x can be used to describe the rate of change of functions along curves through x. In particular, the rate of change of a function $f(x)$ in the direction μ is

$$\frac{d}{d\lambda} f(x_{(\mu)}(\lambda))\Big|_{\lambda=0} = \frac{dx^\nu_{(\mu)}}{d\lambda}\Big|_{\lambda=0} \frac{\partial f}{\partial x^\nu} = \frac{\partial f}{\partial x^\mu} \tag{1.40}$$

We identify the basis tangent vector $e_{(\mu)}$ with the differential operator that gives us the rate of change along the curve $x_{(\mu)}(\lambda)$ i.e. with $\frac{d}{d\lambda}$ of functions evaluated on this curve, leading to (1.39). For an arbitrary curve $x(\lambda)$ we have

$$\frac{d}{d\lambda} f(x(\lambda))\Big|_{\lambda=0} = \frac{dx^\mu}{d\lambda}\Big|_{\lambda=0} \frac{\partial f}{\partial x^\mu} = \frac{dx^\mu}{d\lambda}\Big|_{\lambda=0} e_{(\mu)}(f) \,, \tag{1.41}$$

which defines a tangent vector whose components with respect to the basis $e_{(\mu)}$ are $\frac{dx^\mu}{d\lambda}$.

More abstractly, given the basis $e_{(\mu)}$, then a general vector consists of a linear combination of the $e_{(\mu)}$ and so can be written

$$X = X^\mu e_{(\mu)} \,. \tag{1.42}$$

So it is the components X^μ of the vector X in the coordinate basis given by the $e_{(\mu)}$ are what we were previously thinking of as the vector itself.

The vector X, and not its components X^μ, is independent of the choice of basis. The transformation rule of X^μ under a Lorentz transformation follows from the requirement that

$$X^\mu e_{(\mu)} = X'^\mu e'_{(\mu)} \,, \tag{1.43}$$

where by definition of the coordinate basis

$$e'_{(\mu)} = \frac{\partial}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu} \frac{\partial}{\partial x^\nu} = (\Lambda^{-1})^\nu{}_\mu e_\nu \,. \tag{1.44}$$

We can similarly discuss bases for covectors. In general we will denote the basis for covectors by $\theta^{(\mu)}$. Again, the index here is not a component or coordinate index but merely labels the basis covectors. We want our basis to be dual to the basis $e_{(\mu)}$ in the sense that there should be a natural pairing between vector and covector with

$$\theta^{(\mu)}(e_{(\nu)}) = \delta^\mu_\nu \,. \tag{1.45}$$

A general covector is then expanded as

$$\omega = \omega_\mu \theta^{(\mu)} \,, \tag{1.46}$$

with $\omega_\mu \theta^{(\mu)} = \omega'_\mu \theta'^{(\mu)}$ such that $\theta'^{(\mu)} = \Lambda^\mu{}_\nu \theta^{(\nu)}$, $\omega'_\mu = (\Lambda^{-1})^\nu{}_\mu \omega_\nu$. The basis dual to the coordinate basis is provided by the diffentials of the coordinates:

$$\theta^{(\mu)} = dx^\mu \,, \tag{1.47}$$

which will be clearer later when we give more careful definitions of vectors and covectors.

One can already in special relativity begin to generalise the treatment of vectors and tensors. First of all, although the coordinate basis is convenient, it is not mandatory, and one could take at any point x an arbitrary basis for the tangent space. More generally, one can note that Lorentz transformations are a very special class of coordinate transformations $x^\mu \mapsto x'^\mu = \Lambda^\mu{}_\nu x^\nu$ which preserve the form of the Minkowski metric. (They are therefore called *isometries* of the Minkowski metric: we will see more about this in section 8.) When more properly defined, vectors transform also under arbitrary coordinate transformations, which will generically change the form of the metric. An obvious example would be to switch to spherical spatial coordinates instead of Cartesian coordinates. Physics in Minkowski spacetime can be formulated in such coordinates, however explicit physical expressions will not be as simple as in the special inertial frames related by Lorentz transformations.

2 Gravity and equivalence principles

Andrés Iniesta to* El País *(7th Sep 2010)

Cesc gave me a pass, Newton appeared ... When I control the ball, I know it's going to be a goal. I just had to wait for it to come down to kick it. And why did it come down? Because of the law of gravity.

Although from Iniesta's perspective, "the apple was the ball and Newton's head was my foot," in this section we will realise we can also view the 2010 men's World Cup Final from the point of view of the freely falling ball: which saw Iniesta's boot, and the Earth, accelerating towards it at about 9.8 metres a second squared. This simple change of perspective will lead us to the idea that gravity can be described in terms of a *curved spacetime.*

2.1 Newtonian gravity

Newton's law of gravitation states that the attractive gravitational force of a mass M located at $\vec{x'}$ felt by a mass m located at $\vec{x}$ (see figure 5) is:

$$\vec{F} = -\frac{GMm}{r^2}\frac{\vec{x}-\vec{x}'}{r}\,, \quad r \equiv |\vec{x}-\vec{x'}|\,. \tag{2.1}$$

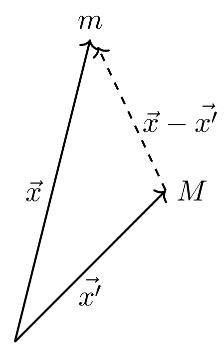


Figure 5: Newtonian gravity

This force is the gradient of a gravitational potential,

$$\vec{F} = -\vec{\nabla}V(\vec{x})\,, \quad V(\vec{x}) \equiv m\Phi(\vec{x})\,, \quad \Phi(\vec{x}) = -\frac{GM}{|\vec{x}-\vec{x'}|}\,. \tag{2.2}$$

Newton's second law is then:

$$m\ddot{\vec{x}} = m\vec{g}\,, \tag{2.3}$$

where the acceleration due to gravity is

$$\vec{g} = -\vec{\nabla}\Phi\,. \tag{2.4}$$

In this case,

$$\vec{g} = -\frac{GM}{r^2}\frac{\vec{x}-\vec{x'}}{r}\,. \tag{2.5}$$

The equations (2.3) and (2.4) hold for general gravitational fields due to different distributions of matter, specified by a matter density ρ such that

$$\nabla^2\Phi(\vec{x}) = 4\pi G\rho(\vec{x})\,. \tag{2.6}$$

2.2 Equivalence principles

A more careful treatment realises that in Newtonian theory, we could choose to distinguish between two types of mass. There is inertial mass, which appears in Newton's second law relating acceleration to force:

$$\vec{F} = m_i\vec{a}\,, \tag{2.7}$$

and gravitational mass, which appears in the equation describing the gravitational force due to a gravitational field $\vec{g}$:

$$\vec{F} = m_g\vec{g}\,. \tag{2.8}$$

In principle, these need not be equal: however the following Principle asserts that they are.

Weak equivalence principle (WEP)

Inertial mass equals gravitational mass,

$$m_i = m_g\,. \tag{2.9}$$

The WEP has been experimentally verified to high accuracy. To be precise, there is a scaling ambiguity on the right-hand-side of (2.8), as we could send $m_g \to \lambda m_g$, $\vec{g} \to \vec{g}/\lambda$. We can therefore fix $m_i = m_g$ for one test mass, thereby defining $\vec{g}$ unambiguously, and then calculate $1 - m_i/m_g$ for all other bodies. The discrepancy here is found to be of order 10^{-12}, confirming the WEP.

As $m_i = m_g$, the motion of a test body in the presence of a gravitational field is given simply by:

$$\ddot{\vec{x}} = \vec{g}(\vec{x}(t), t)\,, \tag{2.10}$$

and is independent of its mass. We can therefore restate the WEP as follows:

Weak equivalence principle (WEP)

The trajectory of a freely falling test body depends only on its initial position and velocity and is independent of its composition.

By *freely falling*, we mean that the gravitational force is the only force acting on the body. By a *test body*, we mean that we are assuming we can neglect the gravitational self-interaction of the body (if it is a composite object rather than some idealised point particle), and that its size is less than the scales on which $\vec{g}$ varies.

Let's define a new frame $(t', \vec{x}')$ which is accelerating with respect to our original coordinates $\vec{x}$:

$$\begin{aligned} t' &= t\,, \\ \vec{x}' &= \vec{x} - \vec{X}(t)\,, \quad \ddot{\vec{X}}(t) = \vec{a}\,. \end{aligned} \tag{2.11}$$

In the new frame, the equation of motion is

$$\ddot{\vec{x}}' = \vec{g} - \vec{a} \equiv \vec{g}\,' \,. \tag{2.12}$$

This means that uniform acceleration and the presence of a gravitational field are equivalent:

- If $\vec{g} \neq 0$, we can take $\vec{a} = \vec{g}$ in our definition of the new frame, to find that in the latter there is no gravitational field, $\vec{g}\,' = 0$.
- If on the other hand there was initially no gravitational field, $\vec{g} = 0$, we find that the new frame has a gravitational field, $\vec{g}\,' = -\vec{a}$.

We should distinguish between whether or not $\vec{g}$ is uniform:

- If $\vec{g}$ is uniform, we can define an inertial frame as that in which the laws of physics are simplest, which in this case is the freely falling frame with $\vec{a} = \vec{g}$.
- If $\vec{g}$ is non-constant, we can approximate it as uniform in a small enough region. Then in this region, we can define a *local inertial frame* $(t, \vec{x})$ using same coordinates we would define in flat Minkowski spacetime, and the laws of physics will be the same as in special relativity.

The Einstein equivalence principle generalises the WEP to encompass not just the motion of test bodies, but all non-gravitational physics.

Einstein equivalence principle (EEP)

1. The WEP holds.
2. In a local inertial frame, the results of all non-gravitational experiments are indistinguishable from the results of the same experiments performed in an inertial frame in Minkowski spacetime.

It is important to understand the local nature of this statement. The claim of the EEP is that in any theory of gravity, it should still always be possible to locally ignore the presence of the gravitational field by defining a freely falling frame as above. However, if we try to extend the region in which this frame is valid, it will in general not be possible to eliminate all gravitational effects. For instance, there can be "tidal effects" caused by the non-uniformity of a gravitational field, which will cause test bodies separated by a distance to fall differently.

2.3 Gravitational time dilation

Time runs slower the deeper you are in a gravitational potential.

Gravitational time dilation from equivalence principle

This is illustrated (literally, if inartistically, in figure 6) by Alice and Bob sitting in a gravitational field $\vec{g} = (0, 0, -g)$ at different positions on the z-axis. We put Bob at $z = 0$ and Alice at $z = h$. Alice sends light signals to Bob at regular intervals $\Delta\tau_A$. These will be received by Bob at intervals $\Delta\tau_B$. How are these intervals related?

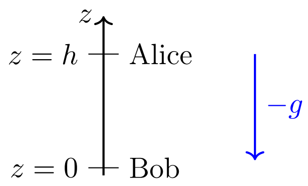


Figure 6: Alice and Bob are in a gravitational field.

We use the equivalence principle to analyse this in the frame where both Alice and Bob are accelerating with acceleration $-\vec{g}$ in Minkowski space. This frame, with the sequence of light transmissions, is shown in figure 7. We can choose this frame such that at $t = 0$ both Alice and Bob are at rest. We will neglect special relativistic effects by assuming that throughout the course of this experiment, our observers do not reach relativistic speeds. As their common velocity in the positive z direction is $v = gt$, this means we assume that gt/c is small.

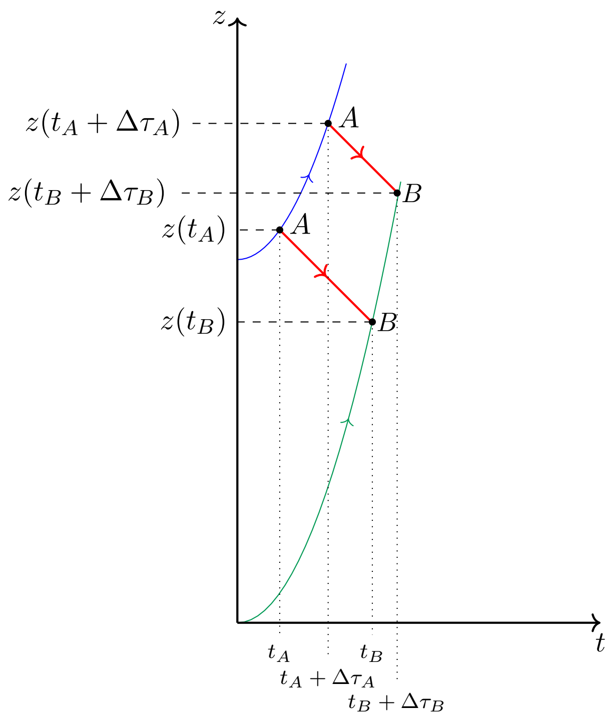


Figure 7: Using an inertial frame, in which Alice and Bob are accelerating. The red lines denote light signals sent from Alice to Bob. The blue and green curves denote their trajectories.

Consider Alice and Bob, travelling along the z-axis with trajectories $z_A(t)$ and $z_B(t)$ respectively. Alice sends light signals to Bob at times $t = t_A$ and $t = t_A + \Delta\tau_A$. The light signal trajectories are:

$$z_1(t) = z_A(t_A) - c(t - t_A)\,, \quad z_2(t) = z_A(t_A + \Delta\tau_A) - c(t - t_A - \Delta\tau_A)\,. \tag{2.13}$$

The first signal is received by Bob at time $t = t_B$ and the second at $t = t_B + \Delta\tau_B$. That is, when

$$z_A(t_A) - c(t_B - t_A) = z_B(t_B)\,, \tag{2.14}$$

$$z_A(t_A + \Delta\tau_A) - c(t_B - t_A + \Delta\tau_B - \Delta\tau_A) = z_B(t_B + \Delta\tau_B)\,. \tag{2.15}$$

Subtracting the former from the latter, we obtain

$$z_B(t_B + \Delta\tau_B) - z_B(t_B) + c\Delta\tau_B = z_A(t_A + \Delta\tau_A) - z_A(t_A) + c\Delta\tau_A \tag{2.16}$$

We have assumed nothing about the trajectories of Alice and Bob so far. Let's assume that the time intervals involved are small, and that we can Taylor expand to first order to find:

$$(c + z_B'(t_B))\Delta\tau_B \approx (c + z_A'(t_A))\Delta\tau_A \tag{2.17}$$

Denoting the velocities of Alice and Bob by $v_A = z_A'$, $v_B = z_B'$, we find:

$$\Delta\tau_B \approx \frac{1 + \frac{v_A(t_A)}{c}}{1 + \frac{v_B(t_B)}{c}} \Delta\tau_A \approx \left(1 + \frac{v_A(t_A) - v_B(t_B)}{c}\right) \Delta\tau_A \,. \tag{2.18}$$

Therefore the perceived time dilation depends on Alice's velocity when she emits the first signal, and on Bob's velocity when he receives the first signal. Note that if $v_A(t) = v_A$ and $v_B(t) = v_B$ are constant, we would find the usual non-relativistic Doppler effect. In the case we are interested in, Alice and Bob have the same acceleration from the same initial velocity, so $v_B(t_B) > v_A(t_A)$. As a result, we inevitably are led to $\Delta\tau_B < \Delta\tau_A$: time is running slower for Bob.

Let's now specialise to the trajectories of Alice and Bob given by:

$$z_A(t) = h + \frac{1}{2}gt^2\,, \quad z_B(t) = \frac{1}{2}gt^2\,. \tag{2.19}$$

Hence, $v_A(t_A) = gt_A$ and $v_B(t_B) = gt_B$. Now, we can approximate $t_B - t_A$ by the time it takes light to travel from $z = h$ to $z = 0$, so $(t_B - t_A) \approx h/c$. Then we can write the result (2.18) as:

$$\Delta\tau_B \approx \left(1 - \frac{gh}{c^2}\right) \Delta\tau_A \,. \tag{2.20}$$

Less time has elapsed for Bob: time runs slower for the observer deeper down the gravitational well.

We can rephrase this result in terms of wavelength, if Alice emits a continuous beam of light. With $\Delta\tau_A = \lambda_A/c$, $\Delta\tau_B = \lambda_B/c$ we find that the light is blueshifted (towards shorter wavelengths):

$$\lambda_B \approx \left(1 - \frac{gh}{c^2}\right) \lambda_A \,. \tag{2.21}$$

Conversely, light emitted by Bob would be redshifted (towards longer wavelengths). This effect has been confirmed experimentally in the Pound-Rebka experiment.

In terms of the gravitational potential, $\Phi = gz$, the general result is:

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right) \Delta\tau_A \,. \tag{2.22}$$

Gravitational time dilation from curved geometry

We have found that in a gravitational field, clocks run at different speeds. This used the equivalence principle to map the gravitational field to a situation with accelerating observers in an inertial frame.

In special relativity, we also found that clocks carried by different observers gave different measurements for the proper time elapsed. This followed from the fundamental properties of Minkowski spacetime, in particular from the definition of the spacetime interval using the Minkowski metric.

Inspiration follows: can we try to account for the gravitational time dilation effect by supposing it arises from having a metric that is not constant? Consider the following definition for the proper time of an observer:

$$c^2 d\tau^2 = \left(1 + \frac{2\Phi(\vec{x})}{c^2}\right) c^2 dt^2 - \left(1 - \frac{2\Phi(\vec{x})}{c^2}\right) d\vec{x}^2 \equiv -g_{\mu\nu}(\vec{x}) dx^\mu dx^\nu \,, \tag{2.23}$$

which replaces the Minkowski metric $\eta_{\mu\nu}$ by

$$g_{\mu\nu}(\vec{x}) = \begin{pmatrix} -1 - \frac{2\Phi(\vec{x})}{c^2} & 0 & 0 & 0 \\ 0 & 1 - \frac{2\Phi(\vec{x})}{c^2} & 0 & 0 \\ 0 & 0 & 1 - \frac{2\Phi(\vec{x})}{c^2} & 0 \\ 0 & 0 & 0 & 1 - \frac{2\Phi(\vec{x})}{c^2} \end{pmatrix} . \tag{2.24}$$

(This particular metric is time independent, but more generally it could also depend on time.) We assume that $\Phi(\vec{x})/c^2$ is small: in fact, this corresponds to a particular weakly curved limit of a genuine background geometry solving the Einstein equation, which we will discover later on.

We now interpret Alice and Bob's lightshow as if they were sitting in the spacetime where distances are described using this metric. Let's put Alice at $\vec{x}_A$ and Bob at $\vec{x}_B$. We don't yet know how to calculate the trajectories of photons in a curved geometry. This does not matter, because the only difference between the first and second signal is the time they are sent at, and the geometry is entirely time independent. Therefore each light signal takes the same path from Alice to Bob, but shifted by a time Δt. We can say that Alice sends signals at t_A and $t_A + \Delta t$, and these are received by Bob at t_B and $t_B + \Delta t$. This is depicted in figure 8.

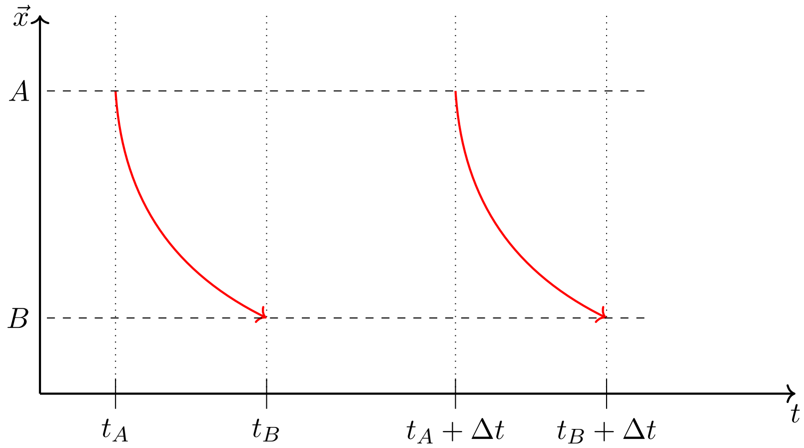


Figure 8: Signals in the curved geometry. The red curves denote the photon paths and are not an accurate depiction: all that matters is that the two separate signals follow the same path at different times.

We calculate the proper times measured in Alice and Bob's frames, using (2.23). The difference in proper time comes from the explicit $\vec{x}$ dependence of this expression. We abbreviate

$\Phi_A \equiv \Phi(\vec{x}_A)$ and $\Phi_B \equiv \Phi(\vec{x}_B)$. We have:

$$(\Delta\tau_A)^2 = \left(1 + \frac{2\Phi_A}{c^2}\right)(\Delta t)^2 \Rightarrow \Delta\tau_A \approx \left(1 + \frac{\Phi_A}{c^2}\right)\Delta t\,, \tag{2.25}$$

$$(\Delta\tau_B)^2 = \left(1 + \frac{2\Phi_B}{c^2}\right)(\Delta t)^2 \Rightarrow \Delta\tau_B \approx \left(1 + \frac{\Phi_B}{c^2}\right)\Delta t \tag{2.26}$$

from which we immediately obtain the same result:

$$\Delta\tau_B \approx \left(1 + \frac{\Phi_B - \Phi_A}{c^2}\right)\Delta\tau_A\,. \tag{2.27}$$

The gravitational time dilation now comes from an intrinsic property of the curved spacetime.

Choice of units

The speed of light in vacuum SI units is $c = 299,792,458$ metres/second. There is nothing special about metres or seconds. We are free to use whatever units we want to measure length and time. What we want to do is choose units such that the speed of light is just $c = 1$. This means our unit of time is the same as our unit of length, so that any time interval Δt is measured by the distance that light travels in Δt. Put differently, we choose to measure time using some unit $T' = L$ and length also using the unit L. In a "conventional" system of units where length is measured by some unit L (for example, the length of a king's forearm), and time is measured by some unit T (for example, the maximum duration of a recording possible to be stored on an audio CD[a]), then we have the conversion $T = cT'$, where c denotes the value of the speed of light in T/L.

From now on, we set $c = 1$.

[a]This is one Beethoven's Ninth, if that helps younger readers.

3 Manifolds, vectors and tensors

The simplest example of a manifold is $\mathbb{R}^n$, a set of points each labelled by n numbers. The choice of the numbers with which we label the points is called a choice of coordinates. We can use Cartesian coordinates, in which points in $\mathbb{R}^n$ are labelled by $x^\mu = (x^1, \ldots, x^n)$, with each $x^\mu \in \mathbb{R}$. Other coordinates are possible: for instance, we could translate or rotate our Cartesian coordinates. We could also use n independent functions of the original Cartesian coordinates instead. This is what we do in order to define polar coordinates in $\mathbb{R}^2$ or spherical coordinates in $\mathbb{R}^3$.

Examples of less trivial manifolds can be generated by considering surfaces embedded in $\mathbb{R}^n$. The most obvious example is the unit sphere, S^{n-1}, defined as the set of points in $\mathbb{R}^n$ obeying (in a choice of standard Cartesian coordinates)

$$(x^1)^2 + \ldots (x^n)^2 = 1\,. \tag{3.1}$$

This is an $(n-1)$-dimensional space, and so we should introduce coordinates on it which are a set of $n-1$ numbers. These coordinates of the sphere correspond to a subset of $\mathbb{R}^{n-1}$. For instance, the unit circle with $n = 2$ can be parametrised using a single coordinate $\theta \in [0, 2\pi)$. (Later on, we will refine this description.) If we like, we can relate this to the coordinates in $\mathbb{R}^2$ by $x^1 = \cos\theta$ and $x^2 = \sin\theta$.

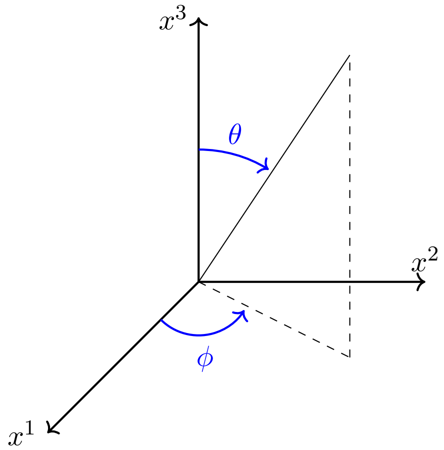


Figure 9: Spherical coordinates

In $n = 3$, we have the unit sphere. As coordinates we most obviously use the pair (θ, ϕ) where now $\theta \in [0, \pi]$ is the polar angle (running from the north to the south pole), and $\phi \in [0, 2\pi)$ is the azimuthal angle (running around the sphere for a fixed value of θ). The embedding in $\mathbb{R}^3$ is given by:

$$x^1 = \sin\theta\cos\phi\,, \quad x^2 = \sin\theta\sin\phi\,, \quad x^3 = \cos\theta\,. \tag{3.2}$$

However, these coordinates do not provide a perfect description of the sphere. When $\theta = 0$ or $\theta = \pi$, $\sin\theta = 0$, and the value of ϕ can be arbitrary. This is problematic if we want a single unambiguous mapping of points in $\mathbb{R}^2$ to describe points on the sphere. (Ultimately this is not possible: note that another way of constructing a sphere is to take the unit disc $(x^1)^2 + (x^2)^2 \leq 1$ in $\mathbb{R}^2$ and to identify all points on its boundary circle with each other.) For instance, if we are taking a limit, or differentiating some function at $\theta = 0$ or $\theta = \pi$, we will have difficulties if any value of ϕ is allowed, for example the derivative $\left.\frac{\partial f}{\partial \phi}\right|_{\theta=0} = \lim_{h\to 0} \frac{1}{h}\left(f(0,\phi+h) - f(0,\phi)\right)$ is naively not well-defined because the points $(0, \phi+h)$ and $(0,\phi)$ cannot be distinguished using these coordinates. In fact, also the derivative with respect to θ is problematic, because for instance $\left.\frac{\partial f}{\partial \theta}\right|_{\theta=\pi} = \lim_{h\to 0} \frac{1}{h}\left(f(\pi+h,\phi) - f(\pi,\phi)\right)$, and there is no point on the sphere with θ coordinate $\pi + h$ for positive h. This problem is present whenever our coordinates take values in a *closed* rather than *open* interval (and so is also present for the circle S^1).

We therefore need to refine our description of coordinates on the sphere. To deal with derivatives in general, we will need to be more careful about the ranges of our coordinates, restricting them to open subsets. This is a mathematical requirement that will allow us to use the theory of calculus on $\mathbb{R}^n$. (Later on, when we start doing physics in curved manifolds we will not always be so particular in worrying about whether we write down the ranges correctly.) To solve the ambiguity at the north and south poles, we will have to abandon the desire to have only one set of coordinates describing our space uniquely.

This is the most important aspect of manifolds: they *require* multiple choices of coordinates. For the sphere, you could object to this requirement by pointing out that points on the sphere are perfectly labelled using the embedding coordinates $(x^1, x^2, x^3) \in \mathbb{R}^3$. However, in general a manifold need not have any description in terms of an embedding as a surface into a higher-dimensional $\mathbb{R}^n$. Our ultimate goal is to describe the full spacetime universe as a manifold, and this is meant to be reality itself, describable without introducing an inaccessible unphysical space in which it is embedded. What we want is to describe manifolds in terms which are purely *intrinsic*.

3.1 Manifolds

A manifold makes mathematically rigourous our intuitive idea of a curved space. This is achieved by regarding a curved space as looking locally like a simpler flat one: the names used below invite you to think of the description of the surface of the Earth – to a good approximation, a sphere – via an *atlas* consisting of individual *charts* which look like regions of flat two-dimensional space. The important mathematics appears in the way that we glue different charts together to build up a complete description of the space we are interested in.

We will define a manifold "backwards", by beginning with the constituents: open sets in $\mathbb{R}^n$, which appear in charts describing some patch of the manifold; these charts are compiled into an atlas to give a complete description covering the whole manifold. When reading these definitions, keep an eye on figure 10, depicting the scene.

We open with a technical definition:

Open

The *open ball* $B_r(y)$ in $\mathbb{R}^n$, with $y \in \mathbb{R}^n$ and $r > 0$ some constant, consists of all points $x \in \mathbb{R}^n$ such that $|x - y| < r$.

An *open set* V in $\mathbb{R}^n$ consists of an arbitrary (potentially infinite) union of open balls: V is open if for all $y \in V$, there exists some $r > 0$ such that $B_r(y) \subset V$.

The reason that we like open sets is that they are needed to define limits and derivatives.

Recall for instance the definition of the derivative of a function $f(x)$ at a point a is given by $\lim_{h\to 0}(f(a+h)-f(a))/h$, which requires being able to choose a point $a+h$ close to the point a in the first place. This will be guaranteed in an open set, but not a closed one.

We now suppose that M is some set, that we wish to view as a manifold by modelling it locally as $\mathbb{R}^n$. The tool for carrying out this modelling is provided by the next definition:

Chart

A *chart* or *coordinate system* on a set M is a subset U of M together with a one-to-one (injective) map $\phi : U \to \mathbb{R}^n$, such that the image $\phi(U)$ is open in $\mathbb{R}^n$. We can then call U an open set in M.

Practically, we will often write $\phi = (x^1, \ldots, x^n)$ or $\phi = (x^\mu)$, where the individual x^μ are the *coordinate functions* in the chart (U, ϕ). If we are being precise, we will denote the coordinates of the point p by $\phi(p) = (x^\mu(p))$ or $\phi(p) = (x^\mu_p)$. Later on, we will begin to drop explicit mention of the map ϕ, and just refer to the coordinates x^μ in some chart. This is how physicists think in practice.

Generically, it is impossible to describe a manifold with a single chart. We need to introduce collections of (overlapping) charts:

Atlas

A (C^∞ or smooth) atlas on a set M is a collection of charts $\{(U_\alpha, \phi_\alpha)\}$, labelled by some index α, such that

1. M is covered by the U_α, that is $\bigcup_\alpha U_\alpha = M$.
2. The transformations between charts are smooth; that is, if two charts overlap, $U_\alpha \cap U_\beta \neq \emptyset$, then the following map:

$$\phi_\alpha \circ \phi_\beta^{-1} : \phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \to \phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n \tag{3.3}$$

   is smooth.

We say that two atlases are compatible if their union is also an atlas: the union of all atlases compatible with a given atlas is called a *complete atlas* (or a maximal atlas).

All these ideas come together to define a manifold:

Manifold

A (C^∞ or smooth) *manifold* is a set M together equipped with a complete (C^∞ or smooth) atlas.

In practical terms, given two charts (U_α, ϕ_α) and (U_β, ϕ_β) as shown in figure 10, we will write $\phi_\alpha = (x^\mu)$ and $\phi_\beta = (x'^\mu)$. Then the map $\phi_\alpha \circ \phi_\beta^{-1}$ relating the two charts can be viewed just as

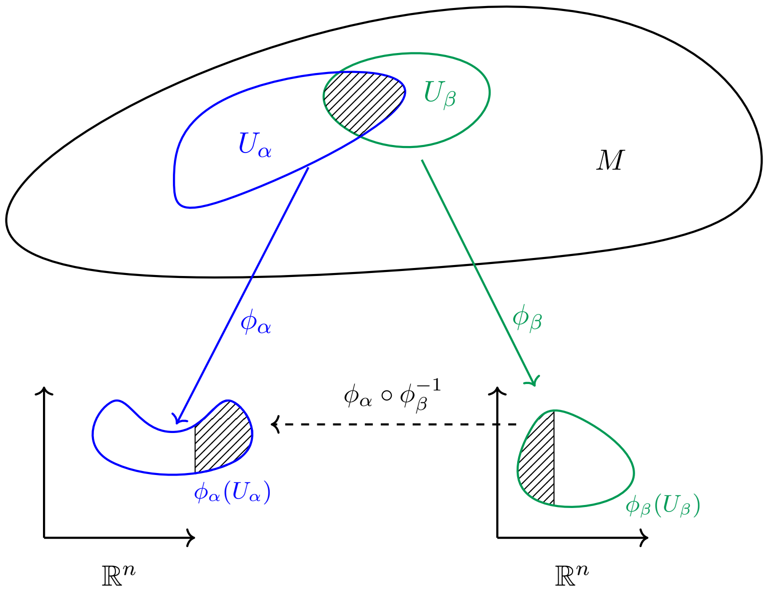


Figure 10: A manifold, showing two charts, and the transformation between their overlap.

a change of coordinates in $\mathbb{R}^n$, $\phi_\alpha \circ \phi_\beta^{-1} = x^\mu(x')$. As this is a map from $\mathbb{R}^n$ to $\mathbb{R}^n$, we can apply all results from calculus, such as the chain rule.

Example: $\mathbb{R}^n$

It is always mathematically appealing to start with a trivial example. So, $\mathbb{R}^n$ is a manifold, which can in fact be covered by a single coordinate chart, $\phi : (x^1, \dots, x^n) \to (x^1, \dots, x^n)$.

Example: S^1

The unit circle is the set of all points (x, y) in $\mathbb{R}^2$ obeying $x^2 + y^2 = 1$. It can be parameterised by $(x, y) = (\cos\theta, \sin\theta)$, with $\theta \in [0, 2\pi)$. However, this is not a good chart, because the interval $[0, 2\pi)$ is not open.

We can define a first chart (U_1, ϕ_1) by letting U_1 be S^1 with the point $(1, 0)$ removed, and letting $\phi_1(p) = \theta_1 \in (0, 2\pi)$.

A second chart (U_2, ϕ_2) involves letting U_2 be S^1 with the point $(-1, 0)$ removed, and letting $\phi_2(p) = \theta_2 \in (-\pi, \pi)$.

Clearly $U_1 \cup U_2 = S^1$, and the overlap $U_1 \cap U_2$ consists of the points $(x, y > 0)$ and $(x, y < 0)$. On the former region, we have $\theta_2 = \phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1$. On the latter, we have $\theta_2 = \phi_2 \circ \phi_1^{-1}(\theta_1) = \theta_1 - 2\pi$. These are obviously smooth.

Example: S^2

The unit sphere is the set of all points (x, y, z) in $\mathbb{R}^3$ obeying $x^2 + y^2 + z^2 = 1$. It can be parameterised by $(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. Again these are problematic, due to not being open, furthermore at $\theta = 0$ or π (the north and south poles) ϕ is arbitrary.

We define a first chart (U_1, φ_1) by restricting to $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$. This means that U_1 is the sphere with the poles plus arc given by $(x > 0, y = 0)$ removed, and $\varphi_1 : U_1 \to (\theta, \phi) \in \{(0, \pi) \times (0, 2\pi)\} \subset \mathbb{R}^2$.

A second chart (U_2, φ_2) compatible with the first, providing a covering of the sphere could then be given by the parametrisation $(x, y, z) = (-\sin\theta' \cos\phi', \cos\theta', \sin\theta' \sin\phi')$ with $\theta' \in (0, \pi)$, $\phi' \in (0, 2\pi)$. This means that U_2 is the sphere with points $(0, \pm 1, 0)$ plus the arc given by $(x < 0, z = 0)$ removed, and $\varphi_2 : U_2 \to (\theta', \phi') \in \{(0, \pi) \times (0, 2\pi)\} \subset \mathbb{R}^2$.

Clearly $S^2 = U_1 \cup U_2$, and $\varphi_1 \circ \varphi_2^{-1}$, $\varphi_2 \circ \varphi_1^{-1}$ can be checked to be smooth on $U_1 \cap U_2$.

3.2 Everything in its right place: vectors and tensors at a point

On a manifold, we introduce coordinates in order to describe its geometry in terms of standard coordinates on (subsets of) $\mathbb{R}^n$. Changes of coordinates give maps from $\mathbb{R}^n$ to $\mathbb{R}^n$, which allow us to apply results from calculus. We will develop these ideas into the theory of differential geometry, which allows us to define functions on manifolds and take their derivatives, leading to an intrinsic definition of vectors and other geometric objects, which can also be differentiated if our manifold admits additional geometric structures.

We will start our study by concentrating on a single point in a manifold, at which differential geometry reduces to linear algebra. It can be argued that the two most important ideas in physics are calculus and linear algebra. Calculus is important because it allows us to make non-linear problems linear; linear algebra is important because it allows us to solve these linearised problems. We can then try to use calculus to build up the solution to the original non-linear problem using the solutions of the linear problem.

In classical physics, for instance, we are fundamentally interested in knowing the future position of a particle at time $t + dt$, given its position at time t. Calculus tells us that, at the linearised level, $\vec{x}(t+dt) \approx \vec{x}(t) + \dot{\vec{x}}(t)dt$. The linear approximation to the position at time $t+dt$ is determined in terms of the velocity vector $\vec{v} \equiv \dot{\vec{x}}$ at time t. If we know all the velocities at each point, we can build up the full trajectory as $\vec{x}(t) = \int^t dt' \vec{v}(t')$.

In flat space, $\mathbb{R}^3$, velocities of particles are again vectors in $\mathbb{R}^3$. When learning classical mechanics, you do not normally distinguish between the two copies of $\mathbb{R}^3$ that appear. Intuitively, it is clear that at each point in $\mathbb{R}^3$ we can put the origin of another $\mathbb{R}^3$ consisting of all possible velocity vectors of particle trajectories passing through that point. Each such space of velocity vectors is however the same (isomorphic) to the original position space $\mathbb{R}^3$.

Now instead of flat space, let's think about the simplest curved manifold to visualise, namely the sphere S^2. Trajectories on the sphere, in terms of the coordinates of an $\mathbb{R}^3$ in which the sphere can be embedded, correspond to $\vec{x}(t)$ such that $\vec{x}(t) \cdot \vec{x}(t) = 1$. Differentiating this condition tells us that $\vec{x}(t) \cdot \dot{\vec{x}}(t) = 0$. The velocity of a particle on a sphere is at each point on its trajectory orthogonal to its position, in the embedding $\mathbb{R}^3$. The possible velocities of particle trajectories through each point on the sphere therefore lie in an orthogonal "tangent plane" touching the sphere at this point. The position space being S^2, a two-dimensional manifold, each space of velocity vectors is then a copy of $\mathbb{R}^2$. In this case, the two spaces are not isomorphic, though they do have the same dimension.

In general, rather than speak of "velocity" vectors, we should think of curves through a manifold, and tangent vectors to these curves. Then at each point p in the manifold M, we can associate an n-dimensional vector space of all possible tangent vectors. This is called the tangent space to M at p, denoted T_pM, with $T_pM \cong \mathbb{R}^n$. This is the description we will now make precise.

Functions and curves

Smooth functions

A function on a manifold is a map $f : M \to \mathbb{R}$, and is *smooth* if and only if for any chart (U, ϕ) the map $F \equiv f \circ \phi^{-1} : \phi(U) \to \mathbb{R}$ is smooth.

The set of all smooth functions on M is denoted by $C^\infty(M)$.

Note that we have introduced some extra notation here to distinguish between the function f on the manifold itself, and the composition of the function with a choice of coordinates, F. If $p \in M$ has coordinates x in some chart, then we have $f(p) = F(x) \in \mathbb{R}$.

Conversely, a curve through the manifold can be viewed as a map from the real numbers to the manifold.

Curves

A *curve* γ on a manifold M is a smooth function $\gamma : I \to M$ where I is an open interval in $\mathbb{R}$. This means that $\phi \circ \gamma$ is a smooth map from I to $\phi(U) \subset \mathbb{R}^n$ for all charts (U, ϕ).

Tangent vectors

Given a function $f : M \to \mathbb{R}$ and a curve $\gamma : I \to M$, then $f \circ \gamma : I \to \mathbb{R}$. Let's parametrise the interval, and hence the curve γ, by a parameter $t \in I$. Then the rate of change of f along the curve γ is given by:

$$\frac{d}{dt}\left[(f \circ \gamma)(t)\right] = \frac{d}{dt} f(\gamma(t)) \,. \tag{3.4}$$

At a point p in M, we can consider all possible curves through p. Then for all possible functions f on M we can compute how they change along these curves. Intuitively, this captures the changes in f in all directions passing through p. These rates of change can be expressed in terms of the *tangent vector* at p. An illustration of these ideas is to be found in figure 11.

Tangent vector

Given a curve $\gamma : I \to M$ with $\gamma(0) = p \in M$, then the tangent vector to γ at p is the linear map $X_p : C^\infty(M) \to \mathbb{R}$ such that

$$X_p(f) = \left[\frac{d}{dt}(f(\gamma(t)))\right]_{t=0} \,. \tag{3.5}$$

This definition is such that the tangent vector, for any functions f and g:

- is linear: $X_p(\alpha f + \beta g) = \alpha X_p(f) + \beta X_p(g)$, where α, β are constants,
- obeys the *Leibniz property* (product rule), $X_p(fg) = X_p(f)g(p) + f(p)X_p(g)$.

A more useful explicit expression for the tangent vector X_p acting on f in a chart ϕ, which we will use below, is the following. We introduce a chart in the neighbourhood of p, which we write as $\phi = (x^1, \ldots, x^n)$. In such a chart, a function f is described by $F = f \circ \phi^{-1}$, by which we mean $f(p) = F(x(p))$. We can write $f \circ \gamma = F \circ \phi \circ \gamma$, and apply the chain rule to the maps $F : \mathbb{R}^n \to \mathbb{R}$ and $\phi \circ \gamma : \mathbb{R} \to \mathbb{R}^n$, in order to write (i.e. using $f(\gamma(t)) = F(x(\gamma(t)))$:

$$X_p(f) = \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} . \tag{3.6}$$

Observe that the first derivative only involves F as a function of the coordinates in the chart, while the second involves the parametrisation of the curve γ in the same chart. Note that by $\left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)}$ we mean $\left(\frac{\partial F(x)}{\partial x^\mu}\right)_{x=x(p)}$ i.e. we evaluate this quantity in the chart (U, ϕ) at the point $\phi(p)$ which has coordinates $x^\mu(p)$.

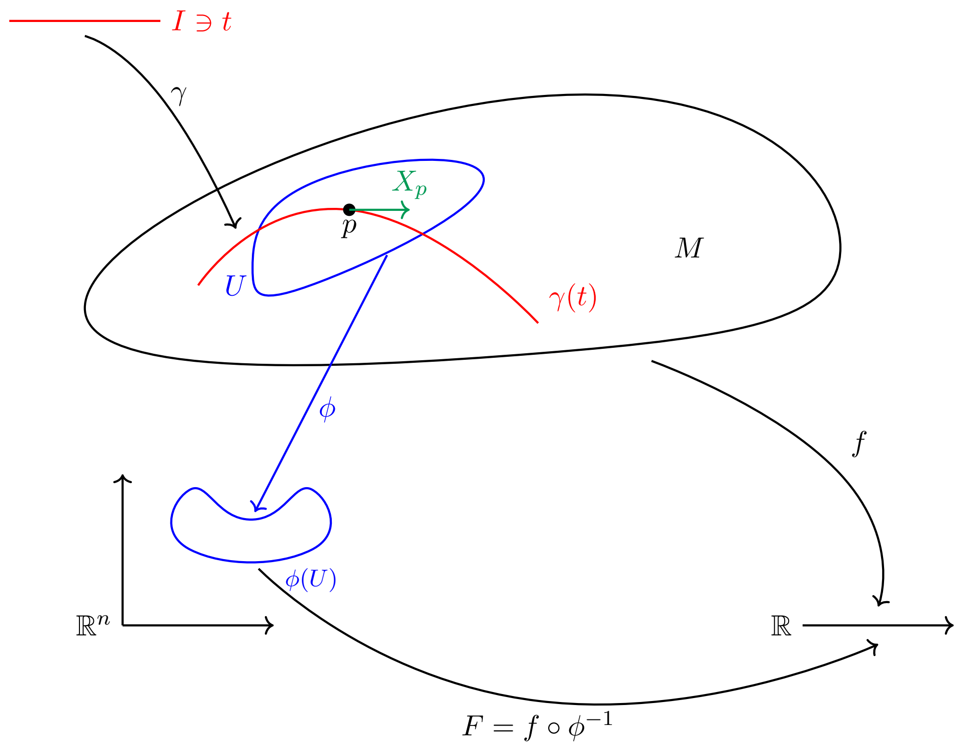


Figure 11: Viewing a tangent vector in a chart

Tangent space

The set of all tangent vectors at p forms an n-dimensional vector space, called the *tangent space* at p, denoted T_pM.

Verifying that the tangent space is a vector space: To confirm that the set of all tangent vectors at p is indeed a valid definition of an n-dimensional vector space, we need to check that the vector space axioms hold, and that we can construct an n-dimensional basis.

For the former, consider γ_1, γ_2 two curves with $\gamma_1(0) = \gamma_2(0) = p$, and corresponding tangent vectors X_p, Y_p. We define addition and multiplication by constants α, β in the obvious way:

$$(\alpha X_p + \beta Y_p)(f) = \alpha X_p(f) + \beta Y_p(f) . \tag{3.7}$$

To confirm that this combination meets our definition of a tangent vector, let $\phi = (x^1, \ldots, x^n)$

be a chart, and define a curve $\nu(t)$ with $\nu(0) = p$ via:

$$\nu(t) = \phi^{-1}\Big(\alpha(\phi(\gamma_1(t)) - \phi(p)) + \beta(\phi(\gamma_2(t)) - \phi(p)) + \phi(p)\Big)\,. \tag{3.8}$$

We can calculate the corresponding tangent vector at p using (3.6):

$$\begin{aligned}
Z_p(f) &\equiv \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{d}{dt}\Big[\alpha(x^\mu(\gamma_1(t)) - x^\mu(p)) + \beta(x^\mu(\gamma_2(t)) - x^\mu(p)) + x^\mu(p)\Big]\right)_{t=0} \\
&= \left(\frac{\partial F(x)}{\partial x^\mu}\right)_{\phi(p)} \left(\alpha\frac{d}{dt}x^\mu(\gamma_1(t)) + \beta\frac{d}{dt}x^\mu(\gamma_2(t))\right)_{t=0} \\
&= \alpha X_p(f) + \beta Y_p(f) \\
&= (\alpha X_p + \beta Y_p)(f)\,.
\end{aligned} \tag{3.9}$$

This true for all smooth functions f, and so we conclude that the definition (3.7) for the usual addition and multiplication by constants of vectors indeed produces another tangent vector at p. We define the zero vector to be given by the tangent vector to the curve $\gamma(t) = p$ for all t.

Next, we need to show that the tangent space is n-dimensional. We will do this by constructing a basis.

Using the same chart $\phi = (x^1, \ldots, x^n)$, define the following n curves:

$$\gamma_{(\mu)}(t) = \phi^{-1}\left(x^1(p), \ldots, x^{\mu-1}(p), x^\mu(p) + t, x^{\mu+1}(p), \ldots, x^n(p)\right)\,, \tag{3.10}$$

i.e. we have $x^\nu(\gamma_{(\mu)}(t)) = (x^1(p), \ldots, x^{\mu-1}(p), x^\mu(p)+t, x^{\mu+1}(p), \ldots, x^n(p))$. Denote the tangent vectors to these curves (temporarily) by $(e_{(\mu)})_p$. We have from (3.6):

$$(e_{(\mu)})_p(f) = \left(\frac{\partial F}{\partial x^\nu}\right)_{\phi(p)} \left(\frac{dx^\nu(\gamma_{(\mu)}(t))}{dt}\right)_{t=0} = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}\,. \tag{3.11}$$

Thus these curves provide tangent vectors which in this chart are just the partial derivatives with respect to the coordinate functions. Let us write these as

$$\left(\frac{\partial}{\partial x^\mu}\right)_p\,, \tag{3.12}$$

and by definition we have

$$\left(\frac{\partial}{\partial x^\mu}\right)_p f = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)}\,. \tag{3.13}$$

The subscript p here on the left-hand side means that we are differentiating functions defined at p in the chart x^μ around p, and evaluating these derivatives at p. The subscript $\phi(p)$ on the right-hand side corresponds to the fact that all quantities on the right-hand side are defined in the chart with coordinates x^μ themselves, and evaluated at the point $\phi(p)$ i.e. at the point $x^\mu(p)$.

The n tangent vectors (3.12) form a basis for the tangent space. This is the *coordinate basis.* To confirm they are indeed a basis, we have to check that they are linearly independent and span the tangent space.

For linear independence, suppose first that there exist constants α^μ such that $\alpha^\mu(\partial/\partial x^\mu)_p = 0$. Then $\alpha^\mu(\partial F(x)/\partial x^\mu)_{\phi(p)} = 0$ for all functions $F = f \circ \phi^{-1}$. However if we take $F = x^\nu$ for each ν, we get $\alpha^\nu = 0$ for all ν. Thus we cannot have a linear combination of the (3.12) equal to zero, and they are therefore linearly independent.

To check that an arbitrary tangent vector can be expressed in this basis, we note that for any function f

$$X_p(f) = \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p (f) \Rightarrow X_p = \left(\frac{dx^\mu(\gamma(t))}{dt}\right)_{t=0} \left(\frac{\partial}{\partial x^\mu}\right)_p . \tag{3.14}$$

Therefore we have shown that the tangent space is a vector space, and found an n-dimensional basis. □

Changes of basis

We constructed the coordinate basis $(\partial/\partial x^\mu)_p$ above. A general basis need not be based on coordinates. Suppose $\{e_{(\mu)} : \mu = 1, \ldots, n\}$ is an arbitrary basis for T_pM. Then $X_p \in T_pM$ can be written as $X_p = X_p^\mu e_{(\mu)}$. We call the X_p^μ the *components* of the tangent vector X_p in the basis $e_{(\mu)}$.

A *change of basis* can be implemented using some arbitrary invertible $n \times n$ matrix $A^\mu{}_\nu$ (an element of the matrix group GL(n)), with

$$e_{(\mu)} \mapsto e'_{(\mu)} = (A^{-1})^\nu{}_\mu e_{(\nu)} , \quad X_p^\mu \mapsto X_p'^\mu = A^\mu{}_\nu X_p^\nu . \tag{3.15}$$

This means that the tangent vector itself is invariant, $X_p = X_p^\mu e_{(\mu)} = X_p'^\mu e'_{(\mu)}$.

A special case is provided by a *change of coordinate basis.* Let $\phi = (x^1, \ldots, x^n)$ and $\phi' = (x'^1, \ldots, x'^n)$ be two charts both defined in a neighbourhood of p. Then, starting with the basis vector (3.12) of the coordinate basis associated to the chart ϕ we can use the chain rule to calculate as follows:

$$\begin{aligned}
\left(\frac{\partial}{\partial x^\mu}\right)_p (f) &= \left(\frac{\partial}{\partial x^\mu} \underbrace{(f \circ \phi^{-1})}_{F}\right)_{\phi(p)} \\
&= \left(\frac{\partial}{\partial x^\mu} \underbrace{(f \circ \phi'^{-1})}_{F'} \circ \underbrace{(\phi' \circ \phi^{-1})}_{x'^\mu(x)}\right)_{\phi(p)} \\
&= \left(\frac{\partial (F'(x'(x)))}{\partial x^\mu}\right)_{\phi(p)} \\
&= \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial F'(x')}{\partial x'^\nu}\right)_{\phi'(p)} = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p (f) .
\end{aligned} \tag{3.16}$$

Therefore for a change of coordinate basis we have:

$$\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)_{\phi(p)} \left(\frac{\partial}{\partial x'^\nu}\right)_p , \quad X_p'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)} X_p^\nu . \tag{3.17}$$

Again, notice the (irritating) subscripts. This is because we have so far only defined vectors at a particular point p. Here x^μ and x'^μ are two different sets of coordinates available in the neighbourhood of the point p, with the coordinate of the point p itself being $\phi(p) \equiv x_p^\mu$ or

$\phi'(p) \equiv x_p'^{\mu}$. Also notice that we have the identity

$$\left(\frac{\partial x'^{\mu}}{\partial x^{\nu}}\right)_{\phi(p)} \left(\frac{\partial x^{\nu}}{\partial x'^{\rho}}\right)_{\phi'(p)} = \delta^{\mu}_{\rho} \tag{3.18}$$

following from the chain rule.

Covectors

The idea now is to import everything we know from linear algebra. Given the tangent space T_pM at p, we can define the cotangent space T_p^*M which is the dual vector space. Elements of T_p^*M are called covectors, and are linear maps from T_pM to $\mathbb{R}$. Extending to multilinear maps from products of T_pM and T_p^*M to $\mathbb{R}$, we can define tensors.

Covectors

The dual space of the tangent space T_pM at $p \in M$ is called the *cotangent space*, denoted by T_p^*M, and consists of all linear maps from T_pM to $\mathbb{R}$. Elements of the cotangent space are called *covectors*. Hence if $\omega_p \in T_p^*M$, $\omega_p : T_pM \to \mathbb{R}$ is defined as a linear map $X_p \mapsto \omega_p(X_p) \in \mathbb{R}$.

Given a basis $\{e_{(\mu)}\}$ of T_pM, let's denote the dual basis of T_p^*M as $\{\theta^{(\mu)}\}$, with $\theta^{(\mu)}(e_{(\nu)}) = \delta^{\mu}_{\nu}$. Then if $X_p = X_p^{\mu} e_{(\mu)} \in T_pM$, we have $\theta^{(\mu)}(X_p) = X_p^{\mu}$, while using linearity

$$\omega_p(X_p) = (\omega_p)_{\mu}\theta^{(\mu)}(X_p^{\nu} e_{(\nu)}) = (\omega_p)_{\mu} X_p^{\nu} \theta^{(\mu)}(e_{(\nu)}) = (\omega_p)_{\mu} X_p^{\mu} \,. \tag{3.19}$$

An example of a covector is the *gradient* of a function.

Gradient

The gradient $(df)_p$ of the smooth function $f : M \to \mathbb{R}$ at the point p is the covector $(df)_p \in T_p^*M$ defined by

$$(df)_p(X_p) = X_p(f)\,, \quad \forall X_p \in T_pM\,. \tag{3.20}$$

To find the basis dual to the coordinate basis for vectors, let's consider the special case where we choose the function f to correspond to one of the components of the coordinate map ϕ, i.e. we take $f = x^{\mu}$.[3] This is obviously a map from M to $\mathbb{R}$. By the definition of the gradient,

$$(dx^{\mu})_p \left(\frac{\partial}{\partial x^{\nu}}\right)_p = \left(\frac{\partial}{\partial x^{\nu}}\right)_p x^{\mu} = \left(\frac{\partial x^{\mu}}{\partial x^{\nu}}\right)_{\phi(p)} = \delta^{\mu}_{\nu}\,. \tag{3.21}$$

Hence, the dual basis consists of the gradients $(dx^{\mu})_p$. In this basis, the components of the gradient of an arbitrary smooth function f can be worked out by applying df to the coordinate

[3] That is, the function $f(p) = x^{\mu}(p)$ gives the μ component of the coordinates defined by $(x^{\mu}) = \phi(p)$. We then have $F(x) = x^{\mu}(\phi^{-1}(x)) = x^{\mu}(p) = x^{\mu}$.

basis at p:

$$((df)_p)_\mu = (df)_p\left(\frac{\partial}{\partial x^\mu}\right)_p = \left(\frac{\partial}{\partial x^\mu}\right)_p(f) = \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} . \tag{3.22}$$

This is just the statement that $df = \partial_\mu f dx^\mu$ by the chain rule. There is an extra layer of convolution here as we are making a distinction between objects defined on the manifold M and how they appear in the chart ϕ.

Under a change of basis, the pairing between the vector basis $e_{(\mu)}$ at p and its dual basis $\theta^{(\mu)}$ must be preserved, so that the value of the number $\omega_p(X_p)$ is invariant under choice of basis. This means that

$$e_{(\mu)} \mapsto e'_{(\mu)} = (A^{-1})^\nu{}_\mu e_{(\nu)}\,, \quad X_p^\mu \mapsto X_p'^\mu = A^\mu{}_\nu X_p^\nu\,. \tag{3.23}$$

and

$$\theta^{(\mu)} \mapsto \theta'^{(\mu)} = A^\mu{}_\nu \theta^{(\nu)}\,, \quad (\omega_p)_\mu \mapsto (\omega_p)'_\mu = (A^{-1})^\nu{}_\mu(\omega_p)_\nu\,. \tag{3.24}$$

For a change of coordinate basis,

$$(dx^\mu)_p \mapsto (dx'^\mu)_p = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right)_{\phi(p)}(dx^\nu)_p\,, \quad (\omega_p)_\mu \mapsto (\omega'_p)_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)_{\phi'(p)}(\omega_p)_\nu\,. \tag{3.25}$$

Tensors

Just as vectors are linear maps from functions on M to $\mathbb{R}$, and covectors are linear maps from vectors on M to $\mathbb{R}$, then a tensor is a linear map from (products of) vectors *and* covectors to $\mathbb{R}$.

Tensors

An (r,s) *tensor* $\mathcal{T}_p$ at $p \in M$, or tensor of type (r,s) at $p \in M$, is a multilinear map from r copies of the cotangent space at p and s copies of the tangent space at p to $\mathbb{R}$:

$$\mathcal{T}_p : \underbrace{T_p^*M \times \cdots \times T_p^*M}_{r} \times \underbrace{T_pM \times \cdots \times T_pM}_{s} \to \mathbb{R}\,. \tag{3.26}$$

Thus, given r covectors $\omega_1, \ldots, \omega_r \in T_p^*M$ and s vectors $X_1, \ldots, X_s \in T_pM$ (we have dropped the subscripts p from these quantities to avoid unnecessary clutter: at the moment everything is living in spaces at the point p), we have a map

$$\mathcal{T}_p : \omega_1, \ldots, \omega_r, X_1, \ldots, X_s \mapsto \mathcal{T}_p(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) \in \mathbb{R} \tag{3.27}$$

which is linear in each entry. In an arbitrary basis $\{e_{(\mu)}\}$ for T_pM and corresponding dual basis $\{\theta^{(\mu)}\}$ for T_p^*M the components of $\mathcal{T}_p$ are:

$$\mathcal{T}_p^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s} \equiv \mathcal{T}_p(\theta^{(\mu_1)}, \ldots, \theta^{(\mu_r)}, e_{(\nu_1)}, \ldots, e_{(\nu_s)})\,, \tag{3.28}$$

such that

$$\mathcal{T}_p(\omega_1, \ldots, \omega_r, X_1, \ldots, X_s) = \mathcal{T}_p^{\mu_1\ldots\mu_r}{}_{\nu_1\ldots\nu_s}\omega_{1\mu_1}\ldots\omega_{r\mu_r}X_1^{\nu_1}\ldots X_s^{\nu_s}\,. \tag{3.29}$$

This should be invariant under choice of basis, hence under an arbitrary change of basis of the tangent space at p, $\theta'^{(\mu)} = A^\mu{}_\nu \theta^{(\nu)}$, $e'_{(\mu)} = (A^{-1})^\nu{}_\mu e_{(\nu)}$ we have

$$\mathcal{T}'^{\mu_1\ldots\mu_r}_p{}_{\nu_1\ldots\nu_s} \equiv A^{\mu_1}{}_{\rho_1}\ldots A^{\mu_r}{}_{\rho_r}(A^{-1})^{\sigma_1}{}_{\nu_1}\ldots(A^{-1})^{\sigma_s}{}_{\nu_s}\mathcal{T}^{\rho_1\ldots\rho_r}_p{}_{\sigma_1\ldots\sigma_s}\,. \tag{3.30}$$

A special case is the change of coordinate basis:

$$\mathcal{T}'^{\mu_1\ldots\mu_r}_p{}_{\nu_1\ldots\nu_s} \equiv \left(\frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\right)_{\phi(p)}\cdots\left(\frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}}\right)_{\phi(p)}\left(\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\right)_{\phi'(p)}\cdots\left(\frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}}\right)_{\phi'(p)}\mathcal{T}^{\rho_1\ldots\rho_r}_p{}_{\sigma_1\ldots\sigma_s}\,. \tag{3.31}$$

There are various operations one can perform on tensors to produce new tensors, which we will next discuss.

Tensor manipulations

Given two tensors, we can take a product.

Outer product

For $\mathcal{S}_p$ an (m,n) tensor at p and $\mathcal{T}_p$ a (r,s) tensor at p, the product $\mathcal{S}_p\otimes\mathcal{T}_p$ is an $(m+r,n+s)$ tensor at p given by:

$$\begin{aligned}&(\mathcal{S}_p\otimes\mathcal{T}_p)(\omega_1,\ldots,\omega_{m+r},X_1,\ldots X_{n+s})\\ &\qquad\qquad= \mathcal{S}_p(\omega_1,\ldots,\omega_m,X_1,\ldots,X_n)\mathcal{T}_p(\omega_{m+1},\ldots,\omega_{m+r},X_{n+1},\ldots,X_{n+s})\,,\end{aligned} \tag{3.32}$$

for arbitrary covectors $\omega_1,\ldots,\omega_{m+r}\in T^*_pM$ and vectors $X_1,\ldots,X_{n+s}\in T_pM$.

Then we can expand a tensor in a vector and dual covector basis p as:

$$\mathcal{T}_p = \mathcal{T}^{\mu_1\ldots\mu_r}_p{}_{\nu_1\ldots\nu_s}e_{(\mu_1)}\otimes\cdots\otimes e_{(\mu_r)}\otimes\theta^{(\nu_1)}\otimes\cdots\otimes\theta^{(\nu_s)}\,. \tag{3.33}$$

For example, in a coordinate basis

$$\mathcal{T}_p = \mathcal{T}^{\mu_1\ldots\mu_r}_p{}_{\nu_1\ldots\nu_s}\left(\frac{\partial}{\partial x^{\mu_1}}\right)_p\otimes\cdots\otimes\left(\frac{\partial}{\partial x^{\mu_r}}\right)_p\otimes(dx^{\nu_1})_p\otimes\cdots\otimes(dx^{\nu_s})_p\,. \tag{3.34}$$

We can thus regard the tensor as being an element of a tensor product vector space:

$$\mathcal{T}_p\in\underbrace{T_pM\otimes\cdots\otimes T_pM}_{r}\otimes\underbrace{T^*_pM\otimes\cdots\otimes T^*_pM}_{s}\,. \tag{3.35}$$

When we gave a preliminary definition of tensors in the context of special relativity, in section 1.3, we mentioned a number of other manipulations one can perform. These can all be defined for tensors at a point in a general manifold. Let us give slightly more formal definitions here.

Contraction

For $\mathcal{T}_p$ a (r,s) tensor at p, then we can produce an $(r-1,s-1)$ tensor by *contraction*. There are rs possible such contractions, obtained by inserting a single basis covector $\theta^{(\mu)}$ and a single basis vector $e_{(\mu)}$ in one of the "slots" of the tensor $\mathcal{T}_p$, producing:

$$\begin{aligned}&(\mathcal{T}_p)'(\omega_1,\ldots,\omega_{r-1},X_1,\ldots X_{s-1})\\&\qquad=\mathcal{T}_p(\omega_1,\ldots,\omega_{i-1},\theta^{(\mu)},\omega_i,\ldots,\omega_{r-1},X_1,\ldots,X_{j-1},e_{(\mu)},X_j,\ldots,X_{s-1})\,,\end{aligned}\tag{3.36}$$

for arbitrary covectors $\omega_1,\ldots,\omega_{r-1}$ and vectors $X_1,\ldots,X_{s-1}$. Here we have contracted in the i^{th} covector slot and j^{th} vector slot.

Exercise 3.1 (Contraction)

Show that the definition of contraction is basis independent.

In components, the above contraction is written as before as:

$$(\mathcal{T}_p')^{\mu_1\ldots\mu_{r-1}}{}_{\nu_1\ldots\nu_{s-1}}=(\mathcal{T}_p)^{\mu_1\ldots\mu_{i-1}\mu\mu_i\ldots\mu_{r-1}}{}_{\nu_1\ldots\nu_{j-1}\mu\nu_j\ldots\nu_{s-1}}\,.\tag{3.37}$$

Next, we can define symmetrisations/antisymmetrisations of tensors. For example, we could define the symmetrisation or antisymmetrisation $S_\pm(\mathcal{T}_p)$ of a $(0,2)$ tensor by

$$S_\pm(\mathcal{T}_p)(X_1,X_2)=\frac{1}{2}\left(\mathcal{T}_p(X_1,X_2)\pm\mathcal{T}_p(X_2,X_1)\right)\tag{3.38}$$

where X_1, X_2 are arbitrary vectors at p. There are obvious generalisations to higher rank tensors by appropriately permuting the vectors and covectors that they act on. Similarly, we can define a tensor to be totally symmetric or totally antisymmetric if it is equal to its symmetrisation (on all arguments) or antisymmetrisation (on all arguments). In components, this corresponds to the definitions we wrote down in section 1.3. We will not repeat these here.

3.3 Vector and tensor fields

We will now liberate ourselves from seeing only a single point in the manifold, and define notions of vector and tensor *fields*, which will be defined on (possibly only subsets of) the manifold as a whole, and which will give at each point a particular vector or tensor defined using the tangent and covector spaces at that point.

Vector fields

Vector field

A *vector field* is a map X taking any point $p \in M$ to a tangent vector X_p in the tangent space T_pM at p.

For any function $f : M \to \mathbb{R}$, given a vector field X we can define a new function $X(f) : M \to \mathbb{R}$ by $X(f)(p) = X_p(f)$. The vector field is smooth if this map is a smooth function for any smooth f.

(This definition assumes that the vector field can be defined on the whole of the manifold. In practice, we may encounter vector fields which are only defined in some subset of the manifold, i.e. not in all coordinate charts. This is not an important distinction for this course.)

In a chart with coordinates x^μ, we define the vector fields $\partial/\partial x^\mu$ by the assignation $\partial/\partial x^\mu : p \mapsto (\partial/\partial x^\mu)_p$, so that

$$\left(\frac{\partial}{\partial x^\mu}\right)(f) : p \mapsto \left(\frac{\partial F}{\partial x^\mu}\right)_{\phi(p)} \tag{3.39}$$

where recall $F \equiv f \circ \phi^{-1}$. This is smooth because f is smooth and hence derivatives of F are smooth. These "coordinate basis vector fields" correspond directly to the partial derivatives with respect to the coordinates in the chart.

We expand an arbitrary vector field in this coordinate chart as:

$$X = X^\mu \left(\frac{\partial}{\partial x^\mu}\right) \tag{3.40}$$

and X is smooth if and only if the components X^μ are smooth functions.

Under a change of coordinates, the transformation rule for vector fields is exactly that of vectors, (3.17), but with the subscripts removed:

$$\left(\frac{\partial}{\partial x^\mu}\right) = \left(\frac{\partial x'^\nu}{\partial x^\mu}\right)\left(\frac{\partial}{\partial x'^\nu}\right), \quad X'^\mu = \left(\frac{\partial x'^\mu}{\partial x^\nu}\right) X^\nu \,. \tag{3.41}$$

By definition, a vector field obeys the linearity and Leibniz properties, in particular $X(fg) = X(f)g + fX(g)$ for functions f, g.

The coordinate basis vector fields provide at each point in a chart a set of basis vectors for the tangent space at that point. In general, a set of n vector fields which are linearly independent at each point in the chart provide a (generally non-coordinate) basis for the tangent space at each point. For instance, for the sphere S^2 in the chart with coordinates (θ, ϕ), the coordinate basis is

$$\frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial\phi}, \tag{3.42}$$

while an equally good basis (recalling that $\theta \in (0, \pi)$) is

$$\frac{\partial}{\partial\theta}, \quad \frac{1}{\sin\theta}\frac{\partial}{\partial\phi} \tag{3.43}$$

(we will see in the next section that this is an *orthonormal* basis when measured in the usual metric on a sphere). In general, it is not guaranteed to be able to find a set of vector fields which provide at all points on the whole manifold a basis for the tangent spaces. Indeed, for the sphere S^2 it is famously impossible to find a vector field which is nowhere vanishing (this is known as the hairy ball theorem), which means in turn that it is impossible to find two vector fields which are everywhere linearly independent, which is what we would need to get a tangent space basis at each point.

If we have some vector fields, we can test whether or not they are the vector fields coming from some coordinate basis using the following idea. Given two vector fields, X and Y, we can construct a new vector field as follows:

Commutator

The *commutator* of two vector fields X and Y is the vector field $[X, Y]$ defined by

$$[X, Y](f) = X(Y(f)) - Y(X(f)), \tag{3.44}$$

for arbitrary smooth function f.

In a coordinate chart, we have

$$\begin{aligned} [X, Y](f) &= X^\mu \frac{\partial}{\partial x^\mu}\left(Y^\nu \frac{\partial F}{\partial x^\nu}\right) - (X \leftrightarrow Y) \\ &= X^\mu Y^\nu \frac{\partial^2 F}{\partial x^\mu \partial x^\nu} + X^\mu \frac{\partial Y^\nu}{\partial x^\mu}\frac{\partial F}{\partial x^\nu} - (X \leftrightarrow Y) \\ &= \left(X^\mu \frac{\partial Y^\nu}{\partial x^\mu} - Y^\mu \frac{\partial X^\nu}{\partial x^\mu}\right)\frac{\partial F}{\partial x^\nu}, \end{aligned} \tag{3.45}$$

from which we see that $[X, Y] = [X, Y]^\mu (\partial/\partial x^\mu)$ with

$$[X, Y]^\mu = \left(X^\nu \frac{\partial Y^\mu}{\partial x^\nu} - Y^\nu \frac{\partial X^\mu}{\partial x^\nu}\right). \tag{3.46}$$

For the basis vector fields, we have

$$\left[\frac{\partial}{\partial x^\mu}, \frac{\partial}{\partial x^\nu}\right] = 0. \tag{3.47}$$

Conversely, if $X_1, \dots X_m$ ($m \leq n$) are commuting vector fields that are linearly independent everywhere, then in a neighbourhood of any point p one can introduce coordinates $(\tilde{x}^1, \dots, \tilde{x}^n)$ such that $X_i = \partial/\partial\tilde{x}^i$ for $i = 1, \dots, m$. This means that if one is given a basis of vector fields, to check whether or not they are a coordinate basis, one just checks the commutators. For example, for the sphere we have for the basis of vector fields in (3.43)

$$\left[\frac{\partial}{\partial\theta}, \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}\right] = -\frac{\cos\theta}{\sin^2\theta}\frac{\partial}{\partial\phi} \tag{3.48}$$

which is not identically zero.

To see how this works, we need to find a way to associate a curve to a vector field.

Integral curves

Integral curve

Given X a vector field on M and $p \in M$, then an *integral curve* of X through p is a curve through p whose tangent vector at every point is given by the vector field X at that point.

In a coordinate chart, the integral curve $\gamma(t)$ is specified by the first order ODE

$$\frac{dx^\mu(\gamma(t))}{dt} = X^\mu(x(\gamma(t)))\,, \quad x^\mu(0) = x^\mu_p\,, \tag{3.49}$$

for which a unique solution always exists locally.

Suppose we have two vector fields, X and Y. Given a point p with coordinates x^μ_p, we can consider the integral curves of both X and Y through that point. Let us use t as the parameter on the integral curves of X and s as the parameter on the integral curves of Y. Then starting at p we can define the integral curves of X and Y passing through points in the neighbourhood of p as shown in figure 12.

Intuitively, we want to see if we can use these integral curves as "coordinate axes" to label the points near p. That is, we would like to say a point has coordinates (t, s) if it is located at the point found by following an integral curve of X a particular parameter distance t from p, and then following an integral curve of Y a parameter distance s. However, for this to make sense, this point better be the same as the one found by first following an integral curve of Y a parameter distance s from p, and then switching to an integral curve of X for a parameter distance t. In this case, it is natural to define the coordinates of this point to be (t, s), i.e. we can change to a coordinate basis in which $X = \frac{\partial}{\partial t}$ and $Y = \frac{\partial}{\partial s}$.

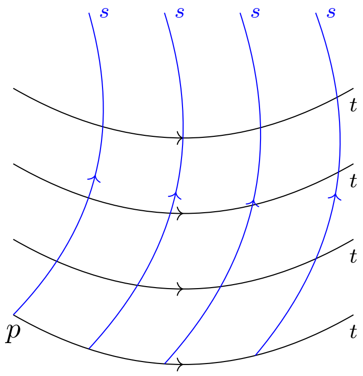


Figure 12: Integral curves in the neighbourhood of a point p.

More precisely (see figure 13), say we follow the integral curve of X an infinitesimal parameter distance δt from p, to a point q, and then follow an integral curve of Y a distance δs from q, so as to reach a point we can call p'. Alternatively, we can start at p, follow an integral curve of Y a parameter distance δs, to a point r, and then follow an integral curve of X a distance δt from r, in order to reach a point p''. The claim is that if X and Y commute, then p' and p'' are the same.

To show this, let $\sigma^\mu_X(t, x)$ denote the (coordinates of the) integral curve of X such that

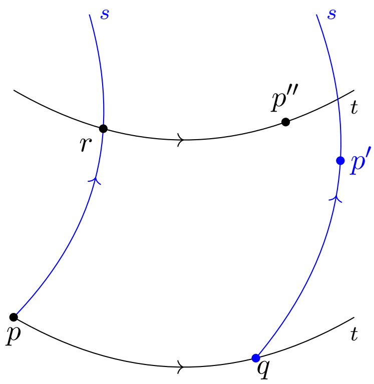


Figure 13: Following integral curves, one way, then the other: first from p to q to p', then from p to r to p''.

$\sigma^\mu_X(t=0,x) = x^\mu$ (and similarly for Y). This obeys by definition

$$\frac{d}{dt}\sigma^\mu_X(t,x) = X^\mu(\sigma_X(t,x)) \tag{3.50}$$

so to first order

$$\sigma^\mu_X(0+\delta t, x) \approx x^\mu + \delta t X^\mu(x)\,. \tag{3.51}$$

The point p' corresponds to

$$\begin{aligned}\sigma^\mu_Y(\delta s, \sigma_X(\delta t, x_p)) &\approx \sigma^\mu_X(\delta t, x_p) + \delta s Y^\mu(\sigma_X(\delta t, x_p)) \\ &\approx x^\mu_p + \delta t X^\mu(x_p) + \delta s Y^\mu(x_p + \delta t X) \\ &\approx x^\mu_p + \delta t X^\mu(x_p) + \delta s Y^\mu(x_p) + \delta s \delta t X^\nu \partial_\nu Y^\mu(x_p)\,.\end{aligned} \tag{3.52}$$

Similarly, we have that p'' has coordinates

$$\sigma^\mu_X(\delta t, \sigma_Y(\delta s, x_p)) \approx x^\mu_p + \delta t X^\mu(x_p) + \delta s Y^\mu(x_p) + \delta s \delta t Y^\nu \partial_\nu X^\mu(x_p)\,. \tag{3.53}$$

The difference between the coordinates of p' and p'' is hence

$$\sigma^\mu_Y(\delta s, \sigma_X(\delta t, x_p)) - \sigma^\mu_X(\delta t, \sigma_Y(\delta s, x_p)) \approx \delta s \delta t [X,Y]^\mu \tag{3.54}$$

This means that our attempted coordinates for the points p' and p'' will not agree unless the vector fields X and Y commute.

Exercise 3.2 (Vector fields)

1. Give an example of two linearly independent, nowhere-vanishing vector fields in $\mathbb{R}^2$ such that their commutator does not vanish. At each point, these provide a basis for the tangent space at that point, however it is not the coordinate basis.

2. Construct and sketch the integral curves for the vector fields you have chosen.

Covector and tensor fields

Covector field

A covector field is a map ω assigning to each point $p \in M$ a covector $\omega_p \in T_p^*M$.

Then $\omega(X)$ defines a function from M to $\mathbb{R}$ for any vector field X, given by $\omega(X)(p) = \omega_p(X_p)$ using the map $\omega_p : T_pM \to \mathbb{R}$ at each p. The covector field will be *smooth* if $\omega(X)$ defines a smooth function from M to $\mathbb{R}$, for any smooth vector field X.

In this language, we can define the gradient as a covector field obeying

$$df(X) = X(f)\,, \tag{3.55}$$

for arbitrary vector fields X. To find the *components* of the gradient in a coordinate basis, let's evaluate df acting on the coordinate basis vector fields:

$$df\left(\frac{\partial}{\partial x^\mu}\right) = \partial_\mu f\,. \tag{3.56}$$

We can further take $f = x^\nu$, as each coordinate x^ν itself in a chart can be viewed as a function from the manifold to $\mathbb{R}$. Then

$$dx^\nu\left(\frac{\partial}{\partial x^\mu}\right) = \partial_\mu x^\nu = \delta^\nu_\mu\,. \tag{3.57}$$

Therefore we see that dx^μ define the basis of covector fields dual to the coordinate basis of vector fields. We can expand:

$$df = \partial_\mu f dx^\mu\,, \tag{3.58}$$

and more generally for a covector field ω,

$$\omega = \omega_\mu dx^\mu\,. \tag{3.59}$$

Under a change of coordinate basis, we have

$$dx^\mu = \left(\frac{\partial x^\mu}{\partial x'^\nu}\right) dx'^\nu\,, \quad \omega'_\mu = \left(\frac{\partial x^\nu}{\partial x'^\mu}\right)\omega_\nu\,. \tag{3.60}$$

Tensor field

A *tensor field* (of type (r,s)) is a map $\mathcal{T}$ assigning to each point $p \in M$ a tensor $\mathcal{T}_p$ (of type (r,s)).

Then a tensor field defines a function from M to $\mathbb{R}$ given any set of r covector fields and s vector fields, with $\mathcal{T}(\omega_1,\ldots,\omega_r,X_1,\ldots,X_s)(p) = \mathcal{T}_p((\omega_1)_p,\ldots,(\omega_r)_p,(X_1)_p,\ldots,(X_s)_p)$. The tensor field will be *smooth* if $\mathcal{T}(\omega_1,\ldots,\omega_r,X_1,\ldots,X_s)$ defines a smooth function from M to $\mathbb{R}$, for any r smooth covector fields $\omega_1,\ldots,\omega_r$ and any s smooth vector fields $X_1,\ldots,X_s$.

In a chart, tensor fields admit the natural expansion in terms of the coordinate basis

$$\mathcal{T} = \mathcal{T}^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}\frac{\partial}{\partial x^{\mu_1}}\otimes\cdots\otimes\frac{\partial}{\partial x^{\mu_r}}\otimes dx^{\nu_1}\otimes\cdots\otimes dx^{\nu_s}\,, \tag{3.61}$$

Note that $\mathcal{T}$ is a smooth tensor if the components $\mathcal{T}^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s}$ in a coordinate basis are smooth functions. Under a change of coordinate basis

$$\mathcal{T}'^{\mu_1\dots\mu_r}{}_{\nu_1\dots\nu_s} \equiv \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}}\cdots\frac{\partial x'^{\mu_r}}{\partial x^{\rho_r}}\frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}}\cdots\frac{\partial x^{\sigma_s}}{\partial x'^{\nu_s}}\mathcal{T}^{\rho_1\dots\rho_r}{}_{\sigma_1\dots\sigma_s}\,. \tag{3.62}$$

Imprecision of language

From now on, we will often write simply vector, covector or tensor instead of vector field, covector field or tensor field. It should be obvious from context whether or not we really mean a tensor at a particular point or not.

Differential forms

An important set of tensors are *differential forms*, which can be defined on any manifold without introducing any extra structure, and then differentiated and integrated. Let's very briefly introduce these important objects.

Differential form

A differential form is a totally antisymmetric $(0,p)$ tensor field, also called simply a p-form.

By definition, functions are 0-forms, and covectors are 1-forms. More generally, if ω is a p-form, we can write it in a coordinate basis as

$$\omega = \omega_{\mu_1\dots\mu_p}dx^{\mu_1}\otimes\cdots\otimes dx^{\mu_p}\,,\quad \omega_{\mu_1\dots\mu_p} = \omega_{[\mu_1\dots\mu_p]}\,. \tag{3.63}$$

Given two differential forms, we can use an antisymmetrisation of the tensor product to define a new differential form.

Wedge product

Given ω a p-form and η a q-form, the wedge product $\omega\wedge\eta$ is a $(p+q)$-form defined in components by

$$(\omega\wedge\eta)_{\mu_1\dots\mu_{p+q}} = \frac{(p+q)!}{p!q!}\omega_{[\mu_1\dots\mu_p}\eta_{\mu_{p+1}\dots\mu_{p+q}]}\,. \tag{3.64}$$

This obeys

$$\omega\wedge\eta = (-1)^{pq}\eta\wedge\omega\,. \tag{3.65}$$

A basis for p-forms is provided by the wedge products of the basis 1-forms; thus

$$\omega = \frac{1}{p!}\omega_{\mu_1\dots\mu_p}dx^{\mu_1}\wedge\cdots\wedge dx^{\mu_p}\,. \tag{3.66}$$

This is just the explicit antisymmetrisation of equation (3.63), note for instance that

$$dx^\mu \wedge dx^\nu \equiv dx^\mu \otimes dx^\nu - dx^\nu \otimes dx^\mu \,. \tag{3.67}$$

We can also differentiate a p-form to get a $(p+1)$-form. This extends the notion of the gradient of a function, i.e. a 0-form, which we saw produced a covector i.e. a 1-form.

Exterior derivative

The exterior derivative d takes p-forms to $(p+1)$-forms, and is *nilpotent*: $d^2 = 0$. For ω a p-form, $d\omega$ is the $(p+1)$-form with components

$$(d\omega)_{\mu_1\dots\mu_{p+1}} = (p+1)\partial_{[\mu_1}\omega_{\mu_2\dots\mu_{p+1}]} \,. \tag{3.68}$$

The simplest example is indeed the gradient of a function, $(df)_\mu = \partial_\mu f$. The most important physical example is electromagnetism. We can view the electromagnetic gauge potential as a 1-form, $A = A_\mu dx^\mu$. Then the field strength is a 2-form, $F = dA$, with components $F_{\mu\nu} = 2\partial_{[\mu}A_{\nu]}$. Because d squares to zero, two of Maxwell's equations follow as an identity, $dF = 0$. To define the other two Maxwell equations however we do need to introduce extra structure, in the form of a metric on the manifold. As it happens (though we will not have much else to say about differential forms in this course), this is what we are going to do next.

Exercise 3.3 (Fun with forms)

Verify explicitly from the above definitions that $d(d\omega) = 0$, for ω a p-form, and that $\omega \wedge \eta = (-1)^{pq}\eta \wedge \omega$ for ω a p-form and η a q-form.

A summary

Stripping away the precision, what are the key points to take away? On a manifold, we do not in general have access to coordinates x^μ which cover the whole space (unlike in $\mathbb{R}^n$ or Minkowski spacetime). We have to use multiple sets of overlapping coordinate charts. So, coordinates are "local". Changes of coordinates $x^\mu \mapsto x'^\mu(x)$ give maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. In particular, we can differentiate x' with respect to x^μ.

A function on the manifold boils down to a map $f(x)$ from $\mathbb{R}^n$ to $\mathbb{R}$ in terms of local coordinates x. This too can be differentiated: the natural derivative operators correspond to vectors on the curved manifold. At a particular point with coordinates x, a general vector can be expanded as $X = X^\mu \partial_\mu$ and $X(f) = X^\mu \partial_\mu f$ gives a directional derivative of the function. A change of coordinates requires $X^\mu \mapsto X'^\mu = \frac{\partial x'^\mu}{\partial x^\nu} X^\nu$. Alongside functions and vectors, we have covectors. A particularly important covector is the gradient $df = \partial_\mu f dx^\mu$, and the one-forms dx^μ give a basis for covectors dual to the coordinate basis ∂_μ of vectors. A general one-form is expanded $\omega = \omega_\mu dx^\mu$, such that given a covector ω and vector X at the point x, $\omega_\mu(x) X^\mu(x)$ is a number. A change of coordinates requires $\omega_\mu \mapsto \omega'_\mu = \frac{\partial x^\nu}{\partial x'^\mu} \omega_\nu$. Building on vectors and covectors, there is a notion of tensors, characterised by a set of components $T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \ldots \nu_s}$ transforming under changes of coordinates in the obvious way.

All of this is formalised by thinking of there being at each point p of the manifold a tangent space T_pM, an n-dimensional vector space in which we place all vectors defined at p; the dual vector space is the cotangent space T_p^*M and is home to the covectors defined at p. Vector fields associate to each point p a vector in T_pM, and in a coordinate chart correspond to what we have intermittently being referring to just as the vector $X^\mu(x)$. (More properly, we can lump together all tangent spaces on M into a "tangent bundle", and then general vector fields should be viewed as coming from this bundle.)

4 Metrics

4.1 The metric tensor

The *raison d'être* of a metric is to allow us to measure the norms of tangent vectors. So a metric is a tensor that takes two vectors and delivers a real number. The natural definition is as follows:

Metric

A *metric tensor* g at $p \in M$ is a symmetric $(0, 2)$ tensor which is non-degenerate. That is,

- $g(X, Y) = g(Y, X)$ for all $X, Y \in T_pM$,
- $g(X, Y) = 0$ for all $Y \in T_pM$ if and only if $X = 0$.

The metric tensor is symmetric, and therefore an orthonormal choice of basis for T_pM exists in which the metric tensor is represented in components by a diagonal matrix with all diagonal entries equal to ± 1 (as it is non-degenerate).

Signature

A metric has signature (t, s) if with respect to an orthonormal basis it has t negative and s positive eigenvalues. By Sylvester's law of inertia, this is basis independent.

Riemannian and Lorentzian signature

For an n-dimensional manifold, a Riemannian metric has signature $(0, n)$ while a Lorentzian metric has signature $(1, n - 1)$.

Riemannian and Lorentzian manifold

A Riemannian/Lorentzian manifold (M, g) is an n-dimensional manifold with a metric tensor field of Riemannian/Lorentzian signature. (A Lorentzian manifold is also called pseudo-Riemannian.)

From now on, we will just say "metric" when we mean a "metric tensor field" on a manifold.

Signature conventions

The overall sign of a metric is a matter of convention. The "East Coast" convention is that Lorentzian signature is "mostly plus". The "West Coast" convention is that Lorentzian signature is "mostly minus", i.e. the four-dimensional Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}\,(+1, -1, -1, -1)$. In these notes, we are using the mostly plus convention, which is obviously correct.

Properly speaking, we should write the metric as a tensor in a coordinate basis as

$$g = g_{\mu\nu} dx^\mu \otimes dx^\nu \,. \tag{4.1}$$

However, in physics it is conventional to write the metric as a "line element", namely:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu \,. \tag{4.2}$$

This is not a mathematically precise expression, but it captures the intuition that a metric allows us to measure the length of curves through the manifold, as we will discuss in section 4.2.

Using this notation, let's write down some examples of metrics:

- The Euclidean metric on $\mathbb{R}^n$ is:

$$ds^2 = (dx^1)^2 + \cdots + (dx^n)^2 \,. \tag{4.3}$$

  This is valid everywhere on the manifold, and is obviously of Riemannian signature. The coordinate basis of vector fields $e_{(\mu)} = \partial/\partial x^\mu$ is an orthonormal basis.

- The Minkowski metric on $\mathbb{R}^n$ is:

$$ds^2 = -(dx^0)^2 + (dx^1)^2 + \cdots + (dx^{n-1})^2 \,. \tag{4.4}$$

  This is valid everywhere on the manifold, and is obviously of Lorentzian signature. The coordinate basis of vector fields $e_{(\mu)} = \partial/\partial x^\mu$ is an orthonormal basis.

- The round metric on a sphere S^2 in a chart with coordinates (θ, ϕ) is:

$$ds^2 = d\theta^2 + \sin^2\theta d\phi^2 \,. \tag{4.5}$$

  This is not invertible at $\theta = 0$, but this is excluded from our coordinate chart by definition. This is the metric in the coordinate basis, which admits (in this chart) the following basis of vector fields:

$$e_{(\theta)} = \partial_\theta \,, \quad e_{(\phi)} = \partial_\phi \,. \tag{4.6}$$

  This is not an orthonormal basis: $g_{\theta\theta} \equiv g(e_{(\theta)}, e_{(\theta)}) = 1$, $g_{\phi\phi} \equiv g(e_{(\phi)}, e_{(\phi)}) = \sin^2\theta$, though the basis vectors are orthogonal $g_{\theta\phi} \equiv g(e_{(\theta)}, e_{(\phi)}) = 0$. An orthonormal basis, which is not a coordinate basis, is provided by

$$\hat{e}_{(\theta)} = \partial_\theta \,, \quad \hat{e}_{(\phi)} = \frac{1}{\sin\theta}\partial_\phi \,. \tag{4.7}$$

  (Note this is well-defined in this coordinate chart as $\theta \in (0, \pi)$.) In this basis, $g(\hat{e}_{(\theta)}, \hat{e}_{(\theta)}) = 1$, $g(\hat{e}_{(\phi)}, \hat{e}_{(\phi)}) = 1$. As expected, this has Riemannian signature.

On a Riemannian/Lorentzian manifold, we can always choose an orthonormal basis $\hat{e}_{(\mu)}$ at a point p such that the metric at p is equal to the Euclidean/Minkowski metric, $g(\hat{e}_{(\mu)}, \hat{e}_{(\nu)}) = \eta_{\mu\nu}$, , $g(\hat{e}_{(\theta)}, \hat{e}_{(\phi)}) = 0$. As is shown by the sphere example above, this will not in general be possible at all points on the manifold simultaneously.

4.2 Curves and geodesics

With a metric in hand, we can begin to investigate the structure of worldlines in a curved manifold. We can say that a non-zero tangent vector $X \in T_pM$ is timelike, null or spacelike if $g(X,X) < 0$, $g(X,X) = 0$ or $g(X,X) > 0$ respectively. It is very important to realise that because the metric now varies over the manifold, the nature of what is timelike, null and spacelike can vary from point to point. Put differently, the causal structure of a general Lorentzian manifold is *local*.

If $\gamma : I \to M$ is a smooth curve, then we say that it is timelike, null or spacelike if its tangent vector is everywhere timelike, null or spacelike. The proper distance along a spacelike curve $\gamma(\lambda)$ with tangent vector X is given by

$$s = \int_I d\lambda \sqrt{g(X,X)|_{\gamma(\lambda)}}\,, \tag{4.8}$$

while the proper time along a timelike curve is given by

$$\tau = \int_I d\lambda \sqrt{-g(X,X)|_{\gamma(\lambda)}}\,. \tag{4.9}$$

In coordinates, the proper time τ can be related to the parameter λ via

$$d\tau^2 = -g_{\mu\nu}dx^\mu dx^\nu \Rightarrow \left(\frac{d\tau}{d\lambda}\right)^2 = -g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda}\,. \tag{4.10}$$

In special relativity, we saw that straight paths in Minkowski spacetime maximised the proper time of any timelike paths between two points. We can find the analogy of a "straight" path, i.e. a path of maximum (or in principle minimum) proper time, by finding the Euler-Lagrange equations which extremise the functional:

$$\tau[\gamma] = \int_0^1 d\lambda\, L(x(\lambda), \dot{x}(\lambda))\,, \quad L(x(\lambda), \dot{x}(\lambda)) = \sqrt{-g_{\mu\nu}(x(\lambda))\dot{x}^\mu \dot{x}^\nu}\,. \tag{4.11}$$

describing the proper time on timelike curves between $\gamma(0) = p$ and $\gamma(1) = q$, for p, q some points in M. Here $\dot{x}^\mu \equiv \frac{dx}{d\lambda}$. We cannot yet use proper time to parametrise these curves because the value of τ at q is different for each path, and this does not lend itself to a valid extremisation problem starting with the integral in (4.11). Varying with respect to x, we have the usual equations

$$\frac{d}{d\lambda}\frac{\partial L}{\partial \dot{x}^\mu} - \frac{\partial L}{\partial x^\mu} = 0\,, \tag{4.12}$$

implying

$$\frac{d}{d\lambda}\left(-\frac{1}{L}g_{\mu\nu}\dot{x}^\nu\right) + \frac{1}{2L}\partial_\mu g_{\nu\rho}\dot{x}^\nu \dot{x}^\rho = 0\,. \tag{4.13}$$

Now, from (4.10) we see that we have $d\tau = L d\lambda$ or $\frac{d}{d\lambda} = L\frac{d}{d\tau}$. At this point it is valid to view our curve as being directly parametrised by proper time. The equation (4.13) becomes

$$\frac{d}{d\tau}\left(g_{\mu\nu}\frac{dx^\nu}{d\tau}\right) - \frac{1}{2}\partial_\mu g_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,, \tag{4.14}$$

or after writing $\frac{d}{d\tau}g_{\mu\nu} = \partial_\rho g_{\mu\nu}\frac{dx^\rho}{d\tau}$,

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\nu\rho}{}^\mu \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,, \tag{4.15}$$

where a certain mysterious combination of metric derivatives has appeared:

$$\Gamma_{\nu\rho}{}^\mu = \frac{1}{2}g^{\mu\lambda}\left(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}\right)\,. \tag{4.16}$$

These are not the components of a tensor! Note that they are symmetric in the lower indices, $\Gamma_{\nu\rho}{}^\mu = \Gamma_{\rho\nu}{}^\mu$. These are known as the *Christoffel symbols*, and they are going to become very important in section 5, where we will discover that they are the components of the Levi-Civita connection.

The equation (4.15) describes the curves on which massive particles travel in Lorentzian manifolds. It is known as the geodesic equation, and we will discuss it in more generality in the next section. In the special case of Minkowski space, we can take $g_{\mu\nu} = \eta_{\mu\nu}$ everywhere, so that $\Gamma_{\nu\rho}{}^\mu = 0$, and we rediscover the equation $\frac{d^2x^\mu}{d\tau^2} = 0$ for a free particle.

A summary

Our basic manifold comes with local coordinates, x^μ, in terms of which we choose to describe functions, vectors, covectors and tensors. We introduce some *extra structure* by focusing on manifolds which are equipped with a metric. A metric is a special tensor that allows us to compute norms of vectors. In a chart, we denote the components of the metric as $g_{\mu\nu}(x)$, and can treat it as a symmetric invertible matrix. The norm of a vector X is $g_{\mu\nu}X^\mu X^\nu$. Spacetime will be described by a manifold with a metric of Lorentzian signature: if we choose an orthonormal basis for tangent vectors, the diagonal form of the metric will be equal to the Minkowski metric, with one -1 and three $+1$s on the diagonal. This means that we can talk of vectors, and curves via their tangent vectors, as being null, timelike and spacelike, and define lightcones at each point.

We write the metric as $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, and can integrate this infinitesimal line element to compute the lengths of curves in the manifold. Geodesics are the special class of curves which extremise the proper time (or distance) between two points, and the equation for a geodesic was:

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma_{\nu\rho}{}^\mu \frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,, \tag{4.17}$$

where the Christoffel symbols $\Gamma_{\nu\rho}{}^\mu$ were:

$$\Gamma_{\nu\rho}{}^\mu = \frac{1}{2}g^{\mu\lambda}\left(\partial_\nu g_{\rho\lambda} + \partial_\rho g_{\nu\lambda} - \partial_\lambda g_{\nu\rho}\right)\,. \tag{4.18}$$

These are a) not components of a tensor, and b) very important!

5 Covariant derivatives

By introducing coordinate charts, and maps between coordinate charts, we found a way to translate the geometry of a curved manifold into a description based on coordinates, and functions, defined on $\mathbb{R}^n$. This allowed us to start doing calculus on manifolds. We have seen for instance that the partial derivatives of a function defined a covector field. Under a change of coordinates,

$$(df)'_\mu = \frac{\partial f}{\partial x'^\mu} = \frac{\partial x^\nu}{\partial x'^\mu}\frac{\partial f}{\partial x^\nu} = \frac{\partial x^\nu}{\partial x'^\mu}(df)_\nu \,, \tag{5.1}$$

which confirms that the gradient transforms as a covector should.

However, if we try to take partial derivatives of vectors, covectors and higher rank tensors, we run into trouble. Consider the partial derivative of a vector. Under a change of coordinates,

$$\frac{\partial V'^\mu}{\partial x'^\nu} = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial}{\partial x^\rho}\left(\frac{\partial x'^\mu}{\partial x^\sigma}V^\sigma\right) = \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial x'^\mu}{\partial x^\sigma}\frac{\partial V^\sigma}{\partial x^\rho} + \frac{\partial x^\rho}{\partial x'^\nu}\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\rho}V^\sigma \,. \tag{5.2}$$

The first term here is the usual tensorial transformation rule, but the presence of the second term means that the partial derivative of a vector does *not* transform like a tensor.

Exercise 5.1 (Problems with partial derivatives)

Show as well that the partial derivative of a covector, $\partial_\mu \omega_\nu$, does not transform as a tensor would under a charge of coordinates $x^\mu \mapsto x'^\mu(x)$. Viewing the covector as a 1-form, show that exterior derivative, $(d\omega)_{\mu\nu} = 2\partial_{[\mu}\omega_{\nu]}$ does transform as a tensor.

Another conceptual issue is that vectors and tensors really live in different vector spaces at different points on the manifold. So how do we compare tensors at two (infinitesimally separated) points? To resolve this and the problem of defining well-behaved derivatives, we introduce some differential geometric technology.

5.1 Covariant derivatives

Differentiation of tensors on a manifold can be defined using a *covariant derivative* or *connection*, which as the name suggests is a derivative operator which transforms covariantly, i.e. like a tensor, and which can be used to connect and hence compare tensor at different points in the manifold.

Let's start by thinking about functions f and vector fields Y. We know that the partial derivative of a function *is* a covector, but the partial derivative of a vector is a disaster. The goal then is to find a derivative operator, which we denote by ∇, which in coordinates replaces the partial derivative. This will be our covariant derivative.

How should this derivative act? Given that the gradient df of a function f is a well-defined covector, we suppose that acting on functions, we do not need to make any modifications, i.e. we define the covariant derivative of a function f by

$$\nabla f \equiv df \,. \tag{5.3}$$

There is nothing more to be said here; this is part of our definition. Now we turn to vectors. What we want is ∇Y to be a $(1,1)$ tensor (based on the fact that the partial derivative has one lower index, or if you like by noting our definition on functions took a $(0,0)$ tensor to $(0,1)$ tensor). Schematically, in a coordinate chart we will have $\nabla = \partial + \Gamma$, where Γ denotes a non-tensorial quantity (the connection components) which makes ∇ into a covariant operator. We would then further like the covariant derivative to have the same properties as a partial derivative:

- linearity acting on sums $Y+Z$, $\nabla(Y+Z) = \nabla(Y) + \nabla(Z)$,
- the Leibniz rule when acting on products of Y with functions, $\nabla(fY) = (\nabla f)Y + f\nabla Y$.

Now, if ∇Y is a $(1,1)$ tensor then by acting on any covector field ω and vector field X we get a function on the manifold, $\nabla Y(\omega, X)$. Alternatively, we can view a $(1,1)$ tensor as allowing us to define a map which takes a vector field and gives back another vector field, by "leaving out" the covector ω from the argument. Therefore given the $(1,1)$ tensor we define a vector field called $\nabla_X Y$ by

$$\nabla_X Y \equiv \nabla Y(\cdot, X)\,, \tag{5.4}$$

i.e. it is defined such that acting on an arbitrary covector ω, we have $\nabla_X Y(\omega) = \nabla Y(\omega, X)$. We call $\nabla_X Y$ the "covariant derivative of Y with respect to X", and it must obey the linearity property $\nabla_{fX+gZ} Y = f\nabla_X Y + g\nabla_Z Y$ if ∇Y is indeed a tensor.

In effect, we are defining here two closely related geometric objects. The first, and most natural to motivate, is the covariant derivative ∇Y which is a recipe for constructing a tensor in a derivative-like fashion, given *one* vector field. The second is what we call the connection, and is how we normally define ∇ itself. This is obtained by looking at ∇ as a way to take *two* vector fields, X and Y, and produce another vector field, $\nabla_X Y$, which can be interpreted as the covariant derivative of one with respect to the other. This second map does *not* define a tensor, because it is not linear under $Y \mapsto fY$, owing to the Leibniz rule. Let's now collect the formal definitions, in which it is more natural to first define the connection.

Connection

A *connection* is a map ∇ sending every pair of smooth vector fields X, Y to the smooth vector field $\nabla_X Y$, such that for arbitrary vector fields X, Y, Z and functions f, g, we have:

1. linearity conditions:
$$\nabla_{fX+gY} Z = f\nabla_X Z + g\nabla_Y Z\,, \quad \nabla_X(Y+Z) = \nabla_X Y + \nabla_X Z \tag{5.5}$$
2. the Leibniz property:
$$\nabla_X(fY) = (\nabla_X f)Y + f\nabla_X Y\,, \tag{5.6}$$
3. and the definition on functions
$$\nabla_X f = X(f)\,. \tag{5.7}$$

Covariant derivative

The *covariant derivative* of a vector field Y is the $(1,1)$ tensor ∇Y defined by

$$\nabla Y : (\omega, X) \mapsto (\nabla_X Y)(\omega) \tag{5.8}$$

for arbitrary vector field X and covector ω.

We can extend our covariant derivative to act on covectors and then arbitrary tensors using the Leibniz property. For instance, given a covector η then $\nabla\eta$ is a $(0,2)$ tensor defined by

$$(\nabla\eta)(X,Y) \equiv (\nabla_X\eta)(Y) = \nabla_X(\eta(Y)) - \eta(\nabla_X(Y)) = X((\eta(Y)) - \eta(\nabla_X(Y))\,. \tag{5.9}$$

Let's begin to demystify the somewhat abstract definition above. Suppose in some chart we introduce a set of vector fields $\{e_{(\mu)}\}$ which are linearly independent and nowhere vanishing, i.e. which gave a basis for the tangent space at each point. We define the action of the connection on the basis vector fields:

$$\nabla_\mu e_{(\nu)} \equiv \nabla_{e_{(\mu)}} e_{(\nu)} = \Gamma_{\mu\nu}{}^{\rho} e_\rho\,. \tag{5.10}$$

Note this is the *definition* of some collection of (non-tensorial) objects $\Gamma_{\mu\nu}{}^{\rho}$. Then for two vector fields $X = X^\mu e_{(\mu)}$ and $Y = Y^\mu e_{(\mu)}$ we find using the Leibniz property

$$\begin{aligned} \nabla_X Y &= \nabla_{X^\mu e_{(\mu)}}(Y^\nu e_{(\nu)}) \\ &= X^\mu \left(\nabla_{e_{(\mu)}}(Y^\nu)e_{(\nu)} + Y^\nu \nabla_{e_{(\mu)}} e_{(\nu)}\right) \\ &= X^\mu \left(e_{(\mu)}(Y^\nu)e_{(\nu)} + Y^\nu \Gamma_{\mu\nu}{}^{\rho} e_{(\rho)}\right) \\ &= X^\mu \left(e_{(\mu)}(Y^\nu) + \Gamma_{\mu\rho}{}^{\nu} Y^\rho\right) e_{(\nu)} \end{aligned} \tag{5.11}$$

hence

$$(\nabla_X Y)^\mu = X^\nu \left(e_{(\nu)}(Y^\mu) + \Gamma_{\nu\rho}{}^{\mu} Y^\rho\right)\,. \tag{5.12}$$

If you understand that $e_\mu(Y^\nu)$ means the action of the basis vector $e_{(\mu)}$ on the components Y^μ, which are themselves functions on the manifold, then you have grasped the essentials of dealing with arbitrary bases.

Exercise 5.2 (Transformation of a connection)

Show that under a change of basis, $e'_{(\mu)} = (A^{-1})^\nu{}_\mu e_{(\nu)}$ we have

$$\Gamma'_{\nu\rho}{}^{\mu} = A^\mu{}_\kappa (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma_{\lambda\sigma}{}^{\kappa} + A^\mu{}_\kappa (A^{-1})^\sigma{}_\nu e_{(\sigma)}((A^{-1})^\kappa{}_\rho)\,. \tag{5.13}$$

This is very much not the transformation of a tensor!

Let's now specialise to a coordinate basis, $e_{(\mu)} = \partial/\partial x^\mu$. We view the covariant derivative as a derivative operator ∇_μ such that on functions we have

$$\nabla_\mu f = \partial_\mu f\,, \tag{5.14}$$

on vectors we have,

$$(\nabla_\mu X)^\nu = \partial_\mu X^\nu + \Gamma_{\mu\rho}{}^\nu X^\rho \,. \tag{5.15}$$

By slight abuse of notation we will usually write $\nabla_\mu X^\nu$ in place of $(\nabla_\mu X)^\nu$.

The transformation rule of the connection components $\Gamma_{\mu\nu}{}^\rho$ under a change of coordinate basis is:

$$\Gamma'_{\nu\rho}{}^\mu = \left(\frac{\partial x'^\mu}{\partial x^\kappa}\right)\left(\frac{\partial x^\lambda}{\partial x'^\nu}\right)\left(\frac{\partial x^\sigma}{\partial x'^\rho}\right)\Gamma_{\lambda\sigma}{}^\kappa - \left(\frac{\partial x^\sigma}{\partial x'^\rho}\right)\left(\frac{\partial x^\kappa}{\partial x'^\nu}\right)\left(\frac{\partial^2 x'^\mu}{\partial x^\sigma \partial x^\kappa}\right) . \tag{5.16}$$

Exercise 5.3 (Covariant derivatives in a coordinate basis)

1. Show explicitly that $\nabla_\mu V^\nu$ transforms as a tensor under a change of coordinates.
2. Show using the Leibniz rule on $\nabla_\mu(\omega_\nu X^\nu)$ that the covariant derivative on covectors is

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma_{\mu\nu}{}^\rho \omega_\rho \,. \tag{5.17}$$

   Hence extrapolate to a general tensor:

$$\begin{aligned}\nabla_\mu T^{\nu_1\ldots\nu_r}{}_{\rho_1\ldots\rho_s} =& \partial_\mu T^{\nu_1\ldots\nu_r}{}_{\rho_1\ldots\rho_s} \\ &+ \Gamma_{\mu\sigma}{}^{\nu_1} T^{\sigma\ldots\nu_r}{}_{\rho_1\ldots\rho_s} + \ldots + \Gamma_{\mu\sigma}{}^{\nu_r} T^{\nu_1\ldots\sigma}{}_{\rho_1\ldots\rho_s} \\ &- \Gamma_{\mu\rho_1}{}^\sigma T^{\nu_1\ldots\nu_r}{}_{\sigma\ldots\rho_s} - \ldots - \Gamma_{\mu\rho_s}{}^\sigma T^{\nu_1\ldots\nu_r}{}_{\rho_1\ldots\sigma} \,.\end{aligned} \tag{5.18}$$

Unlike partial derivatives, the action of covariant derivatives does not commute: $\nabla_\mu\nabla_\nu \neq \nabla_\nu\nabla_\mu$. However, the commutator of two covariant derivatives acting on a tensor will again be a tensor, and will in fact lead to quantities of geometric interest.

The simplest example is to consider a function f. A short calculation shows that

$$\nabla_\mu\nabla_\nu f - \nabla_\nu\nabla_\mu f = -2\Gamma_{[\mu\nu]}{}^\rho \nabla_\rho f \,. \tag{5.19}$$

This antisymmetrisation of the connection coefficients defines a tensor, known as the torsion tensor of the connection.

Torsion

In a coordinate basis, the components of the torsion tensor associated to a connection are:

$$T_{\mu\nu}{}^\rho \equiv 2\Gamma_{[\mu\nu]}{}^\rho \,. \tag{5.20}$$

More generally, we define the torsion T such that for arbitrary vector fields X, Y, $T(X,Y)$ is the vector field given by:

$$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] \,. \tag{5.21}$$

Then the torsion defines a $(1,2)$ tensor such that for η a covector and X, Y vectors, $T(\eta, X, Y) \equiv T(X,Y)(\eta)$.

Torsion-free connection

A connection is said to be *torsion-free* if its associated torsion tensor vanishes.

We have defined connections, and written down how they transform, but not said anything about how to construct them in practice. We are interested in manifolds with metric. In this case, the existence of the metric implies that there is a preferred connection. This is the *Levi-Civita connection*.

Levi-Civita connection

On a manifold M with metric g, there exists a unique torsion-free connection ∇ such that the metric is covariantly constant, $\nabla g = 0$.

In general, a connection which annihilates the metric, $\nabla_\mu g_{\nu\rho} = 0$, is said to be *metric-compatible*, or just metric (as in, a metric connection obeys $\nabla g = 0$).

We will demonstrate the construction of the Levi-Civita connection in a coordinate basis.

Construction of the Levi-Civita connection: Firstly, suppose that there is a torsion-free compatible connection with connection components $\Gamma_{\mu\nu}{}^{\rho}$. As it is metric compatible, we have:

$$0 = \nabla_\mu g_{\nu\rho} = \partial_\mu g_{\nu\rho} - \Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\rho} - \Gamma_{\mu\rho}{}^{\sigma} g_{\sigma\nu}\,, \tag{5.22}$$

$$0 = \nabla_\nu g_{\rho\mu} = \partial_\nu g_{\rho\mu} - \Gamma_{\nu\rho}{}^{\sigma} g_{\sigma\mu} - \Gamma_{\nu\mu}{}^{\sigma} g_{\sigma\rho}\,, \tag{5.23}$$

$$0 = \nabla_\rho g_{\mu\nu} = \partial_\rho g_{\mu\nu} - \Gamma_{\rho\mu}{}^{\sigma} g_{\sigma\nu} - \Gamma_{\rho\nu}{}^{\sigma} g_{\sigma\mu}\,, \tag{5.24}$$

Taking the combination (5.22) + (5.23) - (5.24) and using $\Gamma_{\mu\nu}{}^{\rho} = \Gamma_{\nu\mu}{}^{\rho}$ (as it is torsion-free), we find

$$0 = \partial_\mu g_{\nu\rho} + \partial_\nu g_{\rho\mu} - \partial_\rho g_{\mu\nu} - 2\Gamma_{\mu\nu}{}^{\sigma} g_{\sigma\rho}\,, \tag{5.25}$$

hence

$$\Gamma_{\mu\nu}{}^{\rho} = \frac{1}{2} g^{\rho\sigma} \left(\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\mu\sigma} - \partial_\sigma g_{\mu\nu}\right)\,. \tag{5.26}$$

This defines the components of the Levi-Civita connection, and shows that it is uniquely determined in terms of the metric. □

Exercise 5.4 (Transformation of the Levi-Civita connection)

To be sure that this is indeed a valid connection, check that (5.26) transforms under coordinate transformations as in (5.16).

The connection components (5.26) are the very same as the Christoffel symbols (4.16) that we found in the equation determining the paths of extremal proper time. Let's now show how having a connection allows one an alternative definition of such paths, which go by the general name of geodesics.

5.2 Parallel transport and geodesics again

We said earlier some vague words about a connection allowing us to compare the value of tensors at different points on the manifold. The idea that makes this work in practice is known as *parallel transport*.

What we want to do is to consider taking a vector or tensor at a point p, and transport it along some curve starting at p to another point q. This is trivial to do in flat space: we just keep the components of the tensor fixed, i.e. they obey $\partial_\mu T^{\mu_1 \cdots \mu_r}{}_{\nu_1 \ldots \nu_s} = 0$. To generalise this to an arbitrary manifold with connection, we will replace the partial derivative with a covariant one.

Parallel transport

Given a curve γ with tangent vector X, the parallel transport of a tensor T along γ is defined by the solution to the equation $\nabla_X T = 0$.

A tensor is therefore parallel transported along a path if it is *covariantly constant* along a path. In coordinates, if the curve is $x^\mu(\lambda)$, the tangent vector is then $X^\mu = \frac{dx^\mu}{d\lambda}$, and we define the directional covariant derivative

$$\frac{D}{d\lambda} \equiv \frac{dx^\mu}{d\lambda} \nabla_\mu \,, \tag{5.27}$$

we have

$$\frac{D}{d\lambda} T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = \frac{dx^\mu}{d\lambda} \nabla_\mu T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = 0 \,. \tag{5.28}$$

This is a first-order ODE with as usual a unique local solution once we specify the initial value of the tensor components at some point on the curve.

A hugely important caveat is that parallel transport is *path dependent*: if we parallel transport a tensor from p to q by two separate paths, we need not obtain the same result at q. In fact the result depends on the *curvature* of the manifold, as we will shortly see.

Parallel transport is connection dependent. If we use a metric-compatible connection, the metric is always parallel transported with respect to it. Similarly, the norm $g(X,Y)$ of two vectors is preserved if we parallel transport X and Y between two points.

The condition for curves to extremise the proper time was found to be (4.15), which we repeat:

$$\frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\nu\rho}{}^\mu \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} = 0 \,, \tag{5.29}$$

where the connection appearing is the Levi-Civita one. This is equivalent to:

$$\frac{D}{d\tau} \frac{dx^\mu}{d\tau} = 0 \,, \tag{5.30}$$

i.e. that the tangent vector to the curve is parallel transported along it, using the Levi-Civita connection. This is an example of a *geodesic*.

Geodesic

An affinely parametrised geodesic is a curve along which its own tangent vector is parallel transported (with respect to a particular connection ∇). Equivalently, given a vector field X, an affinely parametrised geodesic is an integral curve of X such that $\nabla_X X = 0$.

By *affinely parametrised* we mean that the geodesic equation is of the form $\nabla_X X = 0$ or

$$\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} = 0\,, \tag{5.31}$$

while for a non-affine parameter we would have $\nabla_X X \propto X$. This follows by considering choosing some other parameter $\alpha(\lambda)$. A short calculation gives the geodesic equation for the tangent vector $\frac{dx^\mu}{d\alpha}$,

$$\begin{aligned}\frac{D}{d\alpha}\frac{dx^\mu}{d\alpha} &= \frac{d}{d\alpha}\left(\frac{d\lambda}{d\alpha}\frac{dx^\mu}{d\lambda}\right) + \Gamma_{\nu\rho}{}^{\alpha}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\left(\frac{d\lambda}{d\alpha}\right)^2 \\ &= \left(\frac{d\lambda}{d\alpha}\right)^2\left(\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}{}^{\alpha}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda}\right) + \frac{dx^\mu}{d\lambda}\frac{d}{d\alpha}\frac{d\lambda}{d\alpha} \\ &= \left(\frac{d\lambda}{d\alpha}\right)^2\frac{D}{d\lambda}\frac{dx^\mu}{d\lambda} - \frac{dx^\mu}{d\alpha}\left(\frac{d^2\alpha}{d\lambda^2}\right)\left(\frac{d\alpha}{d\lambda}\right)^{-2}\end{aligned} \tag{5.32}$$

having used $\frac{d}{d\alpha}\frac{d\lambda}{d\alpha} = \frac{d\lambda}{d\alpha}\frac{d}{d\lambda}\left(\frac{d\alpha}{d\lambda}\right)^{-1}$. Hence one has

$$\frac{D}{d\alpha}\frac{dx^\mu}{d\alpha} = f(\alpha)\frac{dx^\mu}{d\alpha} \tag{5.33}$$

with $f(\alpha) = -\left(\frac{d^2\alpha}{d\lambda^2}\right)\left(\frac{d\alpha}{d\lambda}\right)^{-2}$. There is then a two-parameter freedom of affine parameter, with $\alpha = a\lambda + b$ also affine, for a, b constant.

For timelike curves parametrised using proper time τ, the normalisation $g(X, X) = -1$ fixes $a = 1$. In this case, we can write the geodesic equation equivalently in terms of the four-velocity $U^\mu = \frac{dx^\mu}{d\tau}$ or the four-momentum $p^\mu = mU^\mu$ as

$$U^\nu\nabla_\nu U^\mu = 0 = p^\nu\nabla_\nu p^\mu\,. \tag{5.34}$$

For null curves, we have the full two parameter ambiguity. It is convenient to choose the normalisation of the affine parameter on a null geodesic such that the momentum four-vector is $p^\mu = \frac{dx^\mu}{d\lambda}$ (note there is of course no rest mass in this case). Then for instance an observer with four-velocity U^μ will measure the (intrinsic) energy of the massless particle to be $E = -p_\mu U^\mu$.

In summary, we have uncovered a very important result. Affinely parameterised geodesics obey the equation:

$$\frac{d^2x^\mu}{d\lambda^2} + \Gamma_{\nu\rho}{}^{\mu}\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = 0\,, \tag{5.35}$$

where $\Gamma_{\mu\nu}{}^{\rho}$ are the components of a particular connection. A further special class of geodesics are those which extremise the length functionals

$$S = \int d\lambda\sqrt{|g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu|}\,, \tag{5.36}$$

which are the geodesics obeying (5.35) for the particular choice of the Levi-Civita connection.

Free particles in general relativity

Free particles move on geodesics of the Levi-Civita connection: null geodesics for massless particles and timelike geodesics for massive ones.

The equation (5.35) can also be derived as the Euler-Lagrange equation following from the action

$$S' = \int d\lambda \frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu \, . \tag{5.37}$$

For a given metric, varying S' is perhaps the most direct way to obtain the components of the Levi-Civita connection.

5.3 Local inertial frames

We're now going to connect our discussion of the mathematical description of Lorentzian manifolds back to the physical ideas embodied in the Einstein Equivalence Principle. This claimed that we can find local inertial frames, in which physics behaved exactly as you would expect it to in a genuine inertial frame in Minkowski spacetime. This means that we should be able to construct local coordinates on our manifold in which the metric becomes the Minkowski metric, and the Christoffel symbols vanish.

The coordinates that we want are known as Riemann normal coordinates. To construct these, we first choose an orthonormal basis for the tangent space at a particular point $p \in M$. With this basis, we can define geodesics whose tangent vectors at p are the elements of this basis. We can then define coordinates in the neighbourhood of p according to their parameter distance along these geodesics.

Exponential map

Given a point $p \in M$, the *exponential map* from T_pM to M is defined as the map which sends $X_p \in T_pM$ to the point unit affine parameter distance along the geodesic through p with tangent X_p at p.

This map is locally one-to-one and onto, in a small enough neighbourhood around p. The existence and (local) uniqueness of the exponential map is guaranteed by the usual ODE argument: we are solving the geodesic equation for a curve $x^\mu(\lambda)$, with initial conditions $x^\mu(0) = x^\mu_p$, $\frac{dx^\mu}{d\lambda}(\lambda = 0) = X^\mu_p$.

We now construct the desired coordinates.

Riemann normal coordinates

Given $p \in M$ and a basis $\{e_{(\mu)}\}$ for T_pM, then we define *normal coordinates* in a neighbourhood of p by saying the coordinates of q near p are given by the components of the tangent vector $X^\mu \in T_pM$ which maps to q under the exponential map from T_pM to the neighbourhood of p.

To recapitulate: given $X^\mu e_{(\mu)} \in T_pM$, we construct the geodesic through p whose tangent vector is $X^\mu e_{(\mu)}$. At unit affine parameter distance along this geodesic, we come to the point q. Then we say the coordinates of $q \in M$ in this coordinate system are defined to be $x^\mu(q) = X^\mu$. Note that $x^\mu(p) = 0$ by definition.

From the definition of the exponential map, it follows if X is mapped a unit affine parameter distance along the geodesic through p with tangent X, then tX gets mapped to the point on the same geodesic which is at affine parameter distance t. To see this, consider the curve $x^\mu(\lambda)$ such that $x^\mu(\lambda = 1)$ is the point q. Rescale the parameter λ by t to obtain a curve $x^\mu(t\lambda)$ such that at $\lambda = 1$ it corresponds to the point at affine parameter distance t along the original curve. Then at $\lambda = 0$, $\frac{d}{d\lambda}x^\mu(t\lambda)|_{\lambda=0} = tX^\mu$.

Therefore in normal coordinates, the geodesic itself is parameterised simply as $x^\mu(\lambda) = \lambda X^\mu$. Inserting this into (5.35), we have

$$0 = \frac{d^2x}{d\lambda^2} + \Gamma_{\nu\rho}{}^\mu(x(\lambda))\frac{dx^\nu}{d\lambda}\frac{dx^\rho}{d\lambda} = \Gamma_{\nu\rho}{}^\mu(x(\lambda))X^\nu X^\rho\,. \tag{5.38}$$

Evaluating at $\lambda = 0$ we find $\Gamma_{\nu\rho}{}^\mu(p)X^\nu X^\rho = 0$ which is true for arbitrary X^μ and from which we can conclude that

$$\Gamma_{(\mu\nu)}{}^\rho(p) = 0\,. \tag{5.39}$$

For a torsion-free connection, we can conclude that

$$\Gamma_{\mu\nu}{}^\rho(p) = 0\,, \tag{5.40}$$

so in normal coordinates at p, the components of a torsion-free connection vanish at p. Given this, we also know from the metric compatibility condition $\nabla_\mu g_{\nu\rho} = 0$ evaluated at p that the first derivatives of the metric are zero at p:

$$\partial_\mu g_{\nu\rho}(p) = 0\,. \tag{5.41}$$

We also should check that the basis $\{e_{(\mu)}\}$ coincides with the coordinate basis $\partial/\partial X^\mu$. We note that the integral curve of the vector field $\partial/\partial X^\mu$ through p (i.e. the curve whose tangent vector at each point is $\partial/\partial X^\mu$) is given by $X^\mu(\lambda) = (0, \ldots, 0, \lambda, 0, \ldots, 0)$ and coincides with the geodesic through p with tangent vector $e_{(\mu)}$ at p. Therefore we can identify $e_{(\mu)} = \partial/\partial X^\mu$.

This discussion is valid for any basis $\{e_{(\mu)}\}$ at p. Therefore we can choose an orthonormal basis, so that the metric at p becomes the Minkowski metric: $g(e_{(\mu)}, e_{(\nu)}) = \eta_{\mu\nu}$. This provides us with a mathematical definition of a *local inertial frame*.

Local inertial frame

On a Lorentzian manifold, a *local inertial frame* is a choice of normal coordinates at any point p, such that

$$g_{\mu\nu}(p) = \eta_{\mu\nu}\,, \quad \partial_\mu g_{\nu\rho}(p) = 0\,. \tag{5.42}$$

A summary

To differentiate tensors, we use not partial derivatives but covariant derivatives, defined by

$$\nabla_\mu f = \partial_\mu f\,, \quad \nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma_{\mu\rho}{}^\nu V^\rho\,, \tag{5.43}$$

on functions and vectors, and extended to arbitrary tensors using the Leibniz rule. The *connection components* $\Gamma_{\mu\nu}{}^\rho$ are not the components of a tensor, but transform in a particular way so as to cancel out the non-tensorial part of the transformation of the partial derivative term. A covariant derivative, or a connection, provides an extra structure on a manifold. In particular, on a manifold with a metric, there is a unique covariant derivative annihilating the metric, $\nabla_\mu g_{\nu\rho} = 0$, which is *torsion-free* meaning $\Gamma_{[\mu\nu]}{}^\rho = 0$. This unique covariant derivative is known as the Levi-Civita connection.

A covariant derivative gives a way to *parallel transport* tensors along curves; if a curve $x^\mu(\lambda)$ has tangent vector $X^\mu = \frac{dx^\mu}{d\lambda}$ starting with a tensor T at some point on the curve, we parallel transport it along the curve by solving the requirement of covariant constancy $\nabla_X T = 0$. Parallel transport between two points depends on the precise choice of path between the points (and also on the connection used).

A special class of curves are *geodesics*, defined by the condition that the tangent vector X to the curve is itself parallel transported along the curve $\nabla_X X = 0$. A special class of geodesics are the geodesics of the Levi-Civita connection, which extremise the proper distance (or time) between two points. The components of the Levi-Civita connection are exactly the Christoffel symbols we found in the previous section by finding the condition for this extremisation. In general relativity, free particles move on this class of geodesics.

Geodesics can be used to locally map the structure of the tangent space at a point p to a set of special coordinates in a neighbourhood of p. These are *normal coordinates*, and are such that at p the metric evaluates to the Minkowski metric, and its first derivatives vanish. This realises the idea of a *local inertial frame*: physical laws evaluated at p will agree with those of special relativity.

6 Curvature

6.1 Riemann curvature tensor

We are now going to combine two ideas that were mentioned in the last section: covariant derivatives do not commute, and parallel transport is path dependent. These lead to the appearance of the *Riemann curvature tensor*, which is a local measure of the curvature of a manifold.

Riemann tensor from curvature

We previously calculated the commutator $[\nabla_\mu, \nabla_\nu]f = -T_{\mu\nu}{}^\rho \nabla_\rho f$, where f was a function, finding the torsion tensor $T_{\mu\nu}{}^\rho = 2\Gamma_{[\mu\nu]}{}^\rho$ of the connection used in the covariant derivative. Now let's repeat this calculation on tensors. We have:

$$[\nabla_\mu, \nabla_\nu]V^\rho = R^\rho{}_{\sigma\mu\nu}V^\sigma - T_{\mu\nu}{}^\sigma \nabla_\sigma V^\rho \,, \tag{6.1}$$

where

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}{}^\rho - \partial_\nu \Gamma_{\mu\sigma}{}^\rho + \Gamma_{\mu\lambda}{}^\rho \Gamma_{\nu\sigma}{}^\lambda - \Gamma_{\nu\lambda}{}^\rho \Gamma_{\mu\sigma}{}^\lambda \,. \tag{6.2}$$

The left-hand side of (6.1) is tensorial, and so is the term involving the torsion tensor on the right-hand side. This means that the combination $R^\rho{}_{\sigma\mu\nu}$ must be the components of a tensor. This can indeed be checked, and defines the Riemann curvature tensor.

Exercise 6.1 (Riemann tensor)

1. Verify (6.1) and (6.2).

2. Hence or similarly show that

$$[\nabla_\mu, \nabla_\nu]V_\rho = -R^\lambda{}_{\rho\mu\nu}V_\lambda - T_{\mu\nu}{}^\lambda \nabla_\lambda V_\rho \,, \tag{6.3}$$

and

$$\begin{aligned}
[\nabla_\mu, \nabla_\nu]V^{\rho_1\ldots\rho_r}{}_{\sigma_1\ldots\sigma_s} = &- T_{\mu\nu}{}^\sigma \nabla_\sigma V^{\rho_1\ldots\rho_r}{}_{\sigma_1\ldots\sigma_s} \\
&+ R^{\rho_1}{}_{\lambda\mu\nu}V^{\lambda\ldots\rho_r}{}_{\sigma_1\ldots\sigma_s} + \cdots + R^{\rho_r}{}_{\lambda\mu\nu}V^{\rho_1\ldots\lambda}{}_{\sigma_1\ldots\sigma_s} \\
&- R^\lambda{}_{\sigma_1\mu\nu}V^{\rho_1\ldots\rho_r}{}_{\lambda\ldots\sigma_s} - \cdots - R^\lambda{}_{\sigma_s\mu\nu}V^{\rho_1\ldots\rho_r}{}_{\sigma_1\ldots\lambda} \,.
\end{aligned} \tag{6.4}$$

For completeness, let's give the more abstract definition of the Riemann tensor:

Riemann tensor

The Riemann curvature tensor of a connection ∇ is defined as a map R acting on three vector fields X, Y, Z to define another vector field given by

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]}Z \,. \tag{6.5}$$

The Riemann curvature tensor of the Levi-Civita connection is a function of the second derivatives of the metric. Clearly if our manifold is equipped with the Minkowski metric, the Riemann curvature tensor vanishes everywhere. In general:

- If a coordinate system exists in which the components of the metric are constant, then the Riemann tensor (of the Levi-Civita connection) vanishes.
- If the Riemann tensor vanishes, then we can find a coordinate system in which the metric components are constant.

An outline of the proof of these two statements is found in Carroll.

Parallel transport and curvature

Let's now work out the effect of parallel transporting a vector from one point to another along two separate paths. We will carry out parallel transport according to the picture shown in figure 14. Here X and Y are two commuting and linearly independent vector fields. We choose our coordinates to be $(s, t, \dots)$ with $X = \partial/\partial s$ and $Y = \partial/\partial t$.

We start at $p \in M$ which we take to have coordinates $(0, 0, \dots, 0)$. Then we will consider parallel transport along:

- the curve joining p to the point $q = (\delta s, 0, \dots, 0)$ with tangent X, then
- the curve joining q to the point $r = (\delta s, \delta t, \dots, 0)$ with tangent Y,

and compare it to parallel transport along:

- the curve joining p to the point $u = (0, \delta t, \dots, 0)$ with tangent Y, then
- the curve joining u to the point $r = (\delta s, \delta t, \dots, 0)$ with tangent X.

Here $\delta s, \delta t$ are small.

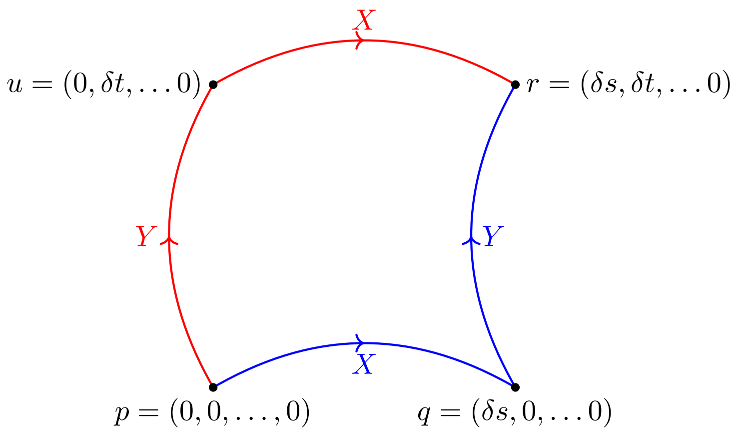


Figure 14: Parallel transport.

We will work in normal coordinates at p, and use indices $\mu, \nu, \dots$ to refer to this chart. Then we view s and t as parameters along the curves with tangent X and Y respectively.

Consider a vector $Z_p \in T_pM$. We will compute the parallel transport of Z_p along pqr to obtain a vector $Z_r \in T_rM$, and compare this to the parallel transport of Z_p along pur, which gives a vector $Z'_r \in T_rM$. We will assume we are using a torsion-free connection.

We start by computing Z_r, in two steps.

- First we parallel transport Z_p along the curve pq with tangent $X = \frac{\partial}{\partial s}$. The parallel transport equation $\nabla_X Z = 0$ is:

$$\frac{dZ^\mu}{ds} = -\Gamma_{\nu\rho}{}^{\mu} Z^\nu X^\rho \Rightarrow \frac{d^2 Z^\mu}{ds^2} = -\partial_\sigma(\Gamma_{\nu\rho}{}^{\mu} Z^\nu X^\rho) X^\sigma \,. \tag{6.6}$$

We use a Taylor expansion to write down

$$\begin{aligned} Z^\mu_q &= Z^\mu_p + \left(\frac{dZ^\mu}{ds}\right)_p \delta s + \frac{1}{2}\left(\frac{d^2Z^\mu}{ds^2}\right)_p \delta s^2 + \dots \\ &= Z^\mu_p - \frac{1}{2}\left(\partial_\sigma(\Gamma_{\nu\rho}{}^{\mu}) Z^\nu X^\rho X^\sigma\right)_p \delta s^2 \,, \end{aligned} \tag{6.7}$$

using the result $(\Gamma_{\mu\nu}{}^{\rho})_p = 0$ in normal coordinates for a torsion-free connection.

- Next we consider the result of parallel transporting Z_q along qr. We have similarly

$$\begin{aligned} Z^\mu_r &= Z^\mu_q + \left(\frac{dZ^\mu}{dt}\right)_q \delta t + \frac{1}{2}\left(\frac{d^2Z^\mu}{dt^2}\right)_q \delta t^2 + \dots \\ &= Z^\mu_q - (\Gamma_{\nu\rho}{}^{\mu} Z^\nu Y^\rho)_q \delta t - \frac{1}{2}\left(\partial_\sigma(\Gamma_{\nu\rho}{}^{\mu} Z^\nu Y^\rho) Y^\sigma\right)_q \delta t^2 \,. \end{aligned} \tag{6.8}$$

We then Taylor expand all the terms at q in δs, and throw away all terms of order δ^3. For instance, $(\Gamma_{\nu\rho}{}^{\mu} Z^\nu Y^\rho)_q \delta t = (\Gamma_{\nu\rho}{}^{\mu} Z^\nu Y^\rho)_p \delta t + (X^\sigma \partial_\sigma(\Gamma_{\nu\rho}{}^{\mu} Z^\nu Y^\rho))_p \delta s \delta t$. This leads to:

$$Z^\mu_r = Z^\mu_p - \left[\frac{1}{2}(\partial_\sigma \Gamma_{\nu\rho}{}^{\mu}) Z^\nu \left(X^\rho X^\sigma \delta s^2 + Y^\rho Y^\sigma \delta t^2 + 2 Y^\rho X^\sigma \delta t \delta s\right)\right]_p + O(\delta^3) \,. \tag{6.9}$$

We can immediately calculate Z'_r by swapping X and Y and s and t.

$$(Z'_r)^\mu = Z^\mu_p - \left[\frac{1}{2}(\partial_\sigma \Gamma_{\nu\rho}{}^{\mu}) Z^\nu \left(X^\rho X^\sigma \delta s^2 + Y^\rho Y^\sigma \delta t^2 + 2 X^\rho Y^\sigma \delta t \delta s\right)\right]_p + O(\delta^3) \,. \tag{6.10}$$

As a result,

$$\begin{aligned} \Delta Z^\mu_r \equiv Z'^\mu_r - Z^\mu_r &= \left[\partial_\sigma \Gamma_{\nu\rho}{}^{\mu} Z^\nu (Y^\rho X^\sigma - X^\rho Y^\sigma)\right]_p \delta s \delta t + O(\delta^3) \\ &= \left[(\partial_\rho \Gamma_{\sigma\nu}{}^{\mu} - \partial_\sigma \Gamma_{\rho\nu}{}^{\mu}) Z^\nu X^\rho Y^\sigma\right]_p \delta s \delta t + O(\delta^3) \end{aligned} \tag{6.11}$$

Remembering we are using normal coordinates, we can identify here the components of the Riemann tensor:

$$\begin{aligned} \Delta Z^\mu_r &= \left[R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma\right]_p \delta s \delta t + O(\delta^3) \\ &= \left[R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma\right]_r \delta s \delta t + O(\delta^3) \end{aligned} \tag{6.12}$$

using the Taylor expansion to $O(\delta)$ to convert the tensorial result at p into one at r. This gives the tensorial result:

$$(R^\mu{}_{\nu\rho\sigma} Z^\nu X^\rho Y^\sigma)_r = \lim_{\delta \to 0} \frac{\Delta Z^\mu_r}{\delta s \delta t} \,. \tag{6.13}$$

Geodesic deviation and curvature

In flat space, geodesics are straight lines. In particular, this means that geodesics that are initially parallel will always remain so. This is the parallel postulate of Euclidean geometry.

In curved space, geodesics may cross, and there is no well-defined notion of what "parallel" should mean. However, we can study what happens to geodesics that are initially nearby.

Let $\gamma_s(t)$ denote a one-parameter family of geodesics with affine parameter t, i.e. for each s $\gamma_s(t)$ is a geodesic. This collection of curves defines a two-dimensional surface parametrised by both s and t. In a chart, we have coordinates $x^\mu(s,t)$, and can write the tangent vector T tangent to the geodesics as well as the *deviation vectors* S as:

$$T^\mu = \frac{\partial x^\mu}{\partial t}\,, \quad S^\mu = \frac{\partial x^\mu}{\partial s}\,. \tag{6.14}$$

Note that $x^\mu(s+\delta s, t) = x^\mu(s,t) + \delta s S^\mu(s,t) + O(\delta s^2)$; hence $\delta s S^\mu$ is the infinitesimal displacement from one geodesic to a neighbouring one. We define the "relative velocity" of geodesics by:

$$V^\mu = (\nabla_T S)^\mu = T^\nu \nabla_\nu S^\mu\,, \tag{6.15}$$

and the "relative acceleration" by:

$$A^\mu = (\nabla_T V)^\mu = T^\nu \nabla_\nu V^\mu\,. \tag{6.16}$$

We can pick our chart such that s and t themselves are our coordinates, hence $T = \partial/\partial t$, $S = \partial/\partial s$ and these commute, $[S,T] = 0$. This means that

$$S^\nu \nabla_\nu T = T^\nu \nabla_\nu S\,. \tag{6.17}$$

We assume we are using a torsion-free connection. Then, we can compute the quantity A^μ. Using (6.17), we write $V^\mu = T^\nu \nabla_\nu S^\mu = S^\nu \nabla_\nu T^\mu$ such that

$$\begin{aligned}(\nabla_T V)^\mu &= T^\nu \nabla_\nu (S^\rho \nabla_\rho T^\mu) \\ &= T^\nu \nabla_\nu S^\rho \nabla_\rho T^\mu + T^\nu S^\rho ([\nabla_\nu, \nabla_\rho] + \nabla_\rho \nabla_\nu) T^\mu \\ &= S^\nu \nabla_\nu T^\rho \nabla_\rho T^\mu + T^\nu S^\rho \nabla_\rho \nabla_\nu T^\mu + R^\mu{}_{\sigma\nu\rho} T^\nu T^\sigma S^\rho\,,\end{aligned} \tag{6.18}$$

but because $T^\nu \nabla_\nu T^\mu = 0$, the first term here is

$$S^\nu \nabla_\nu T^\rho \nabla_\rho T^\mu = \nabla_\nu (S^\nu T^\rho \nabla_\rho T^\mu) - \nabla_\nu S^\nu T^\rho \nabla_\rho T^\mu - S^\nu T^\rho \nabla_\nu \nabla_\rho T^\mu = -S^\nu T^\rho \nabla_\nu \nabla_\rho T^\mu \tag{6.19}$$

which cancels with the second term. Hence we find the *geodesic deviation equation*:

$$A^\mu \equiv \nabla_T \nabla_T S^\mu \equiv \frac{D^2}{dt^2} S^\mu = R^\mu{}_{\nu\rho\sigma} T^\nu T^\rho S^\sigma\,. \tag{6.20}$$

Thus, the Riemann curvature tensor measures how nearby geodesics which are initially close begin to move apart or together. Later on, we will use this equation to describe how the Riemann curvature tensor due to a gravitational wave affects observers at nearby points.

6.2 Properties of the Riemann curvature tensor

The definition of the Riemann tensors implies that it is antisymmetric in its final two indices:

$$R^{\mu}{}_{\nu\rho\sigma} = -R^{\mu}{}_{\nu\sigma\rho} \tag{6.21}$$

The Riemann tensor of a torsion-free connection obeys:

$$R^{\mu}{}_{[\nu\rho\sigma]} = 0\,. \tag{6.22}$$

This follows automatically from antisymmetrising over the indices in (6.2). Note that

$$R^{\mu}{}_{[\nu\rho\sigma]} = \frac{1}{3}\left(R^{\mu}{}_{\nu\rho\sigma} + R^{\mu}{}_{\rho\sigma\nu} + R^{\mu}{}_{\sigma\nu\rho}\right) \tag{6.23}$$

by antisymmetry in the final two indices.

The Riemann tensor of a torsion-free connection also obeys the Bianchi identity:

$$\nabla_{[\mu}R^{\lambda}{}_{|\sigma|\nu\rho]} = 0\,. \tag{6.24}$$

(The bars indicate that the index σ is not to be antisymmetrised.) We can prove this using normal coordinates. At a point p, in normal coordinates we have, remembering that $\Gamma_{\mu\nu}{}^{\rho}(p) = 0$,

$$\nabla_{[\mu}R^{\lambda}{}_{|\sigma|\nu\rho]} = \partial_{[\mu}R^{\lambda}{}_{|\sigma|\nu\rho]} = 2\partial_{[\mu}\partial_{\nu}\Gamma_{\rho]\sigma}{}^{\lambda} = 0\,, \tag{6.25}$$

as partial derivatives commute. This means that we have $\nabla_{[\mu}R^{\lambda}{}_{|\sigma|\nu\rho]} = 0$ in normal coordinates at p. This is a tensorial equation, so it actually holds in any coordinates, and the point p was arbitrary, so in fact it is true everywhere.

For the Levi-Civita connection, the Riemann tensor has further symmetries. Let us define first

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda}R^{\lambda}{}_{\nu\rho\sigma}\,. \tag{6.26}$$

Then we have:

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \tag{6.27}$$

which implies

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma}\,. \tag{6.28}$$

To prove (6.27), we use Riemann normal coordinates at p. In these coordinates,

$$R_{\mu\nu\rho\sigma} = 2g_{\mu\lambda}\partial_{[\rho}\Gamma_{\sigma]\nu}{}^{\lambda} = \frac{1}{2}(\partial_{\nu}\partial_{\rho}g_{\mu\sigma} + \partial_{\mu}\partial_{\sigma}g_{\nu\rho} - \partial_{\nu}\partial_{\sigma}g_{\mu\rho})\,, \tag{6.29}$$

and (6.27) holds by inspection. Now we use the same argument as before, this is a tensorial equation, so if it holds in one basis it holds in any, and the point p used was arbitrary, so (6.27) is true everywhere.

The Riemann tensor and its symmetries

The Riemann tensor of a connection has components

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}{}^\rho - \partial_\nu \Gamma_{\mu\sigma}{}^\rho + \Gamma_{\mu\lambda}{}^\rho \Gamma_{\nu\sigma}{}^\lambda - \Gamma_{\nu\lambda}{}^\rho \Gamma_{\mu\sigma}{}^\lambda \,. \tag{6.30}$$

It obeys:

$$R^\rho{}_{\sigma\mu\nu} = -R^\rho{}_{\sigma\nu\mu} \quad \text{(by definition)}\,, \tag{6.31}$$

$$R^\rho{}_{[\mu\nu\sigma]} = 0 \quad \text{(if torsion-free)}\,, \tag{6.32}$$

$$\nabla_{[\mu} R^\lambda{}_{|\sigma|\nu\rho]} = 0 \quad \text{(if torsion-free)}\,, \tag{6.33}$$

$$R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu} \quad \text{(if Levi-Civita)}\,, \tag{6.34}$$

$$R_{\mu\nu\rho\sigma} = -R_{\nu\mu\rho\sigma} \quad \text{(if Levi-Civita)}\,. \tag{6.35}$$

Finally, let's count the number of independent components of the Riemann tensor of the Levi-Civita connection, on an n-dimensional manifold. We can view it as a tensor $R_{[\mu\nu][\rho\sigma]}$ which is symmetric in the antisymmetric pairs of indices $[\mu\nu]$ and $[\rho\sigma]$. Before taking into account other constraints, this gives

$$\frac{1}{2}\left(\frac{1}{2}n(n-1)\right)\left(\frac{1}{2}n(n-1)+1\right) = \frac{1}{8}(n^4 - 2n^3 + 3n^2 - 2n) \tag{6.36}$$

components. Next, consider the constraint $R_{\mu[\nu\rho\sigma]} = 0$. This implies automatically that $R_{[\mu\nu\rho\sigma]} = 0$. Alternatively, we can require $R_{[\mu\nu\rho\sigma]} = 0$, $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ and $R_{\mu\nu\rho\sigma} = -R_{\mu\nu\sigma\rho}$, which then implies $R_{\mu[\nu\rho\sigma]} = 0$. This is easier to work with. We decompose the Riemann tensor into a part which is totally antisymmetric and a part which is not: $R_{\mu\nu\rho\sigma} = R_{[\mu\nu\rho\sigma]} + X_{\mu\nu\rho\sigma}$ (where $X_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - R_{[\mu\nu\rho\sigma]}$ by definition). The totally antisymmetric part automatically obeys the constraints we have already counted, which therefore apply only to $X_{\mu\nu\rho\sigma}$. We then need to count only the vanishing of this totally antisymmetric part. This gives $\binom{n}{4}$ more constraints (an antisymmetric tensor with p n-dimensional indices has $\binom{n}{p}$ independent components). Subtracting these, we find a total of

$$\frac{1}{12}n^2(n^2-1) \tag{6.37}$$

components.

For $n = 4$, there are 20 components of the Riemann tensor. For comparison, for $n = 3$ there are 6, for $n = 2$ only 1, and $n = 1$ zero.

6.3 Ricci tensor, Ricci scalar and Einstein tensor

Other important tensors can be found by taking contractions of the Riemann tensor.

Ricci tensor

The components of the Ricci tensor, $R_{\mu\nu}$, are defined by:

$$R_{\mu\nu} = R^\rho{}_{\mu\rho\nu}\,. \tag{6.38}$$

Ricci scalar

The Ricci scalar is obtained as the trace of the Ricci tensor:

$$R = g^{\mu\nu} R_{\mu\nu} = R^{\mu}{}_{\mu} = R^{\mu\nu}{}_{\mu\nu} \,. \tag{6.39}$$

For the Levi-Civita connection, the Ricci tensor is automatically symmetric, $R_{\mu\nu} = R_{\nu\mu}$.

Take the Bianchi identity $\nabla_{[\mu} R^{\lambda}{}_{|\sigma|\nu\rho]}$ and contract λ and ρ and then with $g^{\nu\sigma}$. The result is the *contracted Bianchi identity*:

$$\nabla^{\nu} R_{\mu\nu} - \frac{1}{2} \nabla_{\mu} R = 0 \,. \tag{6.40}$$

Einstein tensor

The Einstein tensor is

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,. \tag{6.41}$$

It obeys $\nabla^{\nu} G_{\mu\nu} = 0$.

7 General Relativity

General Relativity (summarised by John Wheeler)

"Spacetime tells matter how to move; matter tells spacetime how to curve."

There isn't a better one sentence summary of a physical theory than that.

Indeed, it's almost a shame to have to convert this into equations. To start doing so, let's go back to the beginning (i.e. the 1660s). The Newtonian gravitational potential, Φ, obeys the equation

$$\nabla^2\Phi = 4\pi G\rho\,, \tag{7.1}$$

where ρ is the density of matter. A test particle subsequently experiences an acceleration due to gravity given by

$$\ddot{\vec{x}} = -\vec{\nabla}\Phi\,. \tag{7.2}$$

We need to explain how to describe the coupling of matter to gravity in general relativity. We have already postulated that free particles on a manifold obey the geodesic equation. We need to extend this discussion to describe the dynamics of all possible types of matter, and then to explain the laws determining the spacetime metric itself. We also need to explain exactly in what sense this generalises (7.1).

7.1 Matter

Special relativity

Let's start by reviewing the types of physics we can describe in special relativistic settings.

- **Individual particles.** Massive free particles with four-momentum $p^\mu = mU^\mu = m\frac{dx^\mu}{d\tau}$ obeyed $\frac{dp^\mu}{d\tau} = 0$, subject to $p^\mu p_\mu = -m^2$, while massless free-particles had null four-momenta, $p^\mu p_\mu = 0$.

- **Fields.** Fields obey particular equations of motion. For instance, a scalar field $\phi(x)$ obeys the Klein-Gordon equation:

  $$\partial_\mu\partial^\mu\phi = m^2\phi\,. \tag{7.3}$$

  The electromagnetic fields $F_{\mu\nu}$ in vacuum obey Maxwell's equations:

  $$\partial^\nu F_{\mu\nu} = 0\,, \quad 3\partial_{[\mu}F_{\nu\rho]} = 0\,. \tag{7.4}$$

- **Fluids.** We can also describe collections of particles as fluids. Then instead of using their individual four-momenta, we use macroscopic quantities such as density, pressure, viscosity, to describe the behaviour of the system as a whole. We can characterise a fluid as having some overall 4-velocity, U^μ. Its properties are further characterised by an energy-momentum tensor, $T^{\mu\nu}$. This encodes the flux of 4-momentum p^μ across surfaces of constant x^ν. For example, T^{00} gives the flux of energy across surfaces of constant time, and so gives the energy density ρ. The components $T^{0i} = T^{i0}$ encode momentum density. The purely spatial components T^{ij} encode the flow of the i^{th} component of momentum in the plane orthogonal to the j directions. The diagonal components T^{ii} (no sum) then give the force exerted per unit area in the i direction, which is just the pressure in the i direction. The off-diagonal components T^{ij} $(i \neq j)$ then describe stresses due to for example viscosity. The energy momentum-tensor is conserved:

$$\partial_\mu T^{\mu\nu} = 0\,. \tag{7.5}$$

This in fact encodes the usual equations of fluid mechanics, in the non-relativistic limit.

- **Perfect fluids.** An idealised example of fluid is the case of a perfect fluid, which is completely specified by its (rest frame) energy density ρ and (rest frame) isotropic (same in all directions) pressure p. Thus in the rest frame we have $T^{\mu\nu} = \mathrm{diag}\,(\rho, p, p, p)$. This can be made Lorentz covariant by writing it as:

$$T^{\mu\nu} = (\rho + p)U^\mu U^\nu + p\eta^{\mu\nu}\,, \tag{7.6}$$

  where U^μ is the four-velocity of the fluid. A perfect fluid is further characterised by an *equation of state* relating the pressure and energy density, $p = p(\rho)$. The three examples which will be relevant later on for cosmology are:

  - ***Dust (matter):*** $p = 0$. This describes a collection of particles at rest with respect to each other, characterised entirely by their common four-velocity U^μ and by the energy density, ρ, which for n particles of the same mass m is given by $\rho = mn$. This can be viewed as a model for galaxies in the universe.
  - ***Radiation:*** $p = \frac{1}{3}\rho$. This describes, for instance, a fluid of massless particles. This is relevant for modelling the very early universe, which was dominated by radiation before atoms began to form.
  - ***Vacuum energy:*** $p = -\rho$. In this case, $T^{\mu\nu} = -\rho\eta^{\mu\nu}$, and this describes the energy-momentum tensor due to the intrinsic energy of the vacuum of the universe. This is believed to be relevant to the universe in the far future, and is related to the cosmological constant.

In fact, we can obtain an energy-momentum tensor for any physical system. For theories described by a Lagrangian, the energy-momentum tensor can be obtained as the conserved quantity associated to spacetime translations (thought it may need to be "improved" by adding an identically conserved divergence to obtain a symmetric tensor). For example, in electromagnetism,

$$T^{\mu\nu} = \frac{1}{4\pi}\left(F_\rho{}^\mu F^{\rho\nu} - \frac{1}{4}F_{\rho\sigma}F^{\rho\sigma}\eta^{\mu\nu}\right)\,. \tag{7.7}$$

Minimal coupling

We have seen that in any Lorentzian manifold, we can choose to go to normal coordinates at a point p, where the metric becomes the Minkowski metric, and its first derivatives vanish. This defines a local inertial frame, in which the laws of physics should take their special relativistic form. This suggests a procedure known as *minimal coupling* in order to extrapolate the known special relativistic laws to equations that hold in general on an arbitrary spacetime manifold. The idea is to reverse what happens in going to normal coordinates.

Minimal coupling

The laws of physics in a Lorentzian manifold are obtained by taking the known equations valid in Minkowski spacetime, writing them in manifestly Lorentz covariant form, and making the substitutions:

$$\eta_{\mu\nu} \to g_{\mu\nu}\,, \quad \partial_\mu \to \nabla_\mu\,, \tag{7.8}$$

where ∇_μ is the Levi-Civita connection.

This procedure works! However it is not unambiguous. For instance, ordinary partial derivatives commute: $\partial_\mu \partial_\nu = \partial_\nu \partial_\mu$, but covariant derivatives do not: $\nabla_\mu \nabla_\nu = \nabla_\nu \nabla_\mu +$ terms involving the Riemann curvature. In going to Minkowski spacetime itself, the curvature is of course zero. Note though that the Riemann curvature does not vanish in normal coordinates at a point. Ultimately, the correct physical equations have to be determined by either mathematical consistency or by matching with experiment. However, minimal coupling works for all the cases we are interested in in this course.

Let's run through the sorts of physical quantities we used in special relativity: particles, fields and fluids. Free particles now move on geodesics of the curved manifold, and can be described by (timelike or null) vector fields U^μ which obey

$$U^\nu \nabla_\nu U^\mu = 0\,. \tag{7.9}$$

The scalar field equation (7.3) becomes

$$g^{\mu\nu} \nabla_\mu \nabla_\nu \Phi = m^2 \Phi\,, \tag{7.10}$$

and electromagnetism in a curved background obeys the equations:

$$g^{\rho\mu} \nabla_\rho F_{\mu\nu} = 0\,, \quad 3\nabla_{[\mu} F_{\nu\rho]} = 0\,. \tag{7.11}$$

The energy-momentum tensor of a perfect fluid is:

$$T^{\mu\nu} = (\rho + p) U^\mu U^\nu + p g^{\mu\nu}\,, \tag{7.12}$$

where now U^μ is a four-velocity vector field. The conservation equation of an energy-momentum tensor becomes:

$$\nabla_\mu T^{\mu\nu} = 0\,. \tag{7.13}$$

7.2 Gravity

The General Theory of Relativity

For the coupling to gravity, we will assume that we can always describe matter by an energy-momentum tensor obeying (7.13). The equation (7.1) for the Newtonian potential had on its right-hand side the matter density ρ, which in the relativistic setting is encoded by T^{00}. This motivates us to find an equation relating the properties of the curved manifold to the energy-momentum tensor. We are looking for some symmetric two-index tensor involving the metric or its derivatives, which we can equate to (some constant times) $T_{\mu\nu}$. Out of respect for Einstein's hard work in getting the wrong answer first and not giving up, it is usual at this point to guess $R_{\mu\nu} = \kappa T_{\mu\nu}$. However, this doesn't work because it requires $\nabla^\nu R_{\mu\nu} = 0$. By the contracted Bianchi identity this then implies that $\nabla_\mu R = 0$, in turn implying $\nabla^\mu T = 0$ (where $T = T^\mu{}_\mu$), so both the Ricci scalar and trace of the energy-momentum tensor have to be constant, which is far too restrictive.

Given that whatever tensor we use has to vanish when contracted with a covariant derivative, by compatibility with (7.13), it is clear that the contracted Bianchi identity tells us that we should be using the Einstein tensor

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} \,, \tag{7.14}$$

which automatically obeys $\nabla^\nu G_{\mu\nu} = 0$. This suggests $G_{\mu\nu} = \kappa T_{\mu\nu}$. This is mathematically self-consistent. The constant κ has to be fixed by matching with the Newtonian limit - we will do this shortly. The result is $\kappa = 8\pi G$.

We are now able to summarise the theory of General Relativity:

General Relativity

1. Spacetime is a four-dimensional Lorentzian manifold with a metric, (M, g), equipped with the Levi-Civita connection.

2. Free particles follow timelike or null geodesics.

3. The field equations governing the geometry are the *Einstein equations*:

$$G_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{7.15}$$

This little box is one of the crowning achievements of 20th century physics, scratch that, all of physics.

The cosmological constant

There is one more thing. What we were looking for was a symmetric tensor, $\tilde{G}_{\mu\nu}$, such that $\nabla^\nu \tilde{G}_{\mu\nu} = 0$, which was a function of the metric and its first and second derivatives. For a four-dimensional Lorentzian theory, the unique answer (up to rescaling) is that $\tilde{G}_{\mu\nu} = G_{\mu\nu} + \Lambda g_{\mu\nu}$, where Λ is a constant. This is known as Lovelock's theorem. So we could also have:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi G T_{\mu\nu} \,. \tag{7.16}$$

The constant Λ is the *cosmological constant*. If Λ is sufficiently large, this causes deviations from the Newtonian theory in the non-relativistic weak field limit. For this reason, it was supposed (modulo some blundering about by Einstein) that we should have $\Lambda = 0$. However, since the mid-1990s, observations have indicated that there is a very small positive cosmological constant present in our universe, which is, however, only relevant on cosmological scales. Explaining the value of the cosmological constant is a major outstanding problem. We will see some cosmological consequences of the cosmological constant later on. Notice as well that we could take the cosmological constant term over to the right-hand side of (7.16), in which case it can be described as a perfect fluid with $-p = \rho = \Lambda/(8\pi G)$.

7.3 The Newtonian limit

The general idea

We are going to take a particular limit of general relativity in order to show how it recovers standard Newtonian theory.

Newtonian limit

We will take the Newtonian limit, assuming:

- The *weak field limit*, namely that the metric is close to the Minkowski metric, $g_{\mu\nu}(x) \approx \eta_{\mu\nu} + h_{\mu\nu}(x)$, where the components $h_{\mu\nu}$ are small (note the metric components are actually dimensionless in our conventions).

- Velocities are *non-relativistic*, so for a test particle on a curve $x^\mu(\tau) = (t(\tau), x^i(\tau))$, $\left|\frac{dx^i}{d\tau}\right| << \left|\frac{dt}{d\tau}\right|$.

- The metric is *static*, i.e. time-independent, $\partial_0 h_{\mu\nu} \approx 0$.

Note that we can be a lot more precise about what we mean by small, however this will not be necessary for our purposes.

It is surprisingly easy to linearise the Levi-Civita connection, Riemann tensor and Ricci tensor. We work to first order in $h_{\mu\nu}$. This means the metric and inverse metric are:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}\,, \quad g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}\,. \tag{7.17}$$

We can think of this as splitting the metric into a "background" Minkowski metric $\eta_{\mu\nu}$ and a small fluctuation $h_{\mu\nu}$. We will use the flat background metric to raise and lower indices in expressions below. We will also define the trace

$$h \equiv \eta^{\mu\nu} h_{\mu\nu}\,. \tag{7.18}$$

The Levi-Civita connection has components

$$\Gamma_{\mu\nu}{}^{\rho} \approx \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right)\,. \tag{7.19}$$

In the Riemann tensor, we can neglect the terms quadratic in the connection components, so that

$$\begin{aligned} R^\rho{}_{\sigma\mu\nu} &\approx \partial_\mu \Gamma_{\nu\sigma}{}^\rho - \partial_\nu \Gamma_{\mu\sigma}{}^\rho \\ &= \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu\partial_\sigma h_{\lambda\nu} + \partial_\nu\partial_\lambda h_{\mu\sigma} - \partial_\mu\partial_\lambda h_{\sigma\nu} - \partial_\nu\partial_\sigma h_{\lambda\mu}\right)\,. \end{aligned} \tag{7.20}$$

Hence the Ricci tensor:

$$R_{\mu\nu} \approx \partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial_\rho\partial^\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h\,, \tag{7.21}$$

and Ricci scalar

$$R \approx \partial^\mu\partial^\nu h_{\mu\nu} - \partial_\rho\partial^\rho h\,. \tag{7.22}$$

The linearised Einstein equation is then:

$$\partial^\rho \partial_{(\mu} h_{\nu)\rho} - \frac{1}{2}\partial_\rho \partial^\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu \partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho \partial^\sigma h_{\rho\sigma} - \partial_\rho \partial^\rho h) = \kappa T_{\mu\nu}\,, \tag{7.23}$$

where we assume $T_{\mu\nu}$ is of the same order as $h_{\mu\nu}$, and write κ for the constant of proportionality which we will now determine from matching to the Newtonian theory.

The geodesic equation in the Newtonian limit

The geodesic equation is:

$$0 = \frac{d^2 x^\mu}{d\tau^2} + \Gamma_{\rho\sigma}{}^\mu \frac{dx^\rho}{d\tau}\frac{dx^\sigma}{d\tau} \approx \frac{d^2 x^\mu}{d\tau^2} + \Gamma_{00}{}^\mu \left(\frac{dt}{d\tau}\right)^2\,, \tag{7.24}$$

neglecting the spatial velocities. As $\Gamma_{00}{}^\mu = -\frac{1}{2}\eta^{\mu\nu}\partial_\nu h_{00}$, we have under our assumptions:

$$\frac{d^2 t}{d\tau^2} = 0 \Rightarrow \frac{dt}{d\tau}\ \text{constant}\,, \tag{7.25}$$

$$\frac{d^2 x^i}{d\tau^2} = \frac{1}{2}\partial^i h_{00}\left(\frac{dt}{d\tau}\right)^2 \Rightarrow \frac{d^2 x^i}{dt^2} = \frac{1}{2}\partial^i h_{00}\,. \tag{7.26}$$

This has the same form as the Newtonian expression (7.2), if we identify $h_{00} = -2\Phi$ (if you like, this is the definition of the Newtonian gravitational potential in terms of a component of a metric that is close to the Minkowski metric).

The Einstein equation in the Newtonian limit

Now consider the Einstein equation, $G_{\mu\nu} = \kappa T_{\mu\nu}$. The trace of this equation gives $R = -\kappa T$, where $T = T^\mu{}_\mu$. Let's assume as well that our matter source is some massive body or bodies, which we can model as dust, with $T_{\mu\nu} = \rho U_\mu U_\nu$. Let's work in the rest frame of this "fluid", with $U^0 = 1 + O(h)$, $U^i = 0$. Then $T_{00} = \rho$ and all other components are zero. Then we have $R = \kappa\rho$, or from (7.22),

$$\partial^k \partial^l h_{kl} - \partial_k \partial^k h = \kappa\rho\,. \tag{7.27}$$

We substitute this into the 00 component of the linearised Einstein equation:

$$-\frac{1}{2}\partial_k \partial^k h_{00} + \frac{1}{2}\kappa\rho = \kappa\rho \Rightarrow \partial_k \partial^k h_{00} = -\kappa\rho \Rightarrow \partial_k \partial^k \Phi = \frac{\kappa}{2}\rho\,. \tag{7.28}$$

To match with the equation (7.1) obeyed in Newtonian theory, we need $\kappa = 8\pi G$ as previously stated.

It is possible to go on and check that the remaining components of the linearised Einstein equation are solved by $h_{ij} = -2\delta_{ij}\Phi$, $h_{0i} = 0$. This can also be more rigourously derived as the solution at this order by a more technical analysis keeping track of the orders of all tensor components, exploiting the symmetries of the linearised theories (which we will need to do when we analyse gravitational wave solutions later on) and using Green's functions. Note that then the metric in this weak field Newtonian limit takes the form:

$$ds^2 = -\left(1 + 2\Phi(\vec{x})\right)dt^2 + \left(1 - 2\Phi(\vec{x})\right)d\vec{x}^2\,, \tag{7.29}$$

which was the very first curved metric we wrote down when analysing gravitational time dilation in section 2.3!

This should hopefully convince you that the Einstein's theory of gravity supersedes and contains Newton's.

We will return to the linearised theory in section 11, in order to study gravitational waves. Our next goal will be to explore full non-linear solutions of the Einstein equation.

8 Symmetries

As is frequently the case in physics, finding complete general solutions of the Einstein equations is difficult. We will focus on solutions which are tractable because they have a high degree of symmetry. First, we need to explain what we mean by a symmetry of a spacetime with metric.

8.1 Diffeomorphisms and isometries

We need a minimal amount of mathematical definitions to get started.

Diffeomorphism

A *diffeomorphism* is a map $\phi : M \to M$ which is smooth, one-to-one, onto and has a smooth inverse. (Smooth here means that $\varphi \circ \phi \circ \varphi'^{-1}$ is smooth for all charts φ, φ' of M.)

We can view a diffeomorphism as a change of coordinates. Let x^μ denote the coordinates of a chart (U, φ) in a neighbourhood of $p \in M$. Under the diffeomorphism, $p \mapsto \phi(p) \in M$. If y^μ denotes the coordinates of a chart (U', φ') in the neighbourhood of $\phi(p)$, then we can view the composition $\varphi' \circ \phi$ as defining a new chart in the neighbourhood of p, with the coordinates $y^\mu(x)$.

If $T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}$ are the components of some tensor T in the original chart, then the components of the tensor in the new chart with coordinates y are denoted $T'^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}$ and these are related by:

$$T'^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}(y(x)) = \left(\frac{\partial y^{\mu_1}}{\partial x^{\rho_1}}\right) \cdots \left(\frac{\partial y^{\mu_r}}{\partial x^{\rho_r}}\right) \left(\frac{\partial x^{\sigma_1}}{\partial y^{\nu_1}}\right) \cdots \left(\frac{\partial x^{\sigma_s}}{\partial y^{\nu_s}}\right) T^{\rho_1 \ldots \rho_r}{}_{\sigma_1 \ldots \sigma_s}(x) \,. \tag{8.1}$$

A diffeomorphism ϕ is a *symmetry* of a tensor if $T'^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s} = T^{\mu_1 \ldots \mu_r}{}_{\nu_1 \ldots \nu_s}$ everywhere. We are particularly interested in symmetries of the metric, which are important enough to warrant their own name.

Isometry

An *isometry* is a symmetry transformation of the metric, i.e. a diffeomorphism such that $g'_{\mu\nu} = g_{\mu\nu}$ everywhere (i.e. in all coordinate charts).

In coordinates, the condition that a diffeomorphism be an isometry is that:

$$\left(\frac{\partial y^\rho}{\partial x^\mu}\right) \left(\frac{\partial y^\sigma}{\partial x^\nu}\right) g_{\rho\sigma}(y(x)) = g_{\mu\nu}(x) \,. \tag{8.2}$$

8.2 Killing vectors

An example of a diffeomorphism is the following. For each vector field X on our manifold M, we can construct the integral curve of X through any point p. Let ϕ_t be the map which sends $p \in M$ to the point parameter distance t along this integral curve. This is a diffeomorphism.

Suppose then we have a vector field K generating a diffeomorphism in this manner. Let ϵ be infinitesimal so that arbitrarily close to x^μ the diffeomorphism is defined by $y^\mu = x^\mu + \epsilon K^\mu + O(\epsilon^2)$.

If we require (8.2), we have:

$$\left(\delta^\rho_\mu + \epsilon \frac{\partial K^\rho}{\partial x^\mu}\right) \left(\delta^\sigma_\nu + \epsilon \frac{\partial K^\sigma}{\partial x^\nu}\right) \left(g_{\rho\sigma}(x) + \epsilon K^\lambda \partial_\lambda g_{\rho\sigma}(x)\right) = g_{\mu\nu}(x) \,. \tag{8.3}$$

Expanding to order ϵ, we find the condition for K^μ to generate an isometry of the metric: it has to be a *Killing vector.*

Killing vector

A Killing vector is a solution of the Killing equation:

$$K^\rho \partial_\rho g_{\mu\nu} + \partial_\mu K^\rho g_{\rho\nu} + \partial_\nu K^\rho g_{\rho\mu} = 0 \,, \tag{8.4}$$

or in terms of a torsion-free metric-compatible connection,

$$\nabla_\mu K_\nu + \nabla_\nu K_\mu = 0 \,. \tag{8.5}$$

This is known as the Killing equation, and solutions K^μ are known as *Killing vector fields.* Each Killing vector field, if it exists, describes an isometry of a metric.

If the metric is independent of some coordinate z, then $K = \frac{\partial}{\partial z}$ is automatically a Killing vector, as follows from the Killing equation in the form (8.4). Conversely, if we do have a Killing vector, then we can always choose coordinates such that the Killing vector has the form $K = \frac{\partial}{\partial z}$ with the metric independent of z.

For each affinely parametrised geodesic, with tangent vector X^μ, then $K_\mu X^\mu$ is constant along the geodesic. This provides constants of motion for test particles. To prove this, we differentiate:

$$\frac{d}{d\lambda}(K_\mu X^\mu) \equiv \nabla_X (K_\mu X^\mu) = X^\nu X^\mu \nabla_\nu K_\mu + K_\mu X^\nu \nabla_\nu X^\mu = 0 \,. \tag{8.6}$$

The second term is zero because this is condition for X^μ to be a geodesic. The first term is zero because we automatically symmetrise and get the Killing equation: $X^\nu X^\mu \nabla_\nu K_\mu = X^\nu X^\mu \nabla_{(\nu} K_{\mu)} = 0$.

Given the energy-momentum tensor, $T_{\mu\nu}$, and a Killing vector K^μ, then $J^\mu = T^\mu{}_\nu K^\nu$ is conserved: $\nabla_\mu J^\mu = 0$. Thus in the presence of Killing vectors we can define a conserved current. If K^μ is a timelike Killing vector, this leads to a definition of a conserved energy, for instance.

Killing vectors of flat space

In Minkowski spacetime, the Killing equation is:

$$\partial_\mu K_\nu + \partial_\nu K_\mu = 0 \,. \tag{8.7}$$

Differentiating again, we have

$$0 = \partial_\mu \partial_\rho K_\nu + \partial_\nu \partial_\rho K_\mu = -2\partial_\mu \partial_\nu K_\rho \,, \tag{8.8}$$

so any solution is at most linear in the coordinates: $K_\mu = a_\mu + \omega_{\mu\nu}x^\nu$, with a_μ, $\omega_{\mu\nu}$ constant. Inserting this ansatz, we are required to have $\omega_{\mu\nu} = -\omega_{\nu\mu}$.

The constant Killing vectors, $K_\mu = a_\mu$, correspond to spacetime translations. There are n of these (4 when $n = 4$). The linear Killing vectors, $K_\mu = \omega_{\mu\nu}x^\nu$, correspond to Lorentz transformations. There are $\frac{1}{2}n(n-1)$ of these (6 when $n = 4$). In general, if an n-dimensional manifold has $\frac{1}{2}n(n+1)$ Killing vectors then it is said to be *maximally symmetric*. This is the case here.

Killing vectors of S^2

The metric on S^2 is:

$$ds^2 = (d\theta)^2 + \sin^2\theta(d\phi)^2 \,. \tag{8.9}$$

It is intuitively clear that the sphere should be invariant under the group $SO(3)$ of rotations in $\mathbb{R}^3$. We expect therefore that there are three Killing vectors, corresponding to rotations about each axis. From our result for the Killing vectors of Minkowski space, we realise that these can be taken to be, in three-dimensional Cartesian coordinates:

$$(K_{(i)})_j = -\epsilon_{ijk}x^k \,, \tag{8.10}$$

so

$$\begin{aligned} K_{(1)} &= y\partial_z - z\partial_y \,, \\ K_{(2)} &= z\partial_x - x\partial_z \,, \\ K_{(3)} &= x\partial_y - y\partial_x \,. \end{aligned} \tag{8.11}$$

In spherical coordinates we find:

$$\begin{aligned} K_{(1)} &= -\sin\phi\partial_\theta - \cot\theta\cos\phi\partial_\phi \,, \\ K_{(2)} &= \cos\phi\partial_\theta - \cot\theta\sin\phi\partial_\phi \,, \\ K_{(3)} &= \partial_\phi \,. \end{aligned} \tag{8.12}$$

These are the Killing vector fields of S^2. As this is a 2-dimensional manifold, and there are three Killing vectors, it is also maximally symmetric.

Exercise 8.1 (Killing vector fields of S^2)

1. Verify that (8.12) are indeed Killing vectors of the sphere.
2. Show that the $K_{(i)}$ obey:

$$[K_{(i)}, K_{(j)}] = -\epsilon_{ijk}K_{(k)} \tag{8.13}$$

under the commutator of vector fields. What is this algebra?

9 Schwarzschild

Schwarzschild to Einstein (December 1915)

"As you can see, the war is friendly with me, allowing me, in spite of fierce gunfire at the quite terrestrial distance, to take this walk in your country of ideas."

Finding solutions to the Einstein equations is not simple. As usual in physics, it is useful to consider situations with high amounts of symmetry. The Schwarzschild solution is the unique spherically symmetric solution to the vacuum Einstein equations (i.e. in the absence of matter, $T_{\mu\nu} = 0$). In fact, even as a vacuum solution it is immediately physically relevant, as it describes the spacetime in the region outside of a spherically symmetric object. Some spherically symmetric sources of gravitational fields include stars and planets. Therefore the Schwarzschild solution can be used to model the motion of planets in orbit around the Sun, and improves on the Newtonian predictions.

Spherically symmetric

A spacetime is said to be *spherically symmetric* if its isometry group contains an $SO(3)$ subgroup, whose orbits are 2-spheres. (The orbit of a point p is the set of all points obtained by acting on p with elements of the isometry group.)

Birkhoff's theorem

The unique spherically symmetric solution of the vacuum Einstein equation is the Schwarzschild metric.

9.1 The Schwarzschild metric

The metric

We will not go through the derivation of the Schwarzschild solution from scratch. The solution in *Schwarzschild coordinates* is as follows.

The Schwarzschild solution

The *Schwarzschild metric* is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \qquad (9.1)$$

where $t \in (-\infty, \infty)$, (θ, ϕ) are the usual coordinates on a sphere, and as $r \to -r$ is equivalent to $M \to -M$ we can assume $r \geq 0$. The parameter M will be interpreted as a mass below, hence we take $M \geq 0$.

It should be clear from the metric (9.1) that surfaces of constant t and r are spheres of area $4\pi r^2$. In this sense r is like a "radial" coordinate (but the form of the metric shows that the proper distance between r_0 and r_1, with $r_1 > r_0$, is not simply $r_1 - r_0$, as it would be if r itself was measuring the distance from some origin). As well as being spherically symmetric, the Schwarzschild spacetime is also invariant under $t \to t + c$, thus it is time-independent. Thus, in addition to the three Killing vectors associated to spherical symmetry, we have a timelike Killing vector $T \equiv \partial/\partial t$.

Now, we know that in practice a coordinate chart may only cover some portion of the manifold it is describing. In the Schwarzschild metric, we have the usual issue regarding needing multiple charts to cover the sphere. But more interestingly, there appear to be singularities at $r = 2GM$ and at $r = 0$. This suggests that either (t, r) are not good coordinates everywhere in Schwarzschild spacetime, or that there is something intrinsically wrong with the latter. We will explore this in great detail later on.

Exercise 9.1 (The Schwarzschild radius)

The *Schwarzschild radius* of a body of mass M is given by $r_s = 2GM$. Work this out for the Sun. (You will need to reinsert some factor of c.) How does it compare to the actual radius of the Sun?

Asymptotics

Another immediate observation is that when $M = 0$, we recover Minkowski spacetime in spherical coordinates. For $M \neq 0$, the Schwarzschild metric resembles Minkowski spacetime as $r \to \infty$, more precisely for $r >> GM$. As a result, we say that the Schwarzschild metric is *asymptotically flat*.

In this asymptotic region, the metric (9.1) can be written

$$ds^2 \approx -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 + \frac{2GM}{r}\right) dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9.2}$$

up to terms of order $(GM/r)^2$. Let us define a new coordinate R by

$$r = R\sqrt{1 + \frac{2GM}{R}}\,. \tag{9.3}$$

The inverse transformation is $R = \sqrt{r^2 + (GM)^2} - GM$ (assuming $R \geq 0$). For $r >> GM$ we have $R \approx r$, so also $R >> GM$. Then

$$\frac{2GM}{r} = \frac{2GM}{R}\frac{1}{\sqrt{1 + \frac{2GM}{R}}} \approx \frac{2GM}{R} + O((GM/R)^2)\,, \tag{9.4}$$

$$dr = \frac{dR}{\sqrt{1 + \frac{2GM}{R}}}\left(1 + \frac{GM}{R}\right) \approx dR + O((GM/R)^2)\,. \tag{9.5}$$

So to leading order the asymptotic metric can be written

$$ds^2 \approx -\left(1 - \frac{2GM}{R}\right) dt^2 + \left(1 + \frac{2GM}{R}\right)\left(dR^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2)\right)\,, \tag{9.6}$$

which is the metric describing a weak gravitational field, of the type we encountered in the Newtonian limit of general relativity. The Newtonian gravitational potential is:

$$\Phi = -\frac{GM}{R}\,, \tag{9.7}$$

from which we see that this describes the gravitational field produced by a spherically symmetric source of mass M. For this reason, we will from now on assume $M > 0$.

Exercise 9.2 (Gravitational redshift)

Suppose that our communicative friends, Alice and Bob, have now made their way to Schwarzschild spacetime. Alice is at rest at $r = r_A$, and Bob is at rest at $r = r_B$. Alice sends signals at intervals $\Delta\tau_A$ to Bob, who receives them at intervals $\Delta\tau_B$. Show that

$$\Delta\tau_B = \left(1 - \frac{2GM}{r_B}\right)^{1/2}\left(1 - \frac{2GM}{r_A}\right)^{-1/2}\Delta\tau_A\,, \tag{9.8}$$

and hence that if Alice positions herself at $r_A = 2GM$ that $\Delta\tau_B \to \infty$: Bob will see Alice as being *frozen* at the Schwarzschild radius (equivalently, there is an infinite red shift).

9.2 Geodesics in Schwarzschild

Particles (representing Alice, Bob, planets, rocketships and suchlike) move on geodesics. To find the geodesics of the Schwarzschild spacetime, it is convenient to start from the action

$$S = \frac{1}{2}\int d\lambda\, g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = \frac{1}{2}\int d\lambda\,\left(-f(r)\dot{t}^2 + f(r)^{-1}\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right)\,, \tag{9.9}$$

where

$$f(r) \equiv 1 - \frac{2GM}{r}\,. \tag{9.10}$$

The equations of motion for t, r, θ, ϕ then correspond to the geodesic equations, from which one can also read off the components of the Levi-Civita connection. For t and ϕ we have

$$\frac{d}{d\lambda}\left(f(r)\dot{t}\right) = 0\,, \tag{9.11}$$

$$\frac{d}{d\lambda}\left(r^2\sin^2\theta\dot{\phi}\right) = 0\,. \tag{9.12}$$

These can be integrated immediately giving

$$f(r)\dot{t} = E\,, \quad r^2\sin^2\theta\dot{\phi} = L\,, \tag{9.13}$$

where E and L are constants. These constants of motion reflect the fact that the Schwarzschild metric, and hence the action (9.9), is independent of t and ϕ. We saw previously that for any Killing vector K^μ that $K_\mu\frac{dx^\mu}{d\lambda}$ is constant on a geodesic $x^\mu(\lambda)$. Hence here E follows from the presence of the timelike Killing vector $T^\mu = (1, 0, 0, 0)$, i.e $T = \partial/\partial t$, and L from the Killing vector $K^\mu_\phi = (0, 0, 0, 1)$ i.e. $K_\phi = \partial/\partial\phi$. For a timelike geodesic, we interpret E as the total *energy* per unit rest mass of the massive particle, and L as the magnitude of angular momentum per unit rest mass. This is based on interpreting E and L for a geodesic in the asymptotic region where $f(r) \approx 1$. Then for instance $E = \frac{dt}{d\tau}$ is the time component of the usual four-velocity.

For a null geodesic, corresponding to a massless particle, we have the freedom to rescale the affine parameter $\lambda \to a\lambda$, which means that the values of E and L are not directly physically meaningful. However, the ratio E/L is invariant under this rescaling, and can be used to characterise null geodesics as we will see below.

There should be two other conserved quantities, corresponding to the remaining two rotational Killing vectors. Let's first write down the θ component of the geodesic equation:

$$\frac{d}{d\lambda}(r^2\dot{\theta}) - r^2 \sin\theta \cos\theta \dot{\phi}^2 = 0\,, \tag{9.14}$$

or

$$\frac{d}{d\lambda}(r^2\dot{\theta}) - \frac{L^2}{r^2}\frac{\cos\theta}{\sin^3\theta} = 0\,, \tag{9.15}$$

We can solve this by supposing we pick coordinates such that $\theta = \frac{\pi}{2}$ at $\lambda = 0$. Then an immediate solution is given by $\theta(\lambda) = \frac{\pi}{2}$. This is guaranteed to be unique by the usual ODE theory. This means we can always pick our coordinates (θ, ϕ) such that the geodesic is confined to the equatorial plane of the S^2 at $\theta = \frac{\pi}{2}$.

We can now interpret this result in terms of the conserved quantities associated to the other Killing vectors. Together the three rotational Killing vectors should lead to conservation of angular momentum. A vector can be specified in terms of its magnitude and its direction. To specify the direction, we need to fix two numbers (as it is given by a three-component unit vector). We can interpret then these two numbers as the remaining conserved quantities, which we have chosen to specify by fixing our geodesic motion to $\theta = \frac{\pi}{2}$. (This reasoning in terms of spatial vectors relies on intuition about conventional angular momentum in flat space, and so strictly speaking we should only really interpret it in this way in the asymptotic region where the Schwarzschild becomes flat.)

Rather than write down the r component of the geodesic equation, it is more convenient to use the condition $g_{\mu\nu}\frac{dx^\mu}{d\lambda}\frac{dx^\nu}{d\lambda} = -\sigma$, where $\sigma = 1$ for timelike geodesics and $\sigma = 0$ for null geodesics. On setting $\theta = \frac{\pi}{2}$ this gives:

$$\frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2\,, \tag{9.16}$$

where

$$V(r) = \frac{1}{2}f(r)\left(\sigma + \frac{L^2}{r^2}\right) = \frac{1}{2}\left(\sigma - \frac{2GM\sigma}{r} + \frac{L^2}{r^2} - \frac{2GML^2}{r^3}\right)\,. \tag{9.17}$$

The equation (9.16) describes effective one-dimensional particle motion in a potential $V(r)$ given by (9.17). Terms involving $1/r$ and $1/r^2$ occur in Newtonian dynamics of a particle moving in a gravitational field. The $1/r^3$ term does not. This provides a relativistic correction to Newtonian gravity.

Null geodesics

Null geodesics in Schwarzschild

The geodesic equations for a null geodesic are:

$$r^2\dot{\phi} = L\,, \qquad \left(1 - \frac{2GM}{r}\right)\dot{t} = E\,, \qquad \frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2\,, \tag{9.18}$$

where

$$V(r) = \frac{L^2}{2r^2} - \frac{GML^2}{r^3}\,. \tag{9.19}$$

Let's consider the form of the potential (9.19). For $r \to \infty$, $V \to 0$, while for $r \to 0$, $V \to -\infty$. We also see that $V(r = 2GM) = 0$. There is a turning point when $V'(r) = 0$. This happens at $r = 3GM$. We get the picture in figure 15).

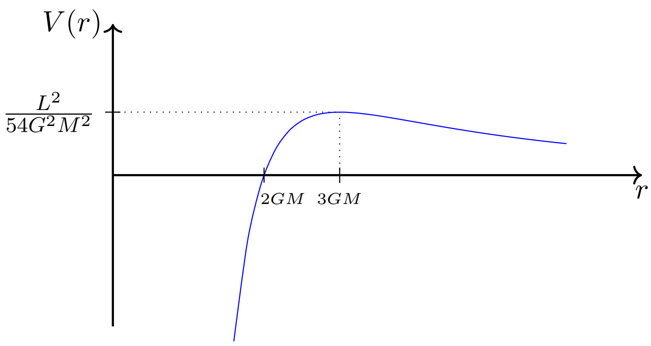


Figure 15: Effective potential, null geodesic

We can analyse the radial behaviour of geodesics using our intuition from Newtonian mechanics. A massless particle incident from $r = \infty$ with $E^2/2$ less than the maximum of the potential, $V_{max} = L^2/(54G^2M^2)$, will hit the potential barrier when $E^2/2 = V(r)$ and bounce back to infinity. A massless particle incident with energy greater than the maximum of the potential will continue all the way to $r = 0$. At $r = 3GM$, where $V'(r) = 0$, we can have circular orbits with $\dot{r} = 0$. These will be unstable.

Define

$$b \equiv \left|\frac{L}{E}\right|\,. \tag{9.20}$$

This quantity is called the *impact parameter*. We can combine the geodesic equations for $\dot{\phi}$ and $\dot{r}$ as follows:

$$\frac{d\phi}{dr} = \frac{\frac{d\phi}{d\lambda}}{\frac{dr}{d\lambda}} = -\frac{\frac{L}{r^2}}{\sqrt{E^2 - 2V(r)}} = -\frac{L}{E}\frac{1}{r^2\sqrt{1 - \frac{b^2}{r^2} + \frac{2GMb^2}{r^3}}}\,, \tag{9.21}$$

assuming for definiteness that $\dot{r} < 0$, i.e. the geodesic is ingoing (we will get the same result for an outgoing geodesic). For large r, we have

$$d\phi \approx -\frac{L}{E}\frac{dr}{r^2}\frac{1}{\sqrt{1 - \frac{b^2}{r^2}}}\,. \tag{9.22}$$

The substitution $u = 1/r$ allows us to solve

$$d\phi \approx \frac{L}{E}\frac{du}{\sqrt{1-b^2u^2}} \Rightarrow \phi - \alpha \approx \frac{L}{E}\frac{1}{b}\sin^{-1}(bu)\,, \tag{9.23}$$

where α is a constant of integration, so that (assuming L/E positive which we can arrange by sending ϕ to $-\phi$ if we have to)

$$r\sin(\phi - \alpha) = b\,. \tag{9.24}$$

What does this describe? In Cartesian coordinates in the plane defined by $\theta = \frac{\pi}{2}$ in which the geodesic motion occurs,

$$b = y\cos\alpha - x\sin\alpha\,, \tag{9.25}$$

which describes a straight line with slope $\tan\alpha$ displaced a distance b from the origin, as demonstrated in figure 16:

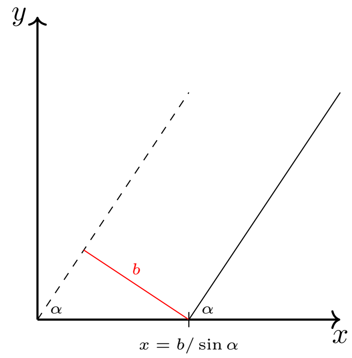


Figure 16: Asymptotic null geodesic

We can therefore characterise null geodesics incident from or returning to infinity in terms of the value of b, which describes how far the geodesic is from a line of constant $\phi = \alpha$ (the dashed line in figure 16).

The condition that the null geodesic has energy allowing it to pass over the potential barrier is

$$\frac{1}{2}E^2 > \frac{L^2}{54G^2M^2} \Leftrightarrow b < \sqrt{27}GM\,. \tag{9.26}$$

So if $b < \sqrt{27}GM$, the null geodesic reaches the Schwarzschild radius, while if $b > \sqrt{27}GM$ it reflects off the barrier and returns to infinity, except that it may now be a distance b from a different line of constant ϕ. This has the physical interpretation that light rays are bent in the gravitational field due to a spherically symmetric body.

Consider a light ray then with $b > \sqrt{27}GM$, which is incident from infinity with impact factor b, and then returns to infinity (see figure 17). By how much is it deflected? Let r_0 denote the turning point of the geodesic, i.e. the point at which $V(r) = \frac{1}{2}E^2$, so the largest real value r_0 solving

$$\frac{1}{2r_0^2} - \frac{GM}{r_0^3} = \frac{1}{2b^2}\,. \tag{9.27}$$

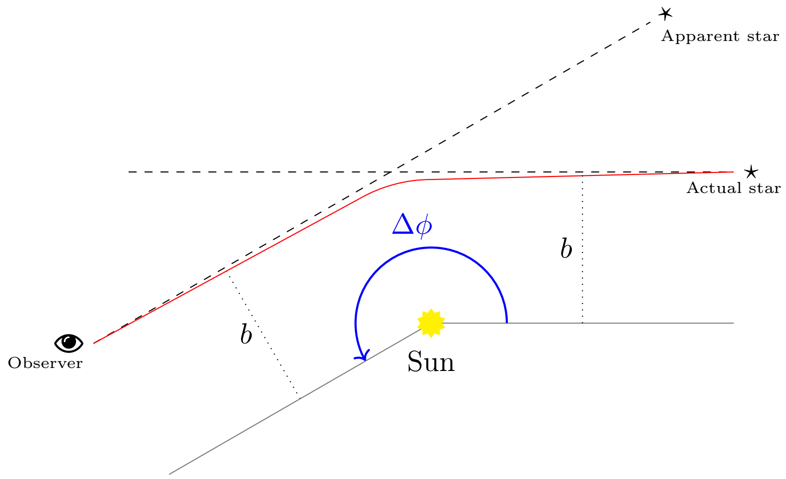


Figure 17: Light bending by the Sun

We have:

$$\begin{aligned}\Delta\phi \equiv \int d\phi &= -\int_{\infty}^{r_0} \frac{b\,dr}{r^2\sqrt{1-\frac{b^2}{r^2}+\frac{2GMb^2}{r^3}}} + \int_{r_0}^{\infty} \frac{b\,dr}{r^2\sqrt{1-\frac{b^2}{r^2}+\frac{2GMb^2}{r^3}}} \\ &= 2\int_0^{u_0} \frac{du}{\sqrt{1-u^2+\frac{2GM}{b}u^3}}\,,\end{aligned} \tag{9.28}$$

after letting $u = b/r$, so $u_0 = b/r_0$. For the Sun, GM/b is of order 10^{-6} for an impact parameter approximately equal to the solar radius. We can therefore evaluate $\Delta\phi$ assuming GM/b is small. The naive method of expanding (9.28) in this quantity is problematic owing to the fact that the combination in the square root vanishes at $u = u_0$, so the term $2GMu^3/b$ cannot be taken to be smaller than the $1-u^2$ term. Furthermore we must take into account that the limit u_0 also depends on the GM/b. For small GM/b, we have $u_0 = 1 + GM/b$. However, one way to proceed is to factorise the square root and then Taylor expand, as follows:

$$\frac{1}{\sqrt{1-u^2+\frac{2GM}{b}u^3}} = \frac{1}{\sqrt{1-\frac{2GM}{b}u}}\frac{1}{\sqrt{\frac{1}{1-\frac{2GM}{b}u}-u^2}} \approx \frac{1+\frac{GM}{b}u}{\sqrt{1+\frac{2GM}{b}u-u^2}}\,. \tag{9.29}$$

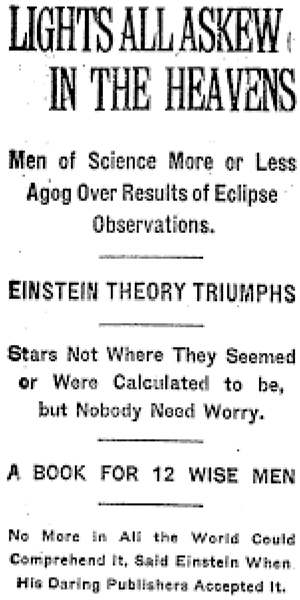
This leaves a straightforward integral which can be computed in terms of elementary functions, giving:

$$\Delta\phi \approx \pi + \frac{4GM}{b}\,. \tag{9.30}$$

Exercise 9.3 (Light deflection)

Verify that you obtain the result (9.30).

This prediction of General Relativity was famously verified by a team led by Eddington during a solar eclipse in 1919, observing the deviation of stars from their known positions when the Sun was positioned between them and the Earth.


LIGHTS ALL ASKEW IN THE HEAVENS

Men of Science More or Less Agog Over Results of Eclipse Observations.

EINSTEIN THEORY TRIUMPHS

Stars Not Where They Seemed or Were Calculated to be, but Nobody Need Worry.

A BOOK FOR 12 WISE MEN

No More in All the World Could Comprehend It, Said Einstein When His Daring Publishers Accepted It.

Figure 18: The rather delirious reporting by the *New York Times*, November 10th, 1919, of Eddington's observations.

Timelike geodesics

Timelike geodesics in Schwarzschild

The geodesic equations for a timelike geodesic are:

$$r^2\dot{\phi} = L\,, \quad \left(1 - \frac{2GM}{r}\right)\dot{t} = E\,, \quad \frac{1}{2}\dot{r}^2 + V(r) = \frac{1}{2}E^2\,, \tag{9.31}$$

where

$$V(r) = \frac{1}{2}\left(1 - \frac{2GM}{r} + \frac{L^2}{r^2} - \frac{2GML^2}{r^3}\right) \tag{9.32}$$

The potential (9.32) goes to $1/2$ as $r \to \infty$ and to $-\infty$ as $r \to 0$. At $r = 2GM$ it vanishes. There will be a turning point (and hence a circular orbit) where $V'(r) = 0$, namely at

$$r_\pm = \frac{L^2 \pm \sqrt{L^4 - 12L^2G^2M^2}}{2GM}\,. \tag{9.33}$$

This means that for $L^2 < 12G^2M^2$ there are no turning points, for $L^2 = 12G^2M^2$ there is exactly one turning point, and for $L^2 > 12G^2M^2$ there is an unstable circular orbit at r_- and a stable circular orbit at r_+. We can again graph the potential (figure 19):

For large L^2, we have

$$r_\pm = \frac{L^2}{2GM}\left(1 \pm \sqrt{1 - \frac{12G^2M^2}{L^2}}\right) \approx \frac{L^2}{2GM}\left(1 \pm 1 \mp \frac{6G^2M^2}{L^2}\right) \Rightarrow r_+ \to \frac{L^2}{GM}\,, r_- \to 3GM\,. \tag{9.34}$$

Therefore the stable orbits obey $3GM < r_- < 6GM < r_+$.

When $L^2 = 12GM$, the circular orbits coincide at $r = 6GM$. This is known as the innermost stable circular orbit (ISCO).

An immediate application of timelike geodesics is to describe the motion of planets in orbit

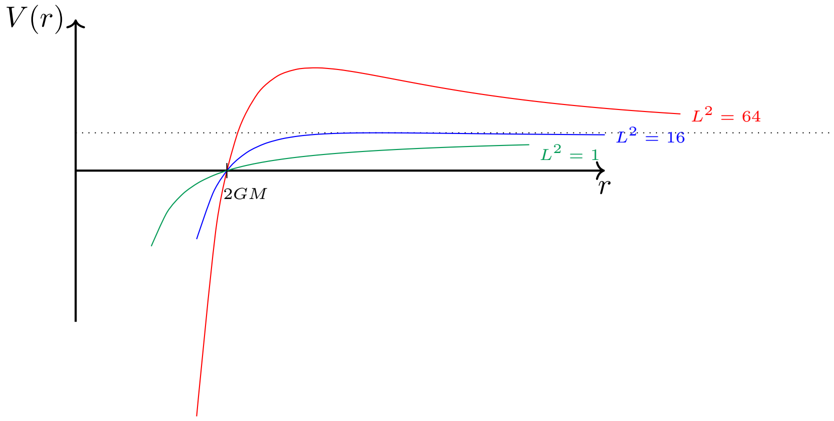


Figure 19: Effective potentials, timelike geodesic, with $GM = 1$.

around the Sun. We can combine the equations for $\dot{\phi}$ and $\dot{r}$ to produce

$$\left(\frac{dr}{d\phi}\right)^2 = \frac{r^4}{L^2}\left(E^2 - 1 + \frac{2GM}{r} - \frac{L^2}{r^2} + \frac{2GML^2}{r^3}\right). \tag{9.35}$$

Let $z = \frac{L^2}{GMr}$. Then, letting $z' \equiv \frac{dz}{d\phi}$, the equation (9.35) becomes

$$(z')^2 = \frac{L^2(E^2-1)}{(GM)^2} + 2z - z^2 + \frac{2(GM)^2}{L^2} z^3 . \tag{9.36}$$

Differentiating this we obtain a second-order ODE:

$$z'' = 1 - z + \frac{3(GM)^2}{L^2} z^2 . \tag{9.37}$$

We will solve this perturbatively in $\alpha \equiv 3(GM/L)^2$. Let $z = z_0 + z_1$, where z_0 is the zeroth order solution to

$$z_0'' = 1 - z_0 , \tag{9.38}$$

and z_1 is the first order solution to

$$z_1'' = -z_1 + \alpha z_0^2 . \tag{9.39}$$

We solve (9.38) by

$$z_0(\phi) = 1 + e\cos\phi . \tag{9.40}$$

This describes a standard Newtonian elliptical orbit.

Ellipses

The equation for an ellipse centred at the origin $(x, y) = (0, 0)$ is $x^2/a^2 + y^2/b^2 = 1$. The eccentricity of the ellipse is $e = \sqrt{1 - \frac{b^2}{a^2}}$. The focal points of the ellipse are at $(0, \pm ea)$. In polar coordinates centred at the focus $(0, ea)$, we have $x = r\cos\phi + ea$ and $y = r\sin\phi$, and the equation for the ellipse can be written as $r(1 + e\cos\phi) = a(1 - e^2)$.

We now substitute the result for z_0 into the equation (9.39). This gives

$$z_1'' = -z_1 + \alpha\left(1 + \frac{1}{2}e^2 + 2e\cos\phi + \frac{1}{2}e^2\cos 2\phi\right), \tag{9.41}$$

and you can check that this is solved by:

$$z_1(\phi) = \alpha\left(1 + \frac{1}{2}e^2 + e\phi\sin\phi - \frac{1}{6}e^2\cos 2\phi\right). \tag{9.42}$$

Our full solution, to the order we are working at, is then

$$\begin{aligned} \frac{L^2}{GM} &= r\left(1 + e\cos\phi + \alpha e\phi\sin\phi\right) + r\alpha\left(1 + \frac{1}{2}e^2 - \frac{1}{6}e^2\cos 2\phi\right) \\ &\approx r\left(1 + e\cos\left(\left[1-\alpha\right]\phi\right)\right) + r\alpha\left(1 + \frac{1}{2}e^2 - \frac{1}{6}e^2\cos 2\phi\right). \end{aligned} \tag{9.43}$$

The crucial result is that this is no longer 2π-periodic in ϕ, and so does not define a closed ellipse. Rather, the orientation of the ellipse *precesses* with each orbit.

Now, the *perihelion* of the orbit is defined as the position of closest approach to the Sun. This occurs whenever

$$e\cos\left(\left[1-\alpha\right]\phi\right) - \alpha e^2\frac{1}{6}\cos 2\phi \tag{9.44}$$

is largest, which to $O(\alpha)$ is at:

$$\phi = \frac{2\pi n}{1-\alpha} \approx 2\pi n(1+\alpha). \tag{9.45}$$

Hence the angular precession of the perihelion is

$$\Delta\phi = 2\pi\alpha = \frac{6\pi(GM)^2}{L^2}. \tag{9.46}$$

Now, the zeroth order solution, which described an ellipse, was

$$\frac{L^2}{GM} \approx r(1 + e\cos\phi), \tag{9.47}$$

which implies that $L^2 \approx GM(1-e^2)a$, where a can be taken to be the semi-major axis of the ellipse. Therefore, we have

$$\Delta\phi = \frac{6\pi GM}{(1-e^2)a}. \tag{9.48}$$

Clearly, a will be smallest, and hence $\Delta\phi$ largest, for the planet Mercury. The numbers work out in this case such that $\Delta\phi$ is 43 arcseconds a century (an arcsecond is 1/60 of a degree). The observed precession is in fact 5601 arcseconds a century. However, Newtonian effects related to the motion of the planet Earth, and the gravitational effects of the other planets on Mercury, account for 5557 arcseconds a century. The prediction of general relativity exactly accounts for the rest!

9.3 The Schwarzschild black hole

Coordinate and curvature singularities

The Schwarzschild metric (9.1) in Schwarzschild coordinates,

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9.49}$$

appears to suffer from singularities at $r = 2GM$ and $r = 0$. We must investigate whether these are intrinsic properties of the geometry, or mere artefacts of the choice of coordinates. Therefore we seek coordinate independent quantities that may contain information about what is happening as r varies. Singularities in the metric would be expected to potentially show up in the curvature of the geometry, as measured by the Riemann curvature tensor. A natural coordinate independent quantity constructed from the Riemann curvature tensor is the Ricci scalar, R. However, the Schwarzschild metric is a vacuum solution of the Einstein equations, so both the Ricci tensor and scalar vanish, $R_{\mu\nu} = 0$, $R = 0$. The simplest non-zero "curvature invariant" that can be constructed is the *Kretschmann scalar*, $K \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$. For Schwarzschild, one finds

$$K = \frac{48G^2M^2}{r^6}\,. \tag{9.50}$$

This is perfectly well-behaved at $r = 2GM$ but blows up as $r \to 0$. This suggests that the metric singularity at $r = 2GM$ is only an apparent singularity, and will not appear in other coordinate choices, whereas the singularity at $r = 0$ is an unavoidable part of the spacetime. The former is called a *coordinate singularity.* The latter is called a *curvature singularity.*

To understand how to extend our charts to past $r = 2GM$, and to understand to what extent we should be concerned about the presence of a singularity in spacetime itself, we need to find new coordinates.

Eddington-Finkelstein coordinates

The starting point is to consider radial null geodesics, obeying

$$\left(1 - \frac{2GM}{r}\right) dt^2 = \frac{dr^2}{1 - \frac{2GM}{r}}\,, \tag{9.51}$$

and hence

$$\frac{dt}{dr} = \pm\left(1 - \frac{2GM}{r}\right)^{-1}\,, \tag{9.52}$$

where the plus sign corresponds to outgoing geodesics (with $dr/d\lambda > 0$) and the minus sign to ingoing geodesics (with $dr/d\lambda < 0$).

A solution to the equation (9.52) is provided by

$$t \mp r_* = \text{constant}\,, \tag{9.53}$$

where

$$r_* = r + 2GM \ln\left|\frac{r}{2GM} - 1\right|\,, \tag{9.54}$$

is known as the *tortoise coordinate*, such that $dr_* = dr/(1 - 2GM/r)$.

The Schwarzschild solution (tortoise coordinate)

The Schwarzschild metric in coordinates (t, r_*, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2GM}{r(r_*)}\right)(dt^2 - dr_*^2) + r^2(r_*)(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9.55}$$

where $t \in (-\infty, \infty)$, $r_* \in (-\infty, \infty)$ and $r(r_*)$ is defined by (9.54).

We next define the *Eddington-Finkelstein* coordinates which are adapted to these ingoing or outgoing null geodesics:

$$v = t + r_*\,, \quad u = t - r_*\,, \tag{9.56}$$

in the sense that $v =$ constant defines ingoing radial null geodesics and $u =$ constant defines outgoing radial null geodesics.

We now have the freedom to choose any two of (t, r, u, v) as coordinates.

The Schwarzschild solution (ingoing Eddington-Finkelstein coordinates)

The Schwarzschild metric in coordinates (v, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9.57}$$

where $v \in (-\infty, \infty)$, $r \in (0, \infty)$.

Although the component g_{vv} of the metric in these coordinates vanishes at $r = 2GM$, the whole metric and its inverse are well-defined at this point. This shows that the apparent singularity at this point was indeed a *coordinate singularity*. In ingoing Eddington-Finkelstein coordinates, there is no coordinate singularity, and we can *extend* the Schwarzschild solution past the point $r = 2GM$.

In the region $r < 2GM$, we could equally well transform to coordinates (t', r) defined by $t' = v - r_*(r)$, using (9.54), such that the Schwarzschild metric takes the same form as in the region $r > 2GM$, namely

$$ds^2 = -\left(1 - \frac{2GM}{r}\right)(dt')^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,. \tag{9.58}$$

However, now $1 - \frac{2GM}{r} < 0$. This means that in these coordinates, r is the timelike direction and t' a spacelike direction!

In Eddington-Finklestein coordinates, ingoing radial null geodesics are described by

$$v = \text{constant}\,, \tag{9.59}$$

and outgoing ones by

$$v - 2r_* = \text{constant} \Rightarrow v = \text{constant} + 2r + 4GM \ln\left|\frac{r}{2GM} - 1\right|\,. \tag{9.60}$$

For $r > 2GM$, these outgoing null geodesics tend towards the expected straight lines given by

$t = \text{constant} + r$ for $r \to \infty$. Conversely, as $r \to 2GM$ (from above), $v \to -\infty$.

For $r < 2GM$, the situation changes substantially. As $r \to 0$, the geodesics reach the point $v = \text{constant}$. As $r \to 2GM$, they tend again to $v \to -\infty$.

This means that for $r > 2GM$ the outgoing geodesics reach infinity. However for $r < 2GM$, the "outgoing" geodesics reach the singularity at $r = 0$. We draw this situation in figure 21.

As r is the timelike direction in the region $r < 2GM$, this singularity in fact lies in the future for any observer. We can see this by considering the future "lightcones" delineated by the curves of the ingoing and outgoing null geodesics. As we approach and pass beyond $r = 2GM$ these become more and more tilted towards the singularity at $r = 0$.

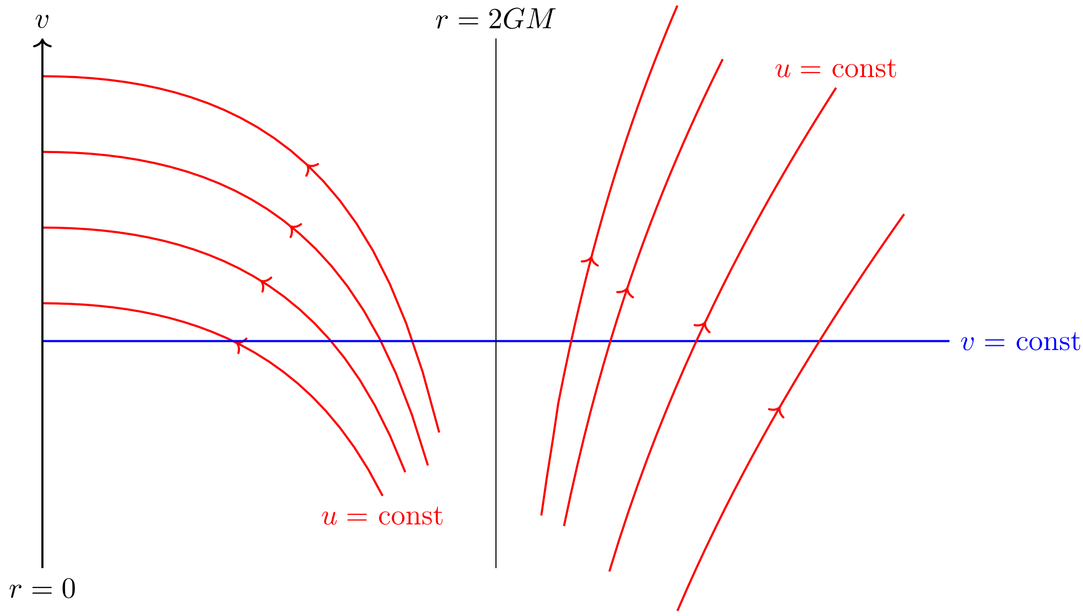


Figure 20: Radial null geodesics in ingoing EF coordinates.

The extension of the Schwarzschild spacetime beyond the $r = 2GM$ coordinate singularity of the original metric reveals that an infalling observer will experience nothing unusual at $r = 2GM$. However, as we have seen at the start of this section, to an observer at $r > 2GM$ they will appear never to reach $r = 2GM$, owing to the infinite time delay (or redshift) of signals sent from $r = 2GM$ to infinity. Furthermore, no signals sent from $r < 2GM$ will ever make it past $r = 2GM$, and once the infalling observer has reached the $r < 2GM$ region they will inexorably hit the singularity in finite proper time: shortly before which they will be ripped apart by infinitely strong tidal forces as the curvature blows up. Alas, poor Alice (or Bob).

So, despite the fact the infalling observer sees nothing unusual at $r = 2GM$, this still marks an important feature of the spacetime. The region with $r < 2GM$ is a *black hole*: a region from which no signals can reach infinity. The surface $r = 2GM$ is called an *event horizon*. This surface prevents observers remaining at $r > 2GM$ from gaining any information about what happens inside the black hole. The existence of an event horizon is a *global* property of spacetime, i.e. one needs to know the complete causal evolution of the geometry to know if there is indeed such a horizon.

Note that everything we have said so far is absolutely true in classical physics. The interplay between black holes and quantum physics leads to a more subtle situation. In particular, quantum mechanics implies that black holes are not absolutely black but in fact radiate. It is an open question to understand exactly how, or indeed more dramatically if, information about matter that has fallen into the black hole is contained in black hole radiation. As one approaches the curvature singularity, the strength of the gravitational field blows up, and it should be expected that "quantum gravity" effects will become imporant and hopefully rid us of this meddlesome infinity. It is another open question as to whether quantum gravity further implies that the structure of a black hole is in fact modified at the horizon itself, as has been argued from various viewpoints.

In short, black holes are very interesting and a sure sign that we do not completely understand gravity. Unfortunately, going into more details is beyond the scope of this course. As a result, we will now dive back into some further changes of coordinates.

We could have chosen to use u instead of v.

The Schwarzschild solution (outgoing Eddington-Finkelstein coordinates)

The Schwarzschild metric in coordinates (u, r, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) du^2 - 2dudr + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9.61}$$

where $u \in (-\infty, \infty)$, $r \in (0, \infty)$.

This gives a *different* extension of Schwarzschild. One way to see this is to realise that now outgoing radial null geodesics with $u =$ constant extend from $r = 0$ out past $r = 2GM$ and to infinity, whereas in the coordinates (t, v) and metric (9.61), the outgoing radial null geodesics could never pass from the $r < 2GM$ to $r > 2GM$ region. Furthermore, ingoing null geodesics that start from $r > 2GM$ never reach $r < 2GM$. Thus in all respects, this extension of Schwarzschild is the opposite of the one previously considered. We could say that it describes a *white hole*, rather than a black hole.

The extension using the ingoing Eddington-Finkelstein coordinates (v, r) in fact follows radial null geodesics into the future, whereas the outgoing Eddington-Finkelstein coordinates (u, r) in fact follows outgoing radial geodesics into the past.

Kruskal-Szekeres coordinates

To understand the complete structure of the Schwarzschild spacetime, we consider first using both u and v as coordinates instead of t and r.

The Schwarzschild solution (ingoing & outgoing Eddington-Finkelstein coordinates)

The Schwarzschild metric in coordinates (u, v, θ, ϕ) is

$$ds^2 = -\left(1 - \frac{2GM}{r(u,v)}\right) dudv + r^2(u,v)(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9.62}$$

where $u, v \in (-\infty, \infty)$, and $r(u, v)$ is defined by $\frac{1}{2}(v - u) = r_*(r)$, using (9.54).

In these coordinates, the point $r = 2GM$ is now at $v = -\infty$ or $u = +\infty$, where the metric degenerates. To bring this point in to a finite coordinate value, and to remove the coordinate

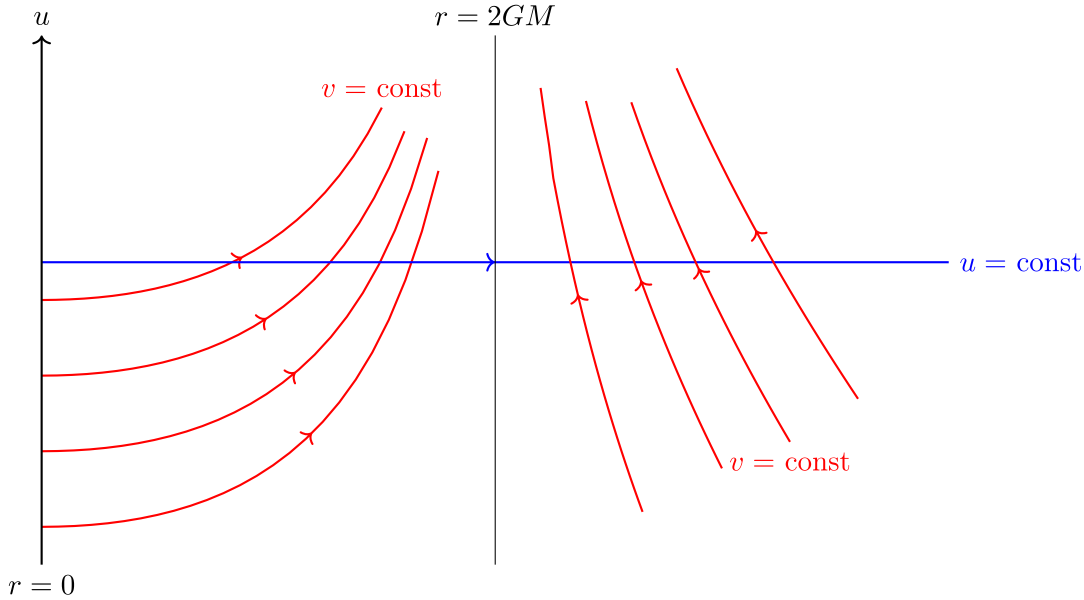


Figure 21: Radial null geodesics in outgoing EF coordinates.

singularity, we define *Kruskal-Szekeres* coordinates

$$U = -e^{-\frac{u}{4GM}}\,, \quad V = e^{\frac{v}{4GM}}\,, \tag{9.63}$$

such that $U < 0$ and $V > 0$. Note that

$$UV = -e^{\frac{r_*}{2GM}} = -e^{\frac{r}{2GM}}\left(\frac{r}{2GM} - 1\right)\,, \tag{9.64}$$

$$\frac{V}{U} = -e^{\frac{t}{2GM}}\,. \tag{9.65}$$

The Schwarzschild solution (Kruskal-Szekeres coordinates)

The Schwarzschild metric in coordinates (U, V, θ, ϕ) is

$$ds^2 = -\frac{32(GM)^3}{r(U,V)}e^{-\frac{r(U,V)}{2GM}}dUdV + r^2(U,V)(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{9.66}$$

where $r(U, V)$ is defined implicitly by equation (9.64).

Given this form of the metric, we extend the coordinate ranges of (U, V) to also include $U \geq 0$ and $V \geq 0$, *defining* the function $r(U, V)$ appearing in the metric by (9.64) for arbitrary values of U and V. The metric (9.66) then describes the *maximal extension* of the original Schwarzschild spacetime geometery.

The coordinate singularity at $r = 2GM$ appears at $UV = 0$, from (9.64), thus either at $U = 0$ or $V = 0$. Meanwhile the genuine curvature singularity at $r = 0$ appears at $UV = 1$, which has two branches (U, V either both positive or both negative).

The maximally extended Schwarzschild spacetime is illustrated in figure 22. This is known as the *Kruskal diagram* of Schwarzschild. We see that the spacetime divides into four regions:

- Region I: $U < 0$, $V > 0$. This is the region covered by the original Schwarzschild coordinates.

- Region II: $U > 0, V > 0$. This is the black hole region discovered by going to ingoing Eddington-Finkelstein coordinates (which cover regions I and II).
- Region III: $U < 0, V < 0$. This is the white hole region discovered by going to outgoing Eddington-Finkelstein coordinates (which cover regions I and III).
- Region IV: $U > 0, V < 0$. This a new region, which is again asymptotically flat, which appears here for the first time. It cannot be reached from the original region I.

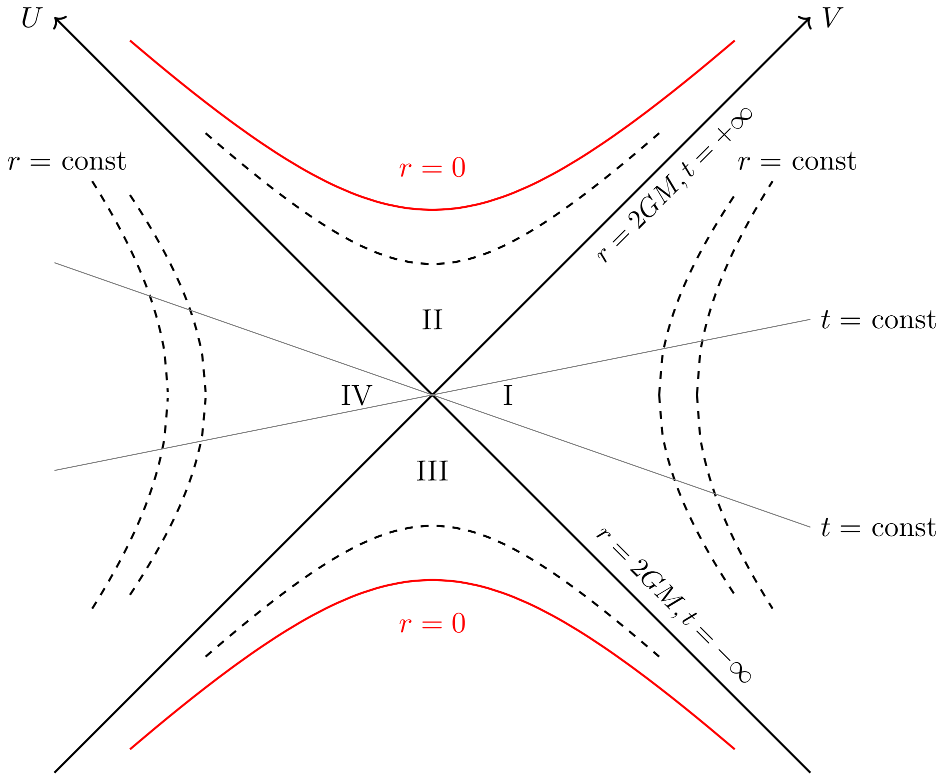


Figure 22: Kruskal diagram of Schwarzschild. Dashed lines are lines of constant r; thin lines are lines of constant t.

Technically, the coordinates U and V are the lightcone version of the Kruskal-Szekeres coordinates, and we could define T and R such that

$$U = T - R\,, \quad V = T + R\,. \tag{9.67}$$

In this case, T would appear as the vertical axis on the diagram 22, and R as the horizontal. Observe that constant V corresponds to ingoing radial null geodesics and constant U to outgoing null geodesics.

Black holes and stellar collapse

The solution depicted in figure 22 is sometimes known as the eternal black hole. It can be viewed as an idealised solution of the vacuum Einstein equations in that it describes an entire "universe" with no matter and with an already intricate causal structure involving the white hole in the far past and the black hole in the far future. We do not expect this full solution to be physically relevant.

Astrophysical black holes are the result of stellar collapse. When stars use up their available fuel, they begin to collapse under the gravitational attraction of their constituent matter. This gravitational attraction can be balanced by pressure.

We depict the modification to the Kruskal diagram corresponding to stellar collapse in figure 23. The dashed line corresponds to the surface of the star, and the yellow region to the star interior. This region is described by some different solution to the Einstein equations: Birkhoff's theorem guarantees that the exterior region is still Schwarzschild.

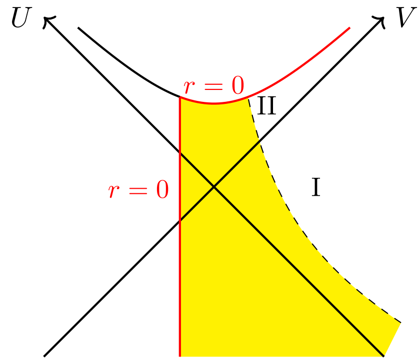


Figure 23: Kruskal diagram of a collapsing star.

10 Cosmology

We now turn to solutions of the Einstein equations that describe the whole universe. Again, we will appeal to symmetry on physically reasonably grounds to help us. The basic idea we will use is the modern version of the Copernican principle, namely, that not only is the position of planet Earth not special, nowhere is special. This means that we observe space to be *isotropic* (looks the same in all directions) and to be *homogeneous* (to have the same metric everywhere).

Isotropic

A manifold with metric (M, g) is *isotropic at the point* $p \in M$ if given (unit vectors) $X, Y \in T_pM$ there is an isometry ϕ such that $\phi(p) = p$ and for which the vector X is *pushed forward* into Y, i.e. $\phi_\star(X) = Y$, where $\phi_\star(X)(f) = X(f \circ \phi)$.

Viewed as a change of coordinates, this means given coordinates x^μ around p, we can find an isometry $\phi : x^\mu \mapsto y^\mu(x)$ such that $X^\mu \mapsto (\partial y^\mu/\partial x^\nu)X^\nu = Y^\mu$.

Homogeneous

A manifold with metric (M, g) is homogeneous if for all points $p, q \in M$, there exists an isometry ϕ such that $\phi(p) = q$.

A manifold which is isotropic at all points p is homogeneous; conversely a manifold which is homogeneous and isotropic at one point is isotropic at all points. In general though there need not be a connection between the two properties.

The spacetime manifold that describes our universe is assumed to be isotropic and homogeneous in the spatial directions only: observations reveal that the universe certainly is not the same in both the past and future directions, for instance, as it is expanding. However, the spacetimes that are fully isotropic and homogeneous are of great interest and importance in cosmology and theoretical physics more generally. Therefore we will first describe these.

10.1 Maximally symmetric spaces

A manifold that is both homogeneous and isotropic everywhere is *maximally symmetric*. This means that it has the same number of isometries as flat space, $\mathbb{R}^n$, with the usual Euclidean metric. The isometries of the latter are n translations and $\frac{1}{2}n(n-1)$ rotations. Intuitively, homogeneity corresponds to invariance under translations, while isotropy corresponds to invariance under rotations.

The Riemann curvature tensor of a maximally symmetric space is:

$$R_{\mu\nu\rho\sigma} = \kappa(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho})\,, \qquad \kappa \equiv \frac{R}{n(n-1)}\,, \tag{10.1}$$

where κ (and hence the Ricci scalar) is constant. Therefore they are also examples of a *space of constant curvature*.

Euclidean maximally symmetric spaces: flat space, spheres and hyperbolic space

The basic examples of such spaces, with Euclidean metric, for $\kappa = 0$, $\kappa < 0$ and $\kappa > 0$, are:

1. ($\kappa = 0$) flat space $\mathbb{R}^n$ with the flat metric.

2. ($\kappa > 0$) the sphere S^n of radius $L > 0$, defined by the equation

$$(x^1)^2 + \ldots (x^{n+1})^2 = L^2 \,, \tag{10.2}$$

   in $\mathbb{R}^{n+1}$. The curvature is given in terms of $\kappa = \frac{1}{L^2}$.

3. ($\kappa < 0$) the hyperbolic space H^n, defined by the equation

$$-(x^0)^2 + (x^1)^2 + \ldots (x^n)^2 = -L^2 \,, \tag{10.3}$$

   in $\mathbb{R}^{n+1}$ with the Minkowski metric. This gives a double-sheeted hyperboloid, and we take H^n to correspond to one of the sheets. The curvature is given in terms of $\kappa = -\frac{1}{L^2}$.

For cosmological applications, we will need to know the $n = 3$ examples. For the three-sphere S^3, define coordinates on the surface (10.2) by

$$\begin{aligned} x^1 &= L \sin\chi \sin\theta \cos\phi \,, \\ x^2 &= L \sin\chi \sin\theta \sin\phi \,, \\ x^3 &= L \sin\chi \cos\theta \,, \\ x^4 &= L \cos\chi \,, \end{aligned} \tag{10.4}$$

where $\chi \in (0, \pi)$, $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$. The flat metric evaluated on these coordinates gives the metric on the three-sphere:

$$ds^2 = L^2 \left(d\chi^2 + \sin^2\chi (d\theta^2 + \sin^2\theta d\phi^2) \right) \,. \tag{10.5}$$

(As for the two-sphere, these coordinates do not cover the whole manifold, but one can easily introduce additional charts.)

For the hyperbolic space H^3, define coordinates on the surface (10.3) by

$$\begin{aligned} x^0 &= L \cosh\chi \,, \\ x^1 &= L \sinh\chi \sin\theta \cos\phi \,, \\ x^2 &= L \sinh\chi \sin\theta \sin\phi \,, \\ x^3 &= L \sinh\chi \cos\theta \,, \end{aligned} \tag{10.6}$$

where $\chi \in (0, \infty)$, $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$. The flat metric evaluated on these coordinates gives the metric on the hyperbolic space:

$$ds^2 = L^2 \left(d\chi^2 + \sinh^2\chi (d\theta^2 + \sin^2\theta d\phi^2) \right) \,. \tag{10.7}$$

(Again, these coordinates do not cover the whole manifold, but one can easily introduce additional charts.)

Lorentzian maximally symmetric spacetimes: Minkowski spacetime, de Sitter and anti-de Sitter

First, let's consider the Einstein equation for a maximally symmetric spacetime (in four dimensions). The Einstein tensor, from contracting (10.1), is $G_{\mu\nu} = -3\kappa g_{\mu\nu}$. Therefore such a space solves the Einstein equation with cosmological constant and vanishing energy-momentum tensor, $G_{\mu\nu} + \Lambda g_{\mu\nu} = 0$, with $\Lambda = 3\kappa$. We conclude that such spacetimes are *vacuum solutions* of the Einstein equations, with a non-zero cosmological constant.

Depending now on the sign of Λ, we have the following basic examples (here we specialise to four-dimensional spacetimes, on grounds of immediate physical applicability, but these can all be defined in general dimensions).

1. ($\Lambda = 0$) flat spacetime $\mathbb{R}^n$ with the flat Minkowski metric

2. ($\Lambda < 0$) anti-de Sitter (adS) spacetime[4], defined by the equation

$$- (x^0)^2 - (x^5)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = -L^2 \,, \tag{10.8}$$

   in (a fictitious) $\mathbb{R}^5$ with indefinite signature $(2, 3)$. This is invariant under $SO(2, 3)$ "Lorentz" transformations of the higher dimensional space. This gives the required $\frac{1}{2}5(5-1) = 10$ isometries of a four-dimensional maximally symmetric spacetime. Define coordinates on this curve (10.8) by

$$\begin{aligned} x^0 &= L \sin t \cosh \rho \,, \\ x^5 &= L \cos t \cosh \rho \,, \\ x^1 &= L \sinh \rho \cos \theta \,, \\ x^2 &= L \sinh \rho \sin \theta \cos \phi \,, \\ x^3 &= L \sinh \rho \sin \theta \sin \phi \,, \end{aligned} \tag{10.9}$$

   with $\rho \in (0, \infty)$, (θ, ϕ) the usual two-sphere coordinates, and naively a periodic time coordinate, $t \in [0, 2\pi]$. This gives however a time independent metric:

$$ds^2 = L^2 \left(- \cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\Omega_2^2 \right) \,. \tag{10.10}$$

   It therefore makes sense to extend the range of the timelike direction to $t \in (-\infty, \infty)$. This avoids encountering the causal difficulty of closed timelike curves. The cosmological constant is $\Lambda = -\frac{3}{L^2}$.

3. ($\Lambda > 0$) de Sitter (dS) spacetime, defined by the equation

$$- (x^0)^2 + (x^1)^2 + \cdots + (x^4)^2 = L^2 \,, \tag{10.11}$$

   in (a fictitious) $\mathbb{R}^5$ with the Minkowski metric. This is invariant under $SO(1, 4)$ Lorentz transformations of the higher dimensional space. This gives the required $\frac{1}{2}5(5-1) = 10$ isometries of a four-dimensional maximally symmetric spacetime. Define coordinates on

[4]The spacetime that launched 17,000 papers (since 1997, when many readers of these notes presumably did not exist).

the curve (10.11) by

$$
\begin{aligned}
x^0 &= L \sinh \tfrac{t}{L} \,, \\
x^1 &= L \cosh \tfrac{t}{L} \sin\chi \sin\theta \cos\phi \,, \\
x^2 &= L \cosh \tfrac{t}{L} \sin\chi \sin\theta \sin\phi \,, \\
x^3 &= L \cosh \tfrac{t}{L} \sin\chi \cos\theta \,, \\
x^4 &= L \cosh \tfrac{t}{L} \cos\chi \,,
\end{aligned}
\tag{10.12}
$$

where $t \in (-\infty, \infty)$, and (χ, θ, ϕ) are S^3 coordinates. The metric of de Sitter spacetime is then:

$$
ds^2 = -dt^2 + L^2 \cosh^2 \tfrac{t}{L} (d\chi^2 + \sin^2\chi d\Omega^2) \,. \tag{10.13}
$$

Note that the part of the metric multiplied by $\cosh^2 \frac{t}{L}$ is the S^3 metric (10.5). The cosmological constant is $\Lambda = \frac{3}{L^2}$.

10.2 Penrose diagrams

The *causal structure* of spacetimes can be analysed using *Penrose diagrams*. The idea here is to find coordinate transformations which map a given metric to an overall scale factor times some simpler metric with the same causal structure – i.e. the same light cones – as the original one. We can express this by writing the simpler (but unphysical) metric as $\bar{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}$, where $g_{\mu\nu}$ is the original metric we are analysing, and $\Omega^2 > 0$ is the scale or conformal factor. Positivity of Ω^2 implies that curves that are timelike, spacelike or null in one metric remain so in the other.

For the dS metric, (10.13), define a new time coordinate η by

$$
\cosh \tfrac{t}{L} = \frac{1}{\cos\eta} \,, \tag{10.14}
$$

with $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Then the metric becomes

$$
ds^2 = \frac{L^2}{\cos^2\eta} \left(-d\eta^2 + d\chi^2 + \sin^2\chi d\Omega^2 \right) \,. \tag{10.15}
$$

Using $\Omega^2 = L^{-2} \cos^2\eta$ we find the unphysical metric

$$
d\bar{s}^2 = -d\eta^2 + d\chi^2 + \sin^2\chi d\Omega^2 \,. \tag{10.16}
$$

In fact, this metric is also a (non-vacuum) solution to the Einstein equations, and with the coordinate range of η extended to $\eta \in (-\infty, \infty)$ is known as the *Einstein static universe.* We have found that de Sitter spacetime is conformal to the patch of the Einstein static universe with $\eta \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

We can depict this patch as a two-dimensional diagram, with axes η and χ, and each point representing a two-sphere of radius $\sin^2\chi$. Lines at forty five degrees represent null paths (at constant θ and ϕ).

So, what can we infer about the causal structure of de Sitter? Let's send in Alice and Bob, as shown in figure 25. Suppose Alice sits (without loss of generality) at $\chi = 0$, so that her worldline is a straight vertical line on the Penrose diagram. Meanwhile Bob follows a similar vertical worldline at a different value of χ.

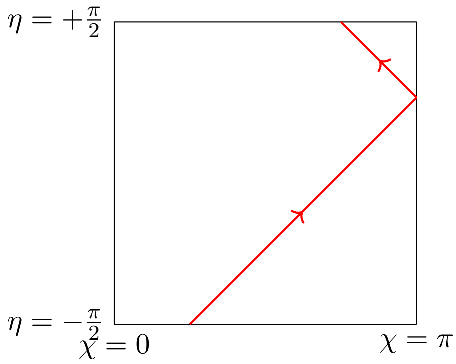


Figure 24: Penrose diagram of de Sitter

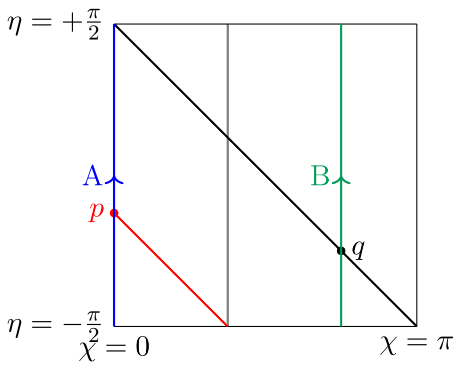


Figure 25: Horizons in de Sitter: the black line is the cosmological horizon of the observer A, and the vertical gray line is the particle horizon at p.

At the point p, Alice can only have received signals from her past lightcone at p, with boundary defined by $\eta = \eta_p - \chi$. We define the *particle horizon* at p as the boundary of the region containing points from a which a signal can be sent to reach p. It is the distance beyond which Alice cannot "see" at p. Note that any signals sent by Bob at sufficiently early times will lie beyond the particle horizon at p; at p Alice has no information about the existence, thoughts, doubts, prayers, hopes, fears or other attributes of Bob.

Furthermore, by the time Alice reaches future infinity at $\eta = +\frac{\pi}{2}$, she can only have received signals from events that occurred within the region bounded by the *cosmological horizon* (the black line at 45 degrees in figure 25, running from $(\eta, \chi) = (-\frac{\pi}{2}, \pi)$ to $(\eta, \chi) = (+\frac{\pi}{2}, 0)$). Alice will receive no signal from events beyond this horizon no matter how long she waits. In particular, any signals sent by Bob from beyond the point q, when his worldline intersects Alice's cosmological horizon, will never reach Alice; and indeed a signal from q will take infinitely long to reach Alice.

Now let's turn to anti-de Sitter spacetime. This time we make the following coordinate transformation:

$$\cosh \rho = \frac{1}{\cos \chi}\,, \tag{10.17}$$

such that the AdS metric (10.10) becomes

$$ds^2 = \frac{L^2}{\cos^2 \chi}\left(-dt^2 + d\chi^2 + \sin^2 \chi d\Omega^2\right)\,. \tag{10.18}$$

Defining $\Omega^2 = L^{-2}\cos^2 \chi$ we see that the AdS metric is also conformal to a patch of the Einstein static universe, but now with $0 < \chi < \frac{\pi}{2}$ and $-\infty < t < \infty$.

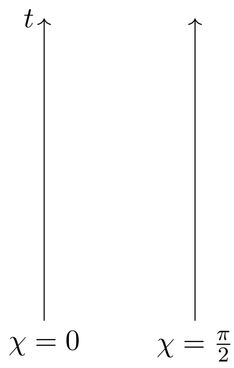


Figure 26: Penrose diagram of anti-de Sitter

The Penrose diagram is illustrated, somewhat minimally, in figure 26. Notice that spatial infinity, at $\chi = \frac{\pi}{2}$, is a timelike surface.

10.3 FLRW solutions

Spacetimes that describe realistic models of the universe can be found by assuming that only spatial hypersurfaces are homogeneous and isotropic. A spatial hypersurface is defined by the requirement that any vector tangent to it is spacelike. The spacetimes we are interested in can be viewed as a foliation $\mathbb{R} \times M_3$, where the time direction is labelled by $\mathbb{R}$ and for each value of time we have a spatial three-dimensional manifold M_3, which is maximally symmetric.

We can choose coordinates such that the metric is

$$ds^2 = -dt^2 + a^2(t)\gamma_{ij}(x)dx^i dx^j \,. \tag{10.19}$$

These are known as *comoving coordinates*. An observer is comoving if their four-velocity never has a component tangent to the spatially homogeneous slices; that is, their four-velocity must be (proportional to) the unit normal to these surfaces. In the comoving coordinates used in the metric (10.19), this unit normal is just $u = \partial/\partial t$. The worldline of a comoving observer is thus specified by $t = \tau$, $x^i =$ constant. Note that it is only comoving observers which see the spatial hypersurfaces as being isotropic. Observers whose worldlines involve non-constant x^i pick out a preferred direction in the spatial hypersurface by virtue of their non-zero velocity within it.

The spatial metric γ_{ij} being a three-dimensional homogeneous and isotropic metric is thus a metric of constant curvature, and can always be (locally) put into the form of either the metric on the three-sphere, hyperbolic space, or flat space. Explicitly,

$$ds^2 = -dt^2 + a^2(t)d\sigma_k^2\,, \quad d\sigma_k^2 \equiv \frac{dr^2}{1-kr^2} + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{10.20}$$

where $k = 1$ corresponds to the three-sphere S^3, $k = 0$ to flat space, and $k = -1$ to the hyperbolic space H^3. To see this, define a new coordinate χ such that

$$d\chi = \frac{dr}{\sqrt{1-kr^2}} \Rightarrow r = \begin{cases} \sin\chi & k = 1\,, \\ \chi & k = 0\,, \\ \sinh\chi & k = -1\,. \end{cases} \tag{10.21}$$

Then the spatial part of the metric (10.20) is seen to correspond to the metrics (10.5) and (10.7) for $k = \pm 1$, and to the three-dimensional flat metric in spherical coordinates for $k = 0$, with the overall scale set by the function $a(t)$. This function is therefore known as the *scale factor*.

The metric (10.20) is called the FLRW universe, after Friedmann, Lemaitre[5], Robertson and Walker.

Exercise 10.1 (FLRW curvature)

Calculate the Christoffel symbols and hence Ricci tensor and Ricci scalar of the FLRW metric (10.20).

We want the FLRW metric to solve the Einstein equations for some realistic energy-momentum tensor. The Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$ follows from the Ricci tensor,

$$R_{00} = -\frac{3\ddot{a}}{a}\,, \quad R_{0i} = 0\,, \quad R_{ij} = (a\ddot{a} + 2\dot{a}^2 + 2k)\gamma_{ij}\,, \tag{10.22}$$

and Ricci scalar

$$R = 6\left(\frac{\ddot{a}}{a} + \left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)\,, \tag{10.23}$$

where a dot denotes the derivative with respect to t. Hence

$$G_{00} = 3\left(\left(\frac{\dot{a}}{a}\right)^2 + \frac{k}{a^2}\right)\,, \quad G_{0i} = 0\,, \quad G_{ij} = -(2a\ddot{a} + \dot{a}^2 + k)\gamma_{ij}\,. \tag{10.24}$$

A natural candidate for the energy-momentum tensor sourcing the FLRW universe is that of a perfect fluid, with

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + p g_{\mu\nu}\,, \tag{10.25}$$

and with the fluid at rest in co-moving coordinates, $U^\mu = (1, 0, 0, 0)$.

This immediately gives the Friedmann equations for the time evolution of the FLRW universe.

Friedmann equations

The Einstein equations for an FLRW metric sourced by a comoving perfect fluid are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{10.26}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{10.27}$$

In practice, we only really need (10.26), as it is sufficient to determine a once we also take into account conservation of the energy-momentum tensor, $\nabla_\nu T^{\mu\nu} = 0$, implying

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0\,, \tag{10.28}$$

[5]Belgian physicist and Catholic priest.

or

$$\frac{d}{dt}(a^3\rho) = -p\frac{d}{dt}(a^3)\,. \tag{10.29}$$

The three main examples of a perfect fluid that are relevant cosmologically are matter, radiation, and vacuum energy. Each of these has an equation of state

$$p = w\rho\,, \quad w = \begin{cases} 0 & \text{matter}\,, \\ \frac{1}{3} & \text{radiation}\,, \\ -1 & \text{vacuum energy}\,. \end{cases} \tag{10.30}$$

Equation (10.28) becomes $\dot\rho/\rho = -3(1+w)\dot a/a$, hence

$$\rho(t) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)}\,, \tag{10.31}$$

where the subscript 0 refers to the quantity at the present time.

As the universe expands, $a(t)$ grows and we see that the energy density of the different example fluids behaves in different ways. For matter, $\rho_m \sim a^{-3}$ (a scaling by the increased volume). For radiation, $\rho_r \sim a^{-4}$ (a scaling by the increased volume plus an extra redshift effect). For vacuum energy, $\rho_\Lambda \sim a^0$. Hence, in the far future ρ_Λ (if present) will dominate over matter and radiation.

The ratio of $\dot a$ to a plays an important role in cosmology. It measure the rate of change of the scale factor.

Hubble parameter

We define the *Hubble parameter* to be

$$H = \frac{\dot a}{a}\,. \tag{10.32}$$

The value of H at the present, H_0, is known as Hubble's constant.

Consider two comoving particles (representing individual galaxies). At time t, their proper distance is $d = a(t)R$, where R denotes their distance measured using the spatial metric γ_{ij}. The rate of change of proper distance is $\dot d = \dot a R = \dot a a R/a = Hd$. Thus the relative velocity of the two galaxies is proportional to their distance, with the proportionality "constant" the Hubble parameter. This is *Hubble's law.*

Another useful quantity involves the density. We define the *density parameter* to be

$$\Omega = \frac{8\pi G}{3H^2}\rho \equiv \frac{\rho}{\rho_{\text{crit}}}\,. \tag{10.33}$$

We can rewrite the Friedmann equation (10.26) as

$$\Omega - 1 = \frac{k}{H^2a^2}\,, \tag{10.34}$$

so that the sign of k, and hence the precise nature of the FLRW metric, is determined by the value of Ω and hence of $\rho_{\text{crit}} = \frac{3H^2}{8\pi G}$. Specifically:

- if $\rho < \rho_{\text{crit}}$ then $\Omega < 1$ and $k = -1$. In this case, we say the universe is *open* (the spatial hypersurfaces are hyperbolic).
- if $\rho = \rho_{\text{crit}}$ then $\Omega = 1$ and $k = 0$. In this case, we say the universe is *flat* (the spatial hypersurfaces are flat).
- if $\rho > \rho_{\text{crit}}$ then $\Omega > 1$ and $k = +1$. In this case, we say the universe is *closed* (the spatial hypersurfaces are spheres).

Current measurements suggest that the current value of Ω is approximately 1.

Let's consider a universe which is dominated by one of matter, radiation or vacuum energy. The Friedmann equation is:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} - \frac{k}{a^2}\,. \tag{10.35}$$

Λ-dominated universes ($\Lambda > 0$)

Let's start in a universe containing only a (positive) cosmological constant. If the universe keeps expanding, this will correspond to the far future of our universe. Then $w = -1$ and we have

$$\dot{a}^2 = C^2\left(a^2 - \frac{k}{C^2}\right)\,, \quad C^2 \equiv \frac{8\pi G\rho_0}{3}\,. \tag{10.36}$$

Let's solve this for a closed universe, $k = 1$. After integrating we have

$$a(t) = \frac{A}{2}\left(e^{Ct} + \frac{1}{C^2A^2}e^{-Ct}\right)\,, \tag{10.37}$$

where A is a constant of integration. Shifting $t \to t - C^{-1}\ln(AC)$ we can write this as

$$a(t) = \frac{1}{C}\cosh(Ct)\,. \tag{10.38}$$

The FLRW metric is thus

$$ds^2 = -dt^2 + \frac{1}{C^2}\cosh^2(Ct)d\sigma^2_{k=1}\,, \tag{10.39}$$

where $d\sigma^2_{k=1}$ is the metric on the sphere S^3. This is the metric of de Sitter spacetime, with $L = 1/C$.

In fact, for $k = 0$ and $k = -1$, the FLRW solution also turns out to correspond to de Sitter, in coordinates giving what are known as the "flat slicing" and "open slicing".

Exercise 10.2 (Flat slicing of de Sitter)

Consider the definition of the de Sitter spacetime as the hyperboloid (10.11). Define the following parametrisation:

$$x^0 = L\sinh\frac{\hat{t}}{L} + \frac{r^2}{2L}e^{\frac{\hat{t}}{L}}\,, \quad x^4 = L\cosh\frac{\hat{t}}{L} - \frac{r^2}{2L}e^{\frac{\hat{t}}{L}}\,, \quad x^i = e^{\frac{\hat{t}}{L}}\hat{x}^i\,, \tag{10.40}$$

where $r^2 \equiv \delta_{ij}\hat{x}^i\hat{x}^j$, $i, j = 1, 2, 3$.

1. Show that these coordinates indeed obey the defining equation (10.11).

2. Show that they give the metric for de Sitter spacetime as

$$ds^2 = -d\hat{t}^2 + e^{2\frac{\hat{t}}{L}}\delta_{ij}d\hat{x}^i d\hat{x}^j \,. \tag{10.41}$$

   This is called the *flat slicing*, because surfaces of constant $\hat{t}$ are flat.

3. Show that the flat FLRW solution for a universe dominated by a positive cosmological constant corresponds to the flat slicing of de Sitter.

4. The coordinates (10.40) are only defined for $x^0 + x^4 \geq 0$. By equating (10.40) to the original coordinates used, (10.12), (using spherical polar coordinates for the $\hat{x}^i$), show that on the Penrose diagram, this corresponds to the left upper triangular region $\eta \geq \chi - \frac{\pi}{2}$.

Exercise 10.3 (Open slicing of de Sitter)

Consider the definition of the de Sitter spacetime as the hyperboloid (10.11). Define the following parametrisation, with (here $\xi \in (0, \infty)$):

$$\begin{aligned}
x^0 &= L \sinh \frac{\tilde{t}}{L} \cosh \xi \,, \quad x^4 = L \cosh \frac{\tilde{t}}{L} \,, \\
x^1 &= L \sinh \frac{\tilde{t}}{L} \sinh \xi \sin\theta \cos\phi \,, \\
x^2 &= L \sinh \frac{\tilde{t}}{L} \sinh \xi \sin\theta \sin\phi \,, \\
x^3 &= L \sinh \frac{\tilde{t}}{L} \sinh \xi \cos\theta \,.
\end{aligned} \tag{10.42}$$

1. Show that these coordinates indeed obey the defining equation (10.11).

2. Show that they give the metric for de Sitter spacetime as

$$ds^2 = -d\tilde{t}^2 + L^2 \sinh^2 \frac{\tilde{t}}{L} \left(d\xi^2 + \sinh^2 \xi \Omega_2^2 \right) \,. \tag{10.43}$$

   This is called the *open slicing*, because surfaces of constant $\tilde{t}$ are the hyperbolic space H^3, as in (10.7).

3. Show that the open FLRW solution for a universe dominated by a positive cosmological constant corresponds to the open slicing of de Sitter.

4. By equating (10.42) to the original coordinates used, (10.12), show that on the Penrose diagram, this corresponds to the left upper triangular region $\eta \geq \chi$.

Flat matter or radiation dominated universes

Now let's consider flat universes, $k = 0$ (which should be a good approximation to our universe at the present time). For $w > -1$, we have

$$\dot{a} = C a_0^{\frac{3}{2}(1+w)} a^{-\frac{1}{2}(1+3w)}, \tag{10.44}$$

hence

$$da\, a^{\frac{1}{2}+\frac{3w}{2}} = C a_0^{\frac{3}{2}(1+w)} dt \tag{10.45}$$

so

$$a(t) = a_0 \left(C\frac{3}{2}(1+w)(t-t_0) + 1 \right)^{\frac{2}{3(1+w)}}. \tag{10.46}$$

Let's run the evolution of this scale factor backwards into the past. When $t = t_0 - \left(C\frac{3}{2}(1+w)\right)^{-1}$, we find that $a = 0$. Using

$$\rho(t) = \rho_0 \left(\frac{a}{a_0}\right)^{-3(1+w)} = \rho_0 \left(C\frac{3}{2}(1+w)(t-t_0) + 1 \right)^{-2}, \tag{10.47}$$

we also see that the energy density blows up at this time. So the size of the universe shrinks to zero, $a \to 0$ and the energy density (which is by definition a scalar function and so invariant in all coordinates) is singular, $\rho \to \infty$. It is convenient to shift the origin of our time coordinate such that this corresponds to $t = 0$. Then the singularity we have found in our solution is known as the *big bang*, and corresponds to the beginning of the universe (as the solution is singular at the time of the big bang, we cannot extend our cosmology further into the past "before" this moment). It has been proven in *cosmological singularity theorems* that any universe which is approximately homogeneous and isotropic must have such a singularity in its past (assuming reasonable conditions on the energy-momentum tensor).

Conformal time and general universes

In order to solve the Friedmann equation for a closed or open radiation or matter dominated universe, and to investigate the general causal structure of FLRW universes, we introduce the *conformal time*.

Conformal time

The *conformal time* η in an FLRW universe is defined by

$$\eta = \int^t \frac{dt'}{a(t')}, \tag{10.48}$$

and so obeys

$$d\eta = \frac{dt}{a}. \tag{10.49}$$

It is convenient to take $\eta = 0$ to correspond to the big bang singularity at $a = 0$.

In terms of conformal time, the Friedmann equation becomes

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G\rho_0}{3} a_0^{3(1+w)} a^{1-3w} - ka^2 \,. \tag{10.50}$$

This is straightforward to integrate for all the cases we are interested in. Once $a(\eta)$ is known then t as a function of η can be found by solving $dt/d\eta = a$. It is simple to do this yourself.

Exercise 10.4 (Matter dominated FLRW universes)

Consider the Friedmann equation (10.50) in terms of conformal time for a universe dominated by matter, $w = 0$. Letting $C^2 = \frac{8\pi G\rho_0}{3} a_0^3$ and taking $t = \eta = 0$ to correspond to the big bang, with $a(\eta = 0) = 0$, show that we have:

$$\begin{array}{lll} \text{Closed } (k=1) & a(\eta) = \frac{1}{2}C^2(1-\cos\eta) & t = \frac{1}{2}C^2(\eta - \sin\eta) \\ \text{Open } (k=-1) & a(\eta) = \frac{1}{2}C^2(\cosh\eta - 1) & t = \frac{1}{2}C^2(\sinh\eta - \eta) \\ \text{Flat } (k=0) & a(\eta) = \frac{1}{4}C^2\eta^2 & t = \frac{1}{12}C^2\eta^3 \,. \end{array} \tag{10.51}$$

Sketch the evolution of a with η in all cases, and observe whether the universe expands indefinitely, or eventually shrinks back to zero size in a "big crunch".

Exercise 10.5 (Radiation dominated FLRW universes)

Consider the Friedmann equation (10.50) in terms of conformal time for a universe dominated by radiation, $w = 1/3$. Letting $C^2 = \frac{8\pi G\rho_0}{3} a_0^4$ and taking $t = \eta = 0$ to correspond to the big bang, with $a(\eta = 0) = 0$, show that we have:

$$\begin{array}{lll} \text{Closed } (k=1) & a(\eta) = C\sin\eta & t = C(1-\cos\eta) \\ \text{Open } (k=-1) & a(\eta) = C\sinh\eta & t = C(\cosh\eta - 1) \\ \text{Flat } (k=0) & a(\eta) = C\eta & t = \frac{1}{2}C\eta^2 \,. \end{array} \tag{10.52}$$

Sketch the evolution of a with η in all cases, and observe whether the universe expands indefinitely, or eventually shrinks back to zero size in a "big crunch".

The FLRW metric in terms of conformal time is:

$$ds^2 = a^2(\eta)(-d\eta^2 + d\sigma_k^2) \,. \tag{10.53}$$

Thus the FLRW metric is conformal to the metric $-d\eta^2 + d\sigma_k^2$ and we can use this simpler metric to explore the causal structure of the universe. For instance, in an expanding universe with a positive cosmological constant, the scale factor goes like $a(t) \sim e^{Ct}$ for large t, when we can neglect matter and radiation. As a result $\eta \sim 1 - e^{-Ct}$ for large t and hence for $t \to \infty$, $\eta \to \eta_\infty$, where η_∞ is some constant value. In such a universe, the causal structure can be sketched as in 27.

This sort of picture is not a full Penrose diagram, as the spatial directions have not been

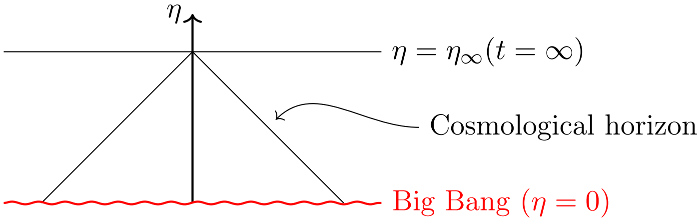


Figure 27: Sketch of causal structure for universe dominated by $\Lambda > 0$ at late times.

(fully) compactified, only the time direction. However, knowing that the universe either ends or starts at a finite value of conformal time allows us to draw interesting physical conclusions, such as the existence of a cosmological horizon for observers at late times in an expanding $\Lambda > 0$ dominated universe (as shown in figure 27).

The horizon problem and inflation

We can also consider the *particle horizons* that are present owing to the fact that only a finite amount of time, conformal or otherwise, has passed since the Big Bang. As past light cones terminate at the Big Bang, the region of space that we can have received signals from at the present time is limited by such a horizon. This leads to a puzzle, because the universe that we observe is spatially homogeneous and isotropic. In particular, we observe a cosmic microwave background (CMB) consisting of photons which stopped interacting with matter (in the form of protons and electrons) at the time of *recombination*, of order 10^5 years after the Big Bang. This background radiation is visible and remarkably uniform (at a temperature of 2.7K with very small deviations) across the whole sky. But the photons that we observe coming from different parts of the sky cannot have been in causal contact at the moment when they decoupled from matter and became part of the background radiation. This situation is drawn in figure 28.

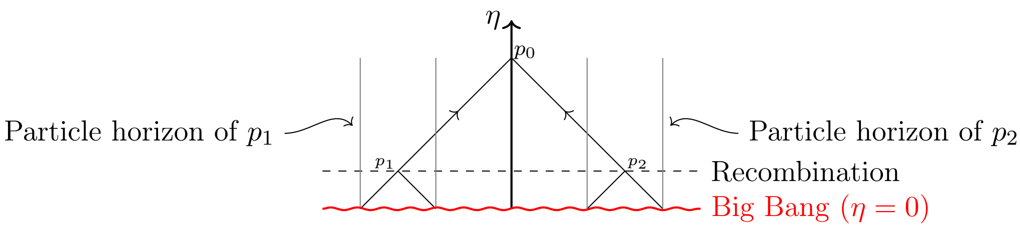


Figure 28: Particle horizons and the CMB. We observe (at p_0) photons emitted from p_1 and p_2, whose particle horizons are completely causally disconnected (no point can send a signal to both p_1 and p_2. Despite this, the photons constituting the CMB are incredibly closely correlated.

The most widely accepted solution to this problem (and a number of related cosmological issues) is *inflation*, which posits that there was a period of rapidly accelerating expansion ($\ddot{a} > 0$) in the very early universe. This period of rapid expansion allows p_1 and p_2 to have initially been in causal contact before being inflated apart, and accounts for the homogeneity and isotropy of the observed universe.

The inflationary period corresponds to a period of evolution of the universe which is not determined by the Friedmann equations. To see how this should work, assume that during inflation the scale factor increases exponentially:

$$a(t) \sim \bar{a} e^{Ht} , \tag{10.54}$$

where H and $\bar{a}$ are constant. The conformal time is then given by

$$\eta - \bar{\eta} = -\frac{1}{H\bar{a}} e^{-Ht} , \tag{10.55}$$

where $\bar{\eta}$ is again a constant. We can now have negative values of conformal time: $\eta \to -\infty$ corresponds to $t \to -\infty$. The scale factor in terms of conformal time is

$$a(\eta) = -\frac{1}{H(\eta - \bar{\eta})} . \tag{10.56}$$

The Big Bang singularity when $a \to 0$ occurs for $\eta \to -\infty$. This scale factor seems to blow up for $\eta \to \bar{\eta}$, however this corresponds to allowing inflation to continue indefinitely into the future, as $\eta \to \bar{\eta}$ corresponds to $t \to +\infty$. Instead we assume that inflation ends (i.e. our assumption that the scale factor takes the form (10.54) ceases to hold) at some point when $\eta \approx 0 < \bar{\eta}$, at which point we segue into standard cosmological evolution determined by the Friedmann equations. The effect is to modify the figure 28 to figure 29, in which we see that the past lightcones of the points p_1 and p_2 now do intersect (the shaded region in 29).

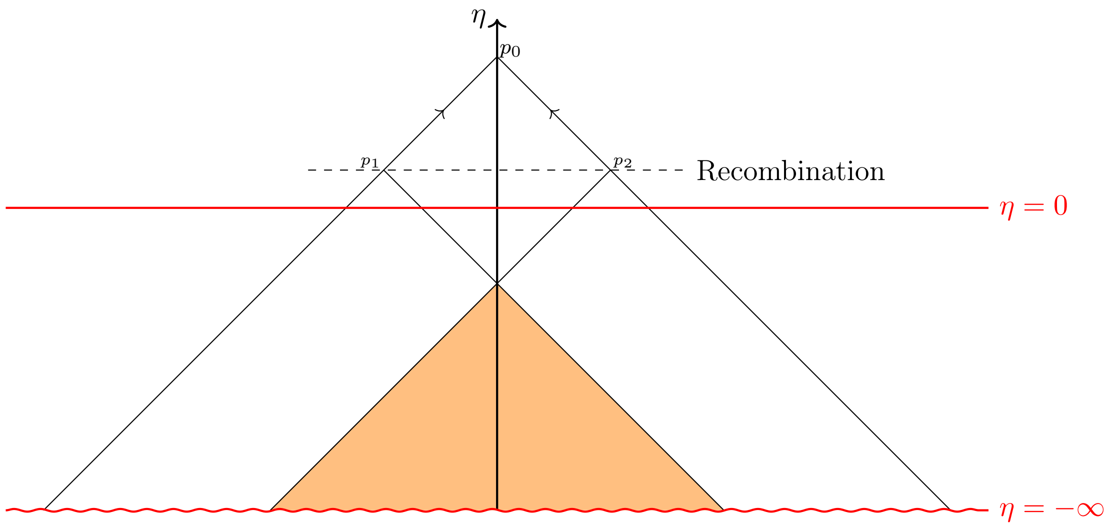


Figure 29: Particle horizons with inflationary period added.

Simple models of inflation can be obtained by introducing matter in the form of a scalar field, whose dynamics drive the accelerated expansion of the universe during the inflationary phase. To say more is beyond the scope of this course.

Cosmological redshifts

Finally, the metric (10.53) allows us to explore null geodesics. Consider the light emitted by some particular galaxy, or Bob, reaching an observer. Suppose a first photon is emitted at conformal time η_e and observed at conformal time η_0. Then a second photon is emitted at $\eta_e + \Delta\eta$ and observed at $\eta_0 + \Delta\eta$. As before, when we thought about gravitational time dilation in a time independent geometry, the paths of the two photons are identical and so the time between emission and observation is the same. However, the proper time at the emitter is $\Delta\tau_e = a(\eta_e)\Delta\eta$, assuming that $\Delta\eta$ is small enough that $a(\eta)$ does not change appreciably between the emissions. The proper time observed is similarly $\Delta\tau_o = a(\eta_o)\Delta\eta$. Consequently,

$$\frac{\Delta\tau_o}{\Delta\tau_e} = \frac{a(\eta_o)}{a(\eta_e)}\,, \tag{10.57}$$

which is greater than 1 for an expanding universe. In terms of wavelengths, $\lambda = c\Delta\tau$, and abbreviating $a(\eta_o) \equiv a_o$, $a(\eta_e) \equiv a_e$, we have

$$\frac{\lambda_o}{\lambda_e} = \frac{a_o}{a_e} > 1\,, \tag{10.58}$$

so the light received has longer wavelength, hence is redshifted. In cosmology, the *redshift* z is defined by

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e}\,. \tag{10.59}$$

If we know z, then the scale factor at emission is related to the scale factor at observation by:

$$a_e = a_o \frac{1}{1+z}\,. \tag{10.60}$$

The redshift is often quoted as a proxy for distance.

11 Gravitational waves

11.1 Linearising

Let's write down again the expressions for the linearisation of general relativity about a flat Minkowski background.

Linearised general relativity

- The metric:

$$g_{\mu\nu} \approx \eta_{\mu\nu} + h_{\mu\nu}\,, \quad g^{\mu\nu} \approx \eta^{\mu\nu} - h^{\mu\nu}\,. \tag{11.1}$$

- The Levi-Civita connection:

$$\Gamma_{\mu\nu}{}^{\rho} \approx \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu h_{\nu\lambda} + \partial_\nu h_{\mu\lambda} - \partial_\lambda h_{\mu\nu}\right)\,. \tag{11.2}$$

- The Riemann tensor:

$$R^{\rho}{}_{\sigma\mu\nu} \approx \frac{1}{2}\eta^{\rho\lambda}\left(\partial_\mu\partial_\sigma h_{\lambda\nu} + \partial_\nu\partial_\lambda h_{\mu\sigma} - \partial_\mu\partial_\lambda h_{\sigma\nu} - \partial_\nu\partial_\sigma h_{\lambda\mu}\right)\,. \tag{11.3}$$

- The Ricci tensor:

$$R_{\mu\nu} \approx \partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial_\rho\partial^\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h\,. \tag{11.4}$$

- The Ricci scalar:

$$R \approx \partial^\mu\partial^\nu h_{\mu\nu} - \partial_\rho\partial^\rho h\,. \tag{11.5}$$

- The linearised Einstein equation:

$$\partial^\rho\partial_{(\mu}h_{\nu)\rho} - \frac{1}{2}\partial_\rho\partial^\rho h_{\mu\nu} - \frac{1}{2}\partial_\mu\partial_\nu h - \frac{1}{2}\eta_{\mu\nu}(\partial^\rho\partial^\sigma h_{\rho\sigma} - \partial_\rho\partial^\rho h) = 8\pi G T_{\mu\nu}\,. \tag{11.6}$$

Recall that under diffeomorphisms, viewed as coordinate transformations $x^\mu \mapsto y^\mu(x)$, the transformation of the metric was given by:

$$\frac{\partial y^\mu}{\partial x^\rho}\frac{\partial y^\nu}{\partial x^\sigma}\tilde{g}_{\mu\nu}(y(x)) = g_{\rho\sigma}(x)\,. \tag{11.7}$$

This provided a gauge symmetry, or redundancy of description, of general relativity. This carries over into the linearised theory. We consider diffeomorphisms which preserve the splitting of the metric into the background Minkowski metric and a perturbation. A general diffeomorphism will not, of course. We can consider those one-parameter families of diffeomorphisms which are infinitesimally close to the identity diffeomorphism, which will certainly preserve this form of the metric to the order we are working at. Consider therefore an infinitesimal diffeomorphism, $y^\mu = x^\mu - \xi^\mu$ (we could replace $\xi^\mu \to \epsilon\xi^\mu$ with ϵ a small parameter if we wanted, but instead let us take ξ^μ to be of the same order as $h_{\mu\nu}$). Letting $\tilde{g}_{\mu\nu} = \eta_{\mu\nu} + \tilde{h}_{\mu\nu}$, we find from (11.7) that

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} + \partial_\mu\xi_\nu + \partial_\nu\xi_\mu\,. \tag{11.8}$$

This transformation

$$h_{\mu\nu} \to h_{\mu\nu} + \delta_\xi h_{\mu\nu}\,, \quad \delta_\xi h_{\mu\nu} \equiv \partial_\mu \xi_\nu + \partial_\nu \xi_\mu \tag{11.9}$$

is the *gauge symmetry* of linearised general relativity. Two perturbations $h_{\mu\nu}$ differing by such a transformation are to be considered physically equivalent.

This gauge symmetry is analogous to the gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$ of the electromagnetic potential, extended in this case to a symmetric "gauge" field, $h_{\mu\nu}$. In that case, the field strength $F_{\mu\nu} = 2\partial_{[\mu} A_{\nu]}$ was gauge invariant.

Exercise 11.1 (Gauge invariance of the Riemann tensor)

Check that $\delta_\xi R_{\mu\nu\rho\sigma} = 0$.

A very useful combination of the metric perturbation and its trace is:

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} h \eta_{\mu\nu}\,. \tag{11.10}$$

For instance, the Einstein equation (11.6) becomes:

$$-\frac{1}{2}\partial^\rho \partial_\rho \bar{h}_{\mu\nu} + \partial^\rho \partial_{(\mu} \bar{h}_{\nu)\rho} - \frac{1}{2}\eta_{\mu\nu}\partial^\rho \partial^\sigma \bar{h}_{\rho\sigma} = 8\pi G T_{\mu\nu}\,, \tag{11.11}$$

when written in terms of $\bar{h}_{\mu\nu}$.

Under the gauge symmetry, we have:

$$\delta_\xi \bar{h}_{\mu\nu} = \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \partial^\rho \xi_\rho \eta_{\mu\nu}\,. \tag{11.12}$$

Now, gauge symmetry implies a redundancy in our description of the physics. It is therefore often convenient to make a "choice of gauge" in which we impose some condition on our fields that "fixes" the gauge. For instance, in electromagnetism, we may choose to impose the Lorenz gauge condition, $\partial^\mu A_\mu = 0$. The analogy of this condition in general relativity is known as the de Donder or harmonic gauge condition:

$$\partial^\nu \bar{h}_{\mu\nu} = 0\,. \tag{11.13}$$

How is the gauge actually fixed? We have

$$\delta_\xi(\partial^\nu \bar{h}_{\mu\nu}) = \partial^\nu \partial_\nu \xi_\mu\,. \tag{11.14}$$

Therefore if we are given any arbitrary $\bar{h}_{\mu\nu}$, we can make a gauge transformation with parameter ξ_μ such that $\partial^\nu \partial_\nu \xi_\mu = -\partial^\nu \bar{h}_{\mu\nu}$. Then the gauge transformed $\bar{h}_{\mu\nu}$ will obey (11.13).

After gauge fixing, we hope to not have any further gauge redundancy left in our description of the physics. This is not yet true in the present case: the gauge condition (11.13) is preserved by gauge transformations ξ_μ with $\partial^\nu \partial_\nu \xi_\mu = 0$. We will use this later on.

In de Donder gauge, the Einstein equation becomes simply:

$$\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}\,. \tag{11.15}$$

This means that each component of the perturbation $\bar{h}_{\mu\nu}$ obeys the wave equation in four-dimensional spacetime, sourced by the energy-momentum tensor. The solutions of this can be written down using Green's function methods (see below).

11.2 Gravitational waves in vacuum

The solutions to $\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = 0$ are:

$$\bar{h}_{\mu\nu}(x) = \mathrm{Re}(H_{\mu\nu} e^{ik\cdot x}) \,, \quad k \cdot x \equiv k_\mu x^\mu \,, \tag{11.16}$$

where $H_{\mu\nu}$ is a constant symmetric complex tensor and k_μ is a null vector, $k_\mu k^\mu = 0$. Thus this solution describes waves propagating at the speed of light in the background Minkowski spacetime. The gauge condition (11.13) requires

$$k^\nu H_{\mu\nu} = 0 \,. \tag{11.17}$$

The residual gauge freedom given by transformations with parameter ξ_μ obeying $\partial^\nu \partial_\nu \xi_\mu = 0$ can be described by writing these explicitly as $\xi_\mu(x) = \mathrm{Re}(X_\mu e^{ik\cdot x})$. Under such a transformation,

$$H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho) \,. \tag{11.18}$$

We can choose X_μ such that:

$$H_{0\mu} = 0 \,, \quad H^\mu{}_\mu = 0 \,. \tag{11.19}$$

Writing out the time and spatial components fully, this means that

$$H_{00} = 0 \,, \quad H_{0i} = 0 \,, \quad H^i{}_i = 0 \,, \tag{11.20}$$

while we also have the constraints (11.17).

Exercise 11.2 (Gauge fixing)

Show that this is indeed possible.

For example, consider a gravitational wave propagating in the z-direction, with $k^\mu = \omega(1,0,0,1)$. This requires that $H_{\mu 0} + H_{\mu 3} = 0$, and hence with this choice of gauge fixing that $H_{i3} = 0$. Therefore the polarisation matrix takes the form:

$$H_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & H_+ & H_\times & 0 \\ 0 & H_\times & -H_+ & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,. \tag{11.21}$$

The fact that $H^\mu{}_\mu = 0$ means that $\bar{h}^\mu{}_\mu = 0 = h^\mu{}_\mu$. Hence in this gauge, $h_{\mu\nu} = \bar{h}_{\mu\nu}$. We therefore have for the particular solution with polarisation matrix (11.21):

$$h_{11} = |H_+| \cos(\omega(t-z) - \alpha_+) = -h_{22} \,, \quad h_{12} = |H_\times| \cos(\omega(t-z) - \alpha_\times) = h_{21} \,, \tag{11.22}$$

having written $H_+ = |H_+|e^{i\alpha_+}$, $H_\times = |H_\times|e^{i\alpha_\times}$ and used $k_\rho x^\rho = -\omega(t-z)$. This particular choice of gauge is known as *transverse traceless gauge*, with the corresponding perturbation denoted by $h^{\rm TT}_{\mu\nu}$, obeying altogether:

$$\partial^\nu h^{\rm TT}_{\mu\nu} = 0\,, \quad h^{\rm TT}_{00} = 0\,, \quad h^{\rm TT}_{0i} = 0\,, \quad \eta^{\mu\nu} h^{\rm TT}_{\mu\nu} = 0\,. \tag{11.23}$$

This gauge reveals the important physical property that a gravitational wave can have two independent polarisations, corresponding to H_+ and $H_\times$.

Effect of gravitational waves on test particles

We now want to understand properly the effect of a gravitational wave perturbation on test particles. Suppose we consider a single particle whose position we describe using the coordinates x^μ corresponding to the transverse traceless gauge. The geodesic equation, to first order in the perturbation, is

$$\frac{d^2x^\mu}{d\tau^2} + \frac{1}{2}\eta^{\mu\lambda}\left(-\partial_\lambda h^{\rm TT}_{\nu\rho} + 2\partial_\nu h^{\rm TT}_{\rho\lambda}\right)\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau} = 0\,. \tag{11.24}$$

Suppose this describes a freely falling particle which is initially at rest, with four-velocity $U^\mu(\tau) = \frac{dx^\mu}{d\tau} = (1,0,0,0)$ at $\tau = 0$. Then we have

$$\frac{d^2x^\mu}{d\tau^2}(\tau=0) + \frac{1}{2}\eta^{\mu\lambda}\left(-\partial_\lambda h^{\rm TT}_{00} + 2\partial_0 h^{\rm TT}_{0\lambda}\right) = 0 \Rightarrow \frac{d^2x^\mu}{d\tau^2}(\tau=0) = 0\,. \tag{11.25}$$

It follows that the geodesic equation is solved by $U^\mu(\tau) = U^\mu(0)$ for all τ. Hence the particle remains at rest in the presence of the metric perturbation.

What is happening here is that the coordinates we are using, which are those adapted to transverse traceless gauge, are in fact coordinates adapted to freely falling particles. The coordinate attached to a particular particle, or point, by definition "move with the wave" i.e. the value of the coordinate at that point is by definition always a certain value. So we cannot see the effect of the wave from considering a single particle. We should consider instead multiple particles.

The simplest coordinate invariant test of the presence of a gravitational wave is to measure the proper distance between two freely falling particles. Suppose we put one at coordinate location $x^i = (0,0,0)$ and the second at coordinate location $x^i = (L,0,0)$. Suppose a gravitational wave propagates in the z direction, described in transverse traceless gauge with the components $h^{\rm TT}_{\mu\nu}(t-z)$. The proper distance between these two particles is

$$\begin{aligned}\Delta s = \int\sqrt{ds^2} &= \int\sqrt{g_{\mu\nu}dx^\mu dx^\nu}\\ &= \int_0^L dx\sqrt{g_{xx}}\\ &= \int_0^L dx\sqrt{1+h^{\rm TT}_{xx}(t)}\\ &\approx L\left(1+\frac{1}{2}h^{\rm TT}_{xx}(t)\right)\,.\end{aligned} \tag{11.26}$$

This proper distance is certainly not constant: it varies in the presence of the gravitational wave. We can write this variation as

$$\frac{\delta L}{L} \approx \frac{1}{2}h^{\rm TT}_{xx}(t)\,. \tag{11.27}$$

This is in fact a gauge independent prediction of gravitational wave theory, and corresponds to the measurements made in gravitational wave detectors. There the distance Δs is measured by bouncing light beams from mirrors at the ends of arms of length L (which works when the metric perturbation varies on a timescale less than the time it takes to travel through the arm).

We can rederive the above result more generally by choosing to work in a different gauge, i.e. with a different set of coordinates. Appropriate coordinates to use are an orthonormal frame associated to one freely falling particle. This can be defined by defining a local inertial frame at one point on the worldline of this particle, and then extending it along the geodesic by parallel transport. In these coordinates, the position of the geodesic is at the origin in the spatial coordinates, and at all points along the geodesic the metric is given by the flat Minkowski metric, and its first derivatives vanish.

Exercise 11.3 (Coordinates adapted to the freely falling observer)

Let (t, x^i) denote the coordinates used in transverse traceless gauge. Define new coordinates by

$$\hat{t} = t\,, \quad \hat{x}^i = x^i + \frac{1}{2}\delta^{ik} h^{\mathrm{TT}}_{kj}(t) x^j\,. \tag{11.28}$$

Given that the positions x^i are unchanged in the presence of the wave, work out the behaviour of test particles described by the new coordinates $\hat{x}^i$, and compare it to the results below coming from solving the geodesic deviation equation.

Show that the metric in terms of these new coordinates is:

$$ds^2 \approx -d\hat{t}^2 + \delta_{ij} d\hat{x}^i d\hat{x}^j \tag{11.29}$$

neglecting terms of second order in the perturbation, and assuming that the wavelength $\lambda \sim 1/\omega$ of the gravitational wave is much bigger than the coordinate distances involved so as to also neglect terms involving $|x^i|/\lambda << 1$.

Given another nearby freely falling particle, we can define a separation vector S^μ which will obey the *geodesic deviation equation*, derived back in section 6 as equation (6.20), which we write here as

$$\nabla_U \nabla_U S^\mu \equiv \frac{D^2}{d\tau^2} S^\mu = R^\mu{}_{\nu\rho\sigma} U^\nu U^\rho S^\sigma\,. \tag{11.30}$$

Here U (which was originally T when we derived this originally) is the tangent vector to the first freely falling geodesic, and τ is the proper time along this geodesic. This equation expresses how the separation S^μ between the two freely falling particles changes as we move along the geodesic of the first particle. The right-hand side of this equation involves the Riemann curvature tensor. We emphasise that in writing (11.30) we are now using new coordinates adapted to the geodesic of the first particle, and not the original coordinates in which the metric perturbation was in transverse traceless gauge $h^{\mathrm{TT}}_{\mu\nu}$. To first order, this change of coordinates is just a change of gauge. The Riemann curvature tensor is gauge invariant to this order. Therefore when we evaluate $R^\mu{}_{\nu\rho\sigma}$ we can use the expression we would get in the transverse traceless gauge, i.e. we will use

$$R^\mu{}_{\nu\rho\sigma} \approx \frac{1}{2}\eta^{\mu\lambda}\left(\partial_\rho\partial_\nu h^{\mathrm{TT}}_{\lambda\sigma} + \partial_\sigma\partial_\lambda h^{\mathrm{TT}}_{\rho\nu} - \partial_\rho\partial_\lambda h^{\mathrm{TT}}_{\nu\sigma} - \partial_\sigma\partial_\nu h^{\mathrm{TT}}_{\lambda\rho}\right)\,. \tag{11.31}$$

The coordinates we are using mean that $U^\mu = (1, 0, 0, 0)$, hence we can use $\tau = t$, and $\Gamma_{\mu\nu}{}^\rho = 0$ along the geodesic. Then the geodesic equation becomes

$$\frac{d^2 S^\mu}{dt^2} = R^\mu{}_{00\nu} S^\nu = \frac{1}{2} \eta^{\mu\rho} \partial_0 \partial_0 h^{\mathrm{TT}}_{\rho\nu} S^\nu \tag{11.32}$$

For a gravitational wave propagating in the z-direction, we have

$$\frac{d^2 S^0}{dt^2} = 0 = \frac{d^2 S^3}{dt^2} \,, \tag{11.33}$$

and we can arrange by a choice of initial conditions that S^0 (the displacement in the time direction) vanishes for all time, and that S^3 is constant. Picking coordinates such that $S^3 = 0$, and considering a ring of particles in xy plane at $z = 0$, the interesting equations are then:

$$\begin{aligned} \frac{d^2 S^1}{dt^2} &= -\frac{1}{2}\omega^2 \left(|H_+| \cos(\omega t - \alpha_+) S^1 + |H_\times| \cos(\omega t - \alpha_\times) S^2 \right) \,, \\ \frac{d^2 S^2}{dt^2} &= -\frac{1}{2}\omega^2 \left(-|H_+| \cos(\omega t - \alpha_+) S^2 + |H_\times| \cos(\omega t - \alpha_\times) S^1 \right) \,, \end{aligned} \tag{11.34}$$

Because $|H_+|$ and $|H_\times|$ are small, we can solve perturbatively. The zeroth order solution is that $S^1(t) = \bar{S}^1$, $S^2(t) = \bar{S}^2$, with $\bar{S}^1, \bar{S}^2$ constants. To find the first order solution, we first substitute the zero order solution into the right-hand side of the equations (11.34), and then solve to get

$$\begin{aligned} S^1(t) &\approx \bar{S}^1 + \frac{1}{2} \left(|H_+| \cos(\omega t - \alpha_+) \bar{S}^1 + |H_\times| \cos(\omega t - \alpha_\times) \bar{S}^2 \right) \,, \\ S^2(t) &\approx \bar{S}^2 + \frac{1}{2} \left(-|H_+| \cos(\omega t - \alpha_+) \bar{S}^2 + |H_\times| \cos(\omega t - \alpha_\times) \bar{S}^1 \right) \,. \end{aligned} \tag{11.35}$$

These represent oscillations in the xy plane. In more detail, consider first the case of a "plus" polarised wave only, $H_+ \neq 0$, $H_\times = 0$. We have

$$\begin{aligned} S^1(t) &\approx \bar{S}^1 + \frac{1}{2} \cos(\omega t - \alpha_+) |H_+| \bar{S}^1 \,, \\ S^2(t) &\approx \bar{S}^2 - \frac{1}{2} \cos(\omega t - \alpha_+) |H_+| \bar{S}^2 \,. \end{aligned} \tag{11.36}$$

If we consider a collection of test particles arranged in a circle in the xy plane, the circle will first expand in the x direction while contracting in the y direction, and vice versa. Snapshots of the arrangement look like a series of ellipses oscillating about the x and y axes, i.e. in the directions indicated by the $+$ symbol. This is illustrated in figure 30.

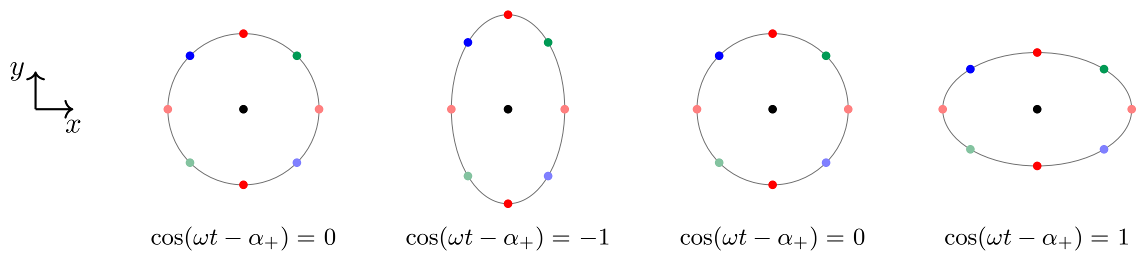


Figure 30: Effect of a plus polarised gravitational wave on a ring of test particles. The particles are coloured to illustrate the individual oscillations.

Next, consider a "cross" polarised wave only, $H_\times \neq 0$, $H_+ = 0$. We have

$$\begin{aligned} S^1(t) &\approx \bar{S}^1 + \frac{1}{2}\cos(\omega t - \alpha_\times)|H_\times|\bar{S}^2\,, \\ S^2(t) &\approx \bar{S}^2 + \frac{1}{2}\cos(\omega t - \alpha_\times)|H_\times|\bar{S}^1\,. \end{aligned} \tag{11.37}$$

If we consider a collection of test particles arranged in a circle in the xy plane, the circle will now expand and contract about axes at 45 degrees to the x and y directions, i.e. in the directions specified by the $\times$ symbol. This is illustrated in figure 31.

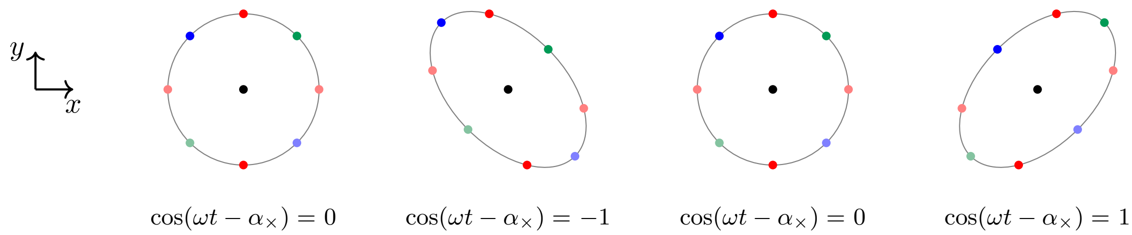


Figure 31: Effect of a cross polarised gravitational wave on a ring of test particles. The particles are coloured to illustrate the individual oscillations.

11.3 Gravitational waves from a source, and their detection

Gravitational waves from a source

Now we return to the linearised Einstein equation in de Donder gauge, equation (11.15), and work out the response to the energy-momentum tensor on the right-hand side. This can be viewed as a standard wave equation in flat space, and the way to solve it is to introduce the Green's function $\mathcal{G}(x)$ obeying

$$\partial_\rho \partial^\rho \mathcal{G}(x-y) = \delta^{(4)}(x-y)\,, \tag{11.38}$$

such that

$$\bar{h}_{\mu\nu}(x) = -16\pi G \int d^4y\, \mathcal{G}(x-y) T_{\mu\nu}(y) \tag{11.39}$$

is automatically a solution to $\partial_\rho \partial^\rho \bar{h}_{\mu\nu} = -16\pi G T_{\mu\nu}$.

Deriving the form of the Green's function should (hopefully) be familiar from earlier courses on electrodynamics. We will use the retarded Green's function,

$$\mathcal{G}(x-y) = -\frac{1}{4\pi|\vec{x}-\vec{y}|}\delta(|\vec{x}-\vec{y}| - (x^0 - y^0))\theta(x^0 - y^0)\,, \tag{11.40}$$

which describes the causal response to a source in the past. Observe we have split $x^\mu = (x^0, \vec{x})$, $y^\mu = (y^0, \vec{y})$. Here $\theta(x^0 - y^0) = 1$ if $x^0 - y^0 > 0$, and is zero otherwise. This gives us our solution for the linearised gravitational field produced by the source with energy-momentum tensor $T_{\mu\nu}$:

$$\bar{h}_{\mu\nu}(t,\vec{x}) = 4G \int d^3y\, \frac{1}{|\vec{x}-\vec{y}|} T_{\mu\nu}(t_r, \vec{y})\,, \tag{11.41}$$

where $t_r \equiv t - |\vec{x}-\vec{y}|$ is the retarded time.

In principle we now take an interesting source, insert into this expression, and integrate.

Let's proceed further by making some reasonable assumptions. Suppose we are dealing with a faraway source at a distance $r \equiv |\vec{x}|$. Further suppose this source is isolated, meaning that it itself is concentrated in a region of space of size $\delta r << r$. See figure 32.

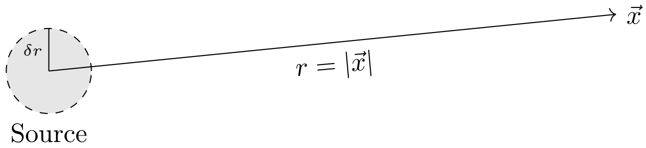


Figure 32: The isolated source (e.g. a spherical cow) is small because it is far away.

Then,

$$|\vec{x} - \vec{y}|^2 = r^2 - 2\vec{x} \cdot \vec{y} + |\vec{y}|^2 = r^2 \left(1 - 2\frac{\vec{x} \cdot \vec{y}}{r^2} + \frac{|\vec{y}|^2}{r^2}\right) \tag{11.42}$$

and inside the integral in (11.41) we assume that we only have contributions for $|\vec{y}| \lesssim \delta r$. Hence

$$|\vec{x} - \vec{y}| = r\,(1 + O(\delta r/r)) \tag{11.43}$$

We can further expand

$$T_{\mu\nu}(t - |\vec{x} - \vec{y}|, \vec{y}) \approx T_{\mu\nu}(t - r, \vec{y}) + \partial_0 T_{\mu\nu} O(\delta r)\,. \tag{11.44}$$

We further assume that the source is varying on a timescale δt, such that $\partial_0 T_{\mu\nu} = T_{\mu\nu}/\delta t$, and that its motion is non-relativistic, so that $\delta r/\delta t << 1$. Then we can neglect the second term in the above Taylor expansion. The result (11.41) for the metric perturbation then simplifies to:

$$\bar{h}_{\mu\nu}(t, \vec{x}) \approx \frac{4G}{r} \int d^3y T_{\mu\nu}(t_r, \vec{y}) \tag{11.45}$$

where now $t_r \equiv t - r$. This is a simpler integral.

We will now proceed to further analyse the spatial components, $\bar{h}_{ij}$. The gauge choice $\partial^\nu \bar{h}_{\mu\nu} = 0$ means that

$$\partial_0 \bar{h}_{i0} = \partial^j \bar{h}_{ij}\,, \tag{11.46}$$

and

$$\partial_0 \bar{h}_{00} = \partial^j \bar{h}_{0j} \tag{11.47}$$

so that once we know $\bar{h}_{ij}$ we can successively determine $\bar{h}_{i0}$ and $\bar{h}_{00}$.

We use the following quite surprising trick. Make use of the seemingly trivial identity $\delta^i_k = \partial_k x^i$ to write (note we suppress the dependence on t_r and here use x^i as the label for our integration variables rather than y^i)

$$\int d^3x T^{ij} = \int d^3x \left(\partial_k (T^{ik} x^j) - \partial_k T^{ik} x^j\right)\,. \tag{11.48}$$

The first term is a total derivative, which we assume vanishes. In the second term, we can use conservation of the energy-momentum tensor (in the background Minkowski spacetime, i.e. to leading order in the perturbation), which implies $\partial_k T^{ik} = -\partial_0 T^{i0}$. Hence,

$$\int d^3x T^{ij} = \int d^3x \partial_0 T^{i0} x^j\,. \tag{11.49}$$

Taking the ∂_0 outside the integral, and noting that we could equally well have swapped i and j in the above calculation, we write:

$$\int d^3x T^{ij} = \partial_0 \int d^3x T^{0(i}x^{j)} \,. \tag{11.50}$$

Next, we repeat the trick: writing

$$\begin{aligned} \int d^3x T^{ij} &= \partial_0 \int d^3x \left(\frac{1}{2}\partial_k (T^{0k}x^i x^j) - \frac{1}{2}\partial_k T^{0k} x^i x^j \right) \\ &= \frac{1}{2}\partial_0 \int d^3x \partial_0 T^{00} x^i x^j \\ &= \frac{1}{2}\partial_0 \partial_0 \int d^3x T^{00} x^i x^j \,. \end{aligned} \tag{11.51}$$

The quantity being time differentiated here is called the *quadrupole moment of the energy density*, and is usually denoted by

$$I_{ij}(t) \equiv \int d^3x T_{00}(t, \vec{x}) x_i x_j \,. \tag{11.52}$$

Reverting to our earliest instincts and writing dots for time derivatives, we can succinctly write

$$\bar{h}_{ij}(t, \vec{x}) \approx \frac{2G}{r} \ddot{I}_{ij}(t - r) \,. \tag{11.53}$$

Gravitational waves from a rotating binary system

We will now study the gravitational waves produced by a rotating binary system. In line with our assumptions above, we will analyse the behaviour of this system using Newtonian physics. This will produce predictions for the induced gravitational wave behaviour that is in line with what has been observed at LIGO. In a more complete analysis, the Newtonian description we will use below can be viewed as the starting point for a "post-Newtonian" expansion, in which we go on to include relativistic effects.

Consider a binary system of two masses, m_1 and m_2, rotating with angular frequency Ω, as shown in figure 33. We will analyse this as a standard Newtonian two-body system, using the reminder provided below. The total separation of the two masses is $R = R_1 + R_2$, with $R_1 = m_2 R/(m_1 + m_2)$, $R_2 = m_1 R/(m_1 + m_2)$, and the reduced mass is $\mu = m_1 m_2/(m_1 + m_2)$. The total energy is:

$$E = \frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 - \frac{G m_1 m_2}{R} = \frac{1}{2}\Omega^2(m_1 R_1^2 + m_2 R_2^2) - \frac{G m_1 m_2}{R} = \frac{1}{2}\mu \Omega^2 R^2 - \frac{G\mu M}{R} \,, \tag{11.54}$$

where $M = m_1 + m_2$, and $v_1 = \Omega R_1$, $v_2 = \Omega R_2$. We can treat this as a single particle of mass μ in the gravitational field of another of mass M.

Reminder of Newtonian two-body physics

Consider two masses m_1 and m_2 with position vectors $\vec{x}_1$ and $\vec{x}_2$ interacting through a potential $V = V(|\vec{x}_1 - \vec{x}_2|)$. Define the centre of mass position, $\vec{X} = (m_1\vec{x}_1 + m_2\vec{x}_2)/(m_1 + m_2)$ and the separation vector $\vec{R} = \vec{x}_1 - \vec{x}_2$. The kinetic energy is

$$E_{\text{kin}} = \frac{1}{2}m_1\dot{\vec{x}_1}^2 + \frac{1}{2}m_2\dot{\vec{x}_2}^2 = \frac{1}{2}M\dot{\vec{X}}^2 + \frac{1}{2}\mu\dot{\vec{R}}^2\,, \tag{11.55}$$

where the total mass M and reduced mass μ are

$$M = m_1 + m_2\,, \quad \mu = \frac{m_1 m_2}{m_1 + m_2}\,. \tag{11.56}$$

In the centre of mass frame, the positions of the bodies are:

$$\vec{x}_1' = \frac{m_2\vec{R}}{m_1 + m_2}\,, \quad \vec{x}_2' = \frac{-m_1\vec{R}}{m_1 + m_2}\,. \tag{11.57}$$

The conserved angular momentum, $\vec{J} = \vec{R}\times\vec{p}$, is always orthogonal to $\vec{R}$. Hence the motion takes place in a plane. For $V(R) = -Gm_1m_2/R = -G\mu M/R$ we can write the Lagrangian in polar coordinates in this plane as:

$$L = \frac{1}{2}\mu\dot{R}^2 + \frac{1}{2}\mu R^2\dot{\theta}^2 + \frac{G\mu M}{R}\,. \tag{11.58}$$

The conserved energy and angular momentum are:

$$E = \frac{1}{2}\mu\dot{R}^2 + \frac{1}{2}\mu R^2\dot{\theta}^2 - \frac{G\mu M}{R}\,, \quad J = \mu R^2\dot{\theta}\,. \tag{11.59}$$

The effective potential is thus

$$V(R) = \frac{J^2}{2\mu R^2} - \frac{G\mu M}{R}\,, \tag{11.60}$$

and circular motion can occur at the minimum of this potential, $V'(R) = 0$, namely when

$$G\mu M = \frac{J^2}{\mu R} \Leftrightarrow \dot{\theta}^2 = \frac{GM}{R^3}\,. \tag{11.61}$$

The energy for the circular orbit is

$$E = -\frac{1}{2}\frac{G\mu M}{R}\,. \tag{11.62}$$

For circular motion, the condition (11.61) becomes $G\mu M/R^2 = \mu\Omega^2 R$ (identifying $\dot{\theta} \equiv \Omega$; also recall that this amounts to balancing gravitational and centripetal force) giving $\Omega^2 = GM/R^3$ (Kepler's third law). This means that:

$$E = -\frac{1}{2}\frac{G\mu M}{R} = -\frac{1}{2}(GM)^{2/3}\mu\Omega^{2/3}\,. \tag{11.63}$$

As these masses orbit, they emit gravitational radiation. This means that they lose energy. We assume that as they lose energy, the radial separation R decreases slowly in such a way that at any given moment the orbit can still be described as being circular, with the above formulae holding. Then we can differentiate to relate the loss in energy to either the decrease in separation or the change in angular frequency:

$$\frac{dE}{dt} = \frac{1}{2}\frac{G\mu M}{R^2}\frac{dR}{dt} = -\frac{1}{3}(GM)^{2/3}\mu\Omega^{-1/3}\frac{d\Omega}{dt}\,. \tag{11.64}$$

Note that as the radius decreases, Ω increases.

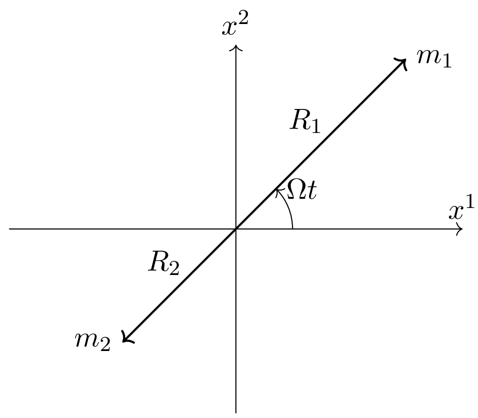


Figure 33: Rotating binary

Now we turn to the part of the calculation involving linearised GR. First, we write down the requisite component of the energy-momentum tensor, corresponding just to the energy (or mass) density of the two particles:

$$T^{00}(t,\vec{x}) = \delta(x^3)\left(m_1\delta(x^1 - R_1\cos\Omega t)\delta(x^2 - R_1\sin\Omega t) + m_2\delta(x^1 + R_2\cos\Omega t)\delta(x^2 + R_2\sin\Omega t)\right)\,. \tag{11.65}$$

We need this in order to calculate the quadrupole moment:

$$\begin{aligned} I_{ij}(t) &= \int d^3x\, x_i x_j T^{00}(t,\vec{x}) \\ &= \frac{1}{2}\mu R^2 \begin{pmatrix} 1+\cos 2\Omega t & \sin 2\Omega t & 0 \\ \sin 2\Omega t & 1-\cos 2\Omega t & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \end{aligned} \tag{11.66}$$

From this we can calculate the metric perturbation according to the result (11.53):

$$\bar{h}_{ij}(t,\vec{x}) = \frac{2G}{r}\ddot{I}_{ij}(t-r) = -\frac{4\Omega^2\mu G R^2}{r}\begin{pmatrix} \cos 2\Omega t_r & \sin 2\Omega t_r & 0 \\ \sin 2\Omega t_r & -\cos 2\Omega t_r & 0 \\ 0 & 0 & 0 \end{pmatrix}\,. \tag{11.67}$$

We observe that $\delta^{ij}\bar{h}_{ij} = 0$, so in vacuum (far from the source) where we can work with transverse traceless gauge, $h_{ij} = \bar{h}_{ij}$. The next thing we need to do is compute the energy carried away from the source as the gravitational radiation. A detailed calculation in second order perturbation theory (whose details we neglect for lack of time) allows us to express the energy loss of the source in terms of the *reduced quadrupole moment*. The latter is defined by:

$$Q_{ij} \equiv I_{ij} - \frac{1}{3}\delta_{ij}\delta^{kl}I_{kl}\,. \tag{11.68}$$

The result for the energy carried away by gravitational waves involves the third time derivatives of the reduced quadrupole moment, squared:

$$\frac{dE_{GW}}{dt} = -\frac{G}{5}\dddot{Q}_{ij}\dddot{Q}^{ij} \,. \tag{11.69}$$

Using (11.66) we find

$$\frac{dE_{GW}}{dt} = -\frac{32}{5}G\mu^2R^4\Omega^6 \,. \tag{11.70}$$

Equating this to the rate of change of the energy of the two-body system, equation (11.64), we find the following formula for the change in Ω:

$$\frac{d\Omega}{dt} = \frac{96}{5}G^{5/3}\mu M^{2/3}\Omega^{11/3} = \frac{96}{5}(G\mathcal{M})^{5/3}\Omega^{11/3} \,, \tag{11.71}$$

where we define the *chirp mass*:

$$\mathcal{M} \equiv (\mu^3M^2)^{1/5} = \frac{(m_1m_2)^{3/5}}{(m_1+m_2)^{1/5}} \,. \tag{11.72}$$

Given Ω and $\frac{d\Omega}{dt}$, we have

$$\mathcal{M} = \frac{1}{G}\left(\frac{5}{96}\Omega^{-11/3}\frac{d\Omega}{dt}\right)^{3/5} \,, \tag{11.73}$$

so if we know the angular frequency and its rate of change we can calculate $\mathcal{M}$.

Detection of gravitational waves

Indirect evidence for the production of gravitational waves was provided by precise measurements of the slowly decaying period of the orbit of a binary pulsar, by Hulse and Taylor. A pulsar is a neutron star emitting a beam of electromagnetic radiation in a particular direction, and a binary pulsar consists of a pulsar in orbit around a second star. The rotation of the pulsar itself means that we observe its emitted electromagnetic radiation as "pulses" at regular intervals. We can use these pulses as a sort of clock to measure properties of the binary system. As the pair lose energy due to gravitational radiation, their orbital frequency increases and hence the period of the orbit decreases in accordance with the calculations made using general relativity.

As described at the beginning of these notes, direct detection of gravitational waves is now possible using gravitational wave interferometers such as LIGO. Here the idea is to bounce two light signals back and forwards between pairs of mirrors in orthogonal "arms" (of length about 4km). A gravitational wave propagating through the apparatus causes tiny shifts in the distance that the light travels in the arms. This is observed when recombining the light signals and looking for a phase shift relative to the original beams.

What is directly measured is the change $\delta L/L$, where L is the length of the arm used in the interferometer. This is related to the metric fluctuation, $\delta L/L \sim \frac{1}{2}h_{ij}$, as follows from our earlier results. For the case of a gravitational wave emitted by a rotating binary, we thus can expect to see fluctuations of the form

$$\frac{\delta L}{L} \sim \frac{2\mu G\Omega(t)^2R^2}{r}\cos(2\Omega(t)t) \,, \tag{11.74}$$

where we emphasise that the angular frequency changes with time. The sort of signal this produces is shown in figure 35.

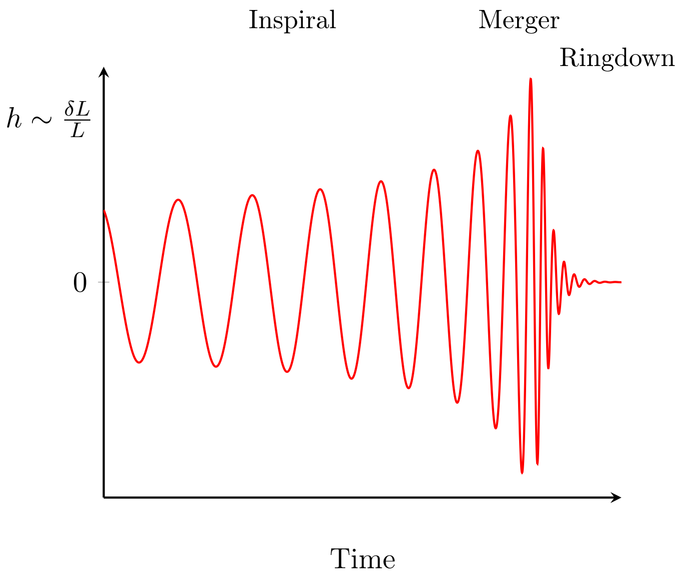


Figure 34: Predicted gravitational wave signal (using numerical relativity data from `https://www.gw-openscience.org/events/GW150914/`).

Initially the orbiting pair are drawing closer to each other, during what is known as the "inspiral phase". We assume that as they do so their motion can still be well-approximated using the Newtonian analysis we carried out above. In that case, we can use the Kepler law $\Omega^2(t) = GM/R(t)^3$, up until the bodies merge, at which point the maximum amplitude of the strain is observed. (A more comprehensive analysis would replace the Newtonian analysis by adding in the leading order relativistic effects, or more practically by solving the Einstein equations for the binary system numerically.) We can then write

$$\frac{\delta L}{L} \sim \frac{2\mu M G^2}{rR} \cos(2\Omega t) \tag{11.75}$$

The actual angular frequency of the observed gravitational wave is $\Omega_{GW} = 2\Omega$. It is convenient to instead work with the frequency $f_{GW} = \frac{\Omega_{GW}}{2\pi} = \frac{\Omega}{\pi}$. This can be crudely estimated from the signal in figure 35 by for instance measuring the time distance Δt between successive crossing points of $\delta L/L = 0$, and setting $f_{GW} = 1/(2\Delta t)$; more sophisticated techniques based on analysis of the waveform are available. Given the time dependence of this frequency, we can obtain the chirp mass from

$$\mathcal{M} = \frac{1}{G}\left(\frac{5}{96}\pi^{-8/3} f_{GW}^{-11/3} \dot{f}_{GW}\right)^{3/5} . \tag{11.76}$$

This provides an estimate of the total mass in the system. If we define $q \equiv m_1/m_2$ to be the mass ratio, then

$$M = (1+q)^{6/5} q^{-3/5} \mathcal{M} , \quad \mu = (1+q)^{-4/5} q^{2/5} \mathcal{M} . \tag{11.77}$$

We can also obtain from (11.76) that

$$f_{GW}^{-8/3}(t) = (G\mathcal{M})^{5/3} \frac{256}{5} \pi^{8/3} (t_c - t) , \tag{11.78}$$

where the constant of integration t_c corresponds to the point at which the binary pair merge. In practice we observe a maximum value of the strain, at a maximum value of the frequency, after which the gravitational wave signal falls off (in the "ringdown" phase). Let $f_{GW}|_{\max}$ denote the maximum value of the frequency. This corresponds to the minimal separation

$$R = \left(\frac{GM}{(\pi f_{GW}|_{\max})^2}\right)^{1/3} . \tag{11.79}$$

If we know this value of R, and have an idea of the mass ratio q appearing in $\mu M = (1+q)^{2/5} q^{-1/5} \mathcal{M}^2$, then from the maximum amplitude of the strain and equation (11.75) we can obtain r, the distance to the source.

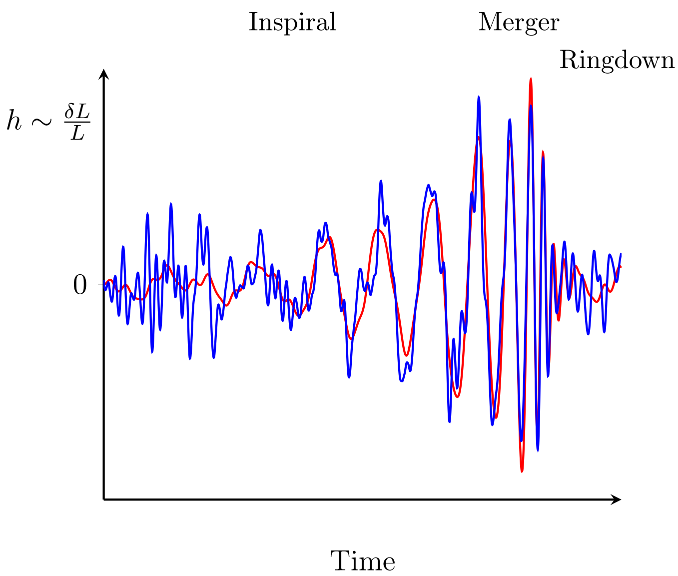


Figure 35: Observed gravitational wave signal (blue) at the LIGO Hanford detector and numerical relativity prediction (red) (using data from `https://www.gw-openscience.org/events/GW150914/`).

Let's review the numbers obtained for the original gravitational wave detection, known by the glamorous name of GW150914. For simplicity, let's first assume the masses of the binary system are equal, $m_1 = m_2 = m$. The maximum frequency obtained is

$$f_{GW}|_{\max} \sim 150\text{Hz} , \tag{11.80}$$

and the chirp mass was worked out to be about:

$$\mathcal{M} \sim 30 M_\odot , \tag{11.81}$$

where $M_\odot$ is the mass of the Sun. The orbital separation at peak amplitude is then

$$R = \left(\frac{G \cdot 2^{6/5} \cdot 30 M_\odot}{(150\pi)^2}\right)^{1/3} \approx 350\text{km} . \tag{11.82}$$

This is a very small distance. Each body involved in the merger has mass about 35 $M_\odot$ – which is quite heavy – with a corresponding Schwarzschild radius of about 100 km. Ordinary stars have radii of the order of hundreds of thousands or millions of kilometres. White dwarf stars are smaller, but still have radii of the order of thousands of kilometres. We therefore expect the merger to involve incredibly compact objects. The candidates are neutron stars or black holes.

The former generally have a radius of about 10 km, but have a maximum possible mass of a few $M_\odot$. This leaves black holes as the only candidate. (It can be shown that for unequal masses, with $q > 1$, the system becomes more compact, and the above conclusions hold. In order to more accurately identify the value of q, we need to go beyond the Newtonian description of the binary, and compare with relativistic predictions for the gravitational wave signal produced.)

The future evolution of gravitational wave detection promises to be very scientifically rich, with new detectors planned in such exciting locations as space (LISA) and, potentially, Limburg[6] (the Einstein telescope).

[6]The Dutch Limburg, that is: the proposal is a joint Belgian/Dutch/German initiative.

12 Einstein-Hilbert action

This section is not examinable, but included for completeness.

The Einstein field equations can be derived from an action.

Einstein-Hilbert action

The Einstein-Hilbert action is:

$$S_{\mathrm{EH}} = \frac{1}{16\pi G} \int d^4x\, \sqrt{-\det g} R\,. \tag{12.1}$$

We will not be precise about the nature of integration on manifolds. The integral in the action (12.1) can be considered to be an integral over the coordinates x^μ in some chart. As our only goal in this section is to derive the local field equations, this will suffice. Nevertheless, we should be sure that under changes of coordinates this is a sensible definition. This is ensured by the invariant measure, or volume form,

$$d^4x \sqrt{-\det g}\,, \tag{12.2}$$

involving the square root of the absolute value of the determinant of the metric. Under a change of coordinates,

$$\begin{aligned} \sqrt{-\det(g'_{\mu\nu})} &= \sqrt{-\det\left(\frac{\partial x^\rho}{\partial x'^\mu}\frac{\partial x^\sigma}{\partial x'^\nu} g_{\rho\sigma}\right)} \\ &= \sqrt{-\det g} \left|\det \frac{\partial x}{\partial x'}\right|\,, \end{aligned} \tag{12.3}$$

while

$$d^4x' = \left|\det \frac{\partial x'}{\partial x}\right| d^4x \tag{12.4}$$

as usual, so that $d^4x'\sqrt{-\det g'} = d^4x\sqrt{-\det g}$. This is compatible with what you already know: for instance consider $\mathbb{R}^3$ in spherical coordinates, with $x'^i = (r,\theta,\phi)$ and $x^i = (x,y,z)$. The metric in the former case has determinant $r^4\sin^2\theta$, and in the latter case has determinant one. Hence the measures are (for a Riemannian metric, we do not need to take minus the determinant as it is already positive):

$$d^3x'\sqrt{\det g'} = dr\, d\theta\, d\phi\, r^2 \sin\theta = dx\, dy\, dz = d^3x\sqrt{\det g}\,. \tag{12.5}$$

Convinced of our righteousness, we can now vary the action (12.1) with respect to the metric. We first of all need to show that:

$$\delta \det g = \det g\, g^{\mu\nu}\delta g_{\mu\nu}\,. \tag{12.6}$$

This follows from writing:

$$\det(g+\delta g) = \det\big(g(I + g^{-1}\delta g)\big) = \det g \det(I + g^{-1}\delta g)\,. \tag{12.7}$$

In general, if ϵ is a matrix whose components are small,

$$\begin{aligned}
\det(I+\epsilon) &= \det\begin{pmatrix} 1+\epsilon_{11} & \epsilon_{12} & \dots & \\ \epsilon_{21} & 1+\epsilon_{22} & & \\ & & \ddots & \\ & & & 1+\epsilon_{nn} \end{pmatrix} \\
&= (1+\epsilon_{11})(1+\epsilon_{22})\dots(1+\epsilon_{nn}) + O(\epsilon^2) \\
&= 1 + \sum_i \epsilon_{ii} + O(\epsilon^2) \\
&= 1 + \mathrm{tr}\epsilon + O(\epsilon^2)\,,
\end{aligned} \tag{12.8}$$

so to first order in the variation we have

$$\det(I + g^{-1}\delta g) = 1 + \mathrm{tr}\, g^{-1}\delta g\,, \tag{12.9}$$

and hence

$$\det(g+\delta g) - \det g = 1 + \mathrm{tr}\, g^{-1}\delta g \tag{12.10}$$

and this gives the result (12.6). It also follows that

$$\delta\sqrt{-\det g} = \frac{1}{2\sqrt{-\det g}}(-\delta\det g) = \frac{-\det g}{2\sqrt{-\det g}} g^{\mu\nu}\delta g_{\mu\nu} = \frac{1}{2}\sqrt{-\det g} g^{\mu\nu}\delta g_{\mu\nu}\,. \tag{12.11}$$

Next, we need to vary the Ricci scalar. Let's start with the Riemann tensor. The trick is to first vary the Riemann tensor at some point p, using Riemann normal coordinates. As $\Gamma_{\mu\nu}{}^{\rho}(p) = 0$ in these coordinates, we have

$$\delta R^{\rho}{}_{\sigma\mu\nu} = 2\partial_{[\mu}\delta\Gamma_{\nu]\sigma}{}^{\rho} = 2\nabla_{[\mu}\delta\Gamma_{\nu]\sigma}{}^{\rho}\,, \tag{12.12}$$

where we again use Riemann normal coordinates to be able to replace the partial derivatives with covariant ones. We also use the fact that the difference of two connections is a tensor, which means that $\delta\Gamma_{\mu\nu}{}^{\rho}$ is a tensor. Now we argue that we have obtained a tensorial equation, which must hold in all coordinates at p, and then must also hold for arbitrary p. So, the exact result is $\delta R^{\rho}{}_{\sigma\mu\nu} = 2\nabla_{[\mu}\delta\Gamma_{\nu]\sigma}{}^{\rho}$. It follows that

$$\delta R_{\mu\nu} = \delta R^{\rho}{}_{\mu\rho\nu} = 2\nabla_{[\rho}\delta\Gamma_{\nu]\mu}{}^{\rho}\,, \tag{12.13}$$

and that

$$\delta R = \delta g^{\mu\nu} R_{\mu\nu} + 2\nabla_{\rho}\left(g^{\mu[\nu}\delta\Gamma_{\nu\mu}{}^{\rho]}\right) \equiv \delta g^{\mu\nu}R_{\mu\nu} + 2\nabla_{\rho}X^{\rho}\,, \tag{12.14}$$

letting $X^{\rho} \equiv 2g^{\mu[\nu}\delta\Gamma_{\nu\mu}{}^{\rho]}$. We now vary the Einstein-Hilbert action, using (12.11) and (12.14):

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G}\int d^4x\,\sqrt{|g|}\left(\delta g^{\mu\nu}(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) + \nabla_{\mu}X^{\mu}\right) \tag{12.15}$$

Note that we used $g^{\mu\rho}\delta g_{\rho\nu} = -\delta g^{\mu\rho}g_{\rho\nu}$. We see the Einstein tensor appearing. We just need to dispose of the term involving X^μ. This is possible because it is in fact a total derivative:

$$
\begin{aligned}
\sqrt{-\det g}\nabla_\mu X^\mu &= \sqrt{-\det g}(\partial_\mu X^\mu + \Gamma_{\mu\nu}{}^\mu X^\nu) \\
&= \sqrt{-\det g}(\partial_\mu X^\mu + \frac{1}{2}g^{\mu\rho}(-\partial_\rho g_{\mu\nu} + \partial_\mu g_{\rho\nu} + \partial_\nu g_{\rho\mu})X^\nu) \\
&= \sqrt{-\det g}(\partial_\mu X^\mu + \frac{1}{2}g^{\nu\rho}\partial_\mu g_{\nu\rho}X^\mu) \\
&= \partial_\mu(\sqrt{-\det g}X^\mu)\,.
\end{aligned}
\tag{12.16}
$$

(The rule for the derivative of a determinant follows the rule for the variation of one.)

Let's also add matter. The matter action is

$$
S_{\text{matter}} = \int d^4x\,\sqrt{-\det g}\mathcal{L}_{\text{matter}}
\tag{12.17}
$$

and a priori can be anything we want as long as $\mathcal{L}_{\text{matter}}$ is a scalar. We now *define* the energy-momentum tensor to be whatever we get when we vary this action with respect to the spacetime metric:

$$
T_{\mu\nu} \equiv -\frac{2}{\sqrt{-\det g}}\frac{\delta S_{\text{matter}}}{\delta g^{\mu\nu}}\,.
\tag{12.18}
$$

This is obviously symmetric, and remarkably turns out to obey the conservation law $\nabla_\nu T^{\mu\nu} = 0$ automatically, as a consequence of invariance under diffeomorphisms. We will not show this here.

We can then vary, dropping total derivatives, to get

$$
\delta(S_{\text{EH}} + S_{\text{matter}}) = \frac{1}{16\pi G}\int d^4x\,\sqrt{-\det g}\delta g^{\mu\nu}\left(G_{\mu\nu} - 8\pi G T_{\mu\nu}\right)\,,
\tag{12.19}
$$

and we clearly see the Einstein equation follows as the equation of motion for the metric.

Exercise 12.1 (Action with cosmological constant)

Show that the cosmological constant can be incorporated by modifying the Einstein-Hilbert action to:

$$
S_{\text{EH},\Lambda} = \frac{1}{16\pi G}\int d^4x\,\sqrt{-\det g}(R - 2\Lambda)\,.
\tag{12.20}
$$