General Relativity 2021/22 - Homework 3

1. *Toy models of black holes.* The following metric is a (non-vacuum) solution to a two-dimensional version of general relativity coupled to a scalar field:

$$ds^2 = -\tanh^2 r\, dt^2 + dr^2 \tag{1}$$

   (a) *(1 pt)* Recall (from your exercise classes, or from Carroll section 2.8) that the Levi-Civita symbol in n-dimensions, $\tilde{\epsilon}_{\mu_1\ldots\mu_n}$, is related to the Levi-Civita tensor, $\epsilon_{\mu_1\ldots\mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\ldots\mu_n}$. Let $n = 2$. Explain why any rank 2 antisymmetric tensor in n-dimensions is necessarily proportional to the Levi-Civita tensor. Hence explain why the Riemann tensor (of the Levi-Civita connection) in two-dimensions must be of the form

$$R_{\mu\nu\rho\sigma} \propto R g_{\mu[\rho} g_{\sigma]\nu}\,, \tag{2}$$

   where R is the Ricci scalar, and determine the constant of proportionality.

   Solution: an antisymmetric tensor $A_{\mu_1\ldots\mu_n}$ in n dimensions can have only one independent component, say $A_{01\ldots n-1}$, with all others related by signs depending on the permutation. Then we can write such a tensor as $A_{\mu_1\ldots\mu_n} = A_{01\ldots n-1}\tilde{\epsilon}_{\mu_1\ldots\mu_n}$ or as $A_{\mu_1\ldots\mu_n} = A_{01\ldots n-1}\frac{1}{\sqrt{|g|}}\epsilon_{\mu_1\ldots\mu_n}$.

   The Riemann curvature tensor of the Levi-Civita tensor is antisymmetric on both its first and last pair of indices:

$$R_{\mu\nu\rho\sigma} = R_{[\mu\nu][\rho\sigma]} \tag{3}$$

   Hence for $n = 2$ it has only 1 independent degree of freedom and must be of the form

$$R_{\mu\nu\rho\sigma} = R_{0101}\frac{1}{|\det g|}\epsilon_{\mu\nu}\epsilon_{\rho\sigma} \tag{4}$$

   We can also note $2g_{\mu[\rho}g_{\sigma]\nu}$ is likewise antisymmetric on both $\mu\nu$ and $\rho\sigma$, so must itself be proportional to $\epsilon_{\mu\nu}\epsilon_{\rho\sigma}$. (Indeed this antisymmetrisation is just the determinant of $g_{\mu\nu}$.) Thus we must have

$$R_{\mu\nu\rho\sigma} = C g_{\mu[\rho} g_{\sigma]\nu} \tag{5}$$

   and we can relate C to the Ricci scalar via the definition

$$R = R^{\mu\nu}{}_{\mu\nu} = \tfrac{C}{2} g^{\mu\rho} g^{\nu\sigma}(g_{\mu\rho}g_{\nu\sigma} - g_{\mu\sigma}g_{\nu\rho}) = \tfrac{C}{2}(4-2) = C \tag{6}$$

   so in fact

$$R_{\mu\nu\rho\sigma} = R g_{\mu[\rho} g_{\sigma]\nu} \tag{7}$$

(b) *(2 pts)* For the metric (1), compute the Christoffel symbols, Ricci tensor, and Ricci scalar. Is $r = 0$ a coordinate or curvature singularity?

Solution: let $f(r) = \tanh^2 r$, with derivatives $f'(r) = 2\frac{\sinh r}{\cosh^3 r}$, $f''(r) = \frac{2}{\cosh^4 r}(1 - 2\sinh^2 r)$. The Christoffel symbols are most easily derived from the geodesic equations via

$$L = -\frac{1}{2}f(r)\dot{t}^2 + \frac{1}{2}\dot{r}^2 \tag{8}$$

which are

$$\ddot{r} + \frac{1}{2}f'\dot{t}^2 = 0\,, \quad \ddot{t}f(r) + f'(r)\dot{r}\dot{t} = 0 \tag{9}$$

hence the only non-zero Christoffel symbols are:

$$\Gamma_{tt}{}^{r} = \frac{1}{2}f' = \frac{\sinh r}{\cosh^3 r}\,, \quad \Gamma_{rt}{}^{t} = \Gamma_{tr}{}^{t} = \frac{1}{2}\frac{f'}{f} = \frac{1}{\sinh r\cosh r} \tag{10}$$

Now, the definition of the Riemann curvature tensor is:

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu\Gamma_{\nu\sigma}{}^{\rho} - \partial_\nu\Gamma_{\mu\sigma}{}^{\rho} + \Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\nu\sigma}{}^{\lambda} - \Gamma_{\nu\lambda}{}^{\rho}\Gamma_{\mu\sigma}{}^{\lambda} \tag{11}$$

The Ricci tensor is by definition

$$R_{\mu\nu} = R^{\rho}{}_{\mu\rho\nu} = \partial_\rho\Gamma_{\nu\mu}{}^{\rho} - \partial_\nu\Gamma_{\rho\mu}{}^{\rho} + \Gamma_{\rho\lambda}{}^{\rho}\Gamma_{\nu\mu}{}^{\lambda} - \Gamma_{\nu\lambda}{}^{\rho}\Gamma_{\rho\mu}{}^{\lambda} \tag{12}$$

and so

$$R = g^{\mu\nu}\partial_\rho\Gamma_{\mu\nu}{}^{\rho} - g^{\mu\nu}\partial_\mu\Gamma_{\nu\rho}{}^{\rho} + g^{\mu\nu}\Gamma_{\rho\lambda}{}^{\rho}\Gamma_{\mu\nu}{}^{\lambda} - g^{\mu\nu}\Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\rho\nu}{}^{\lambda} \tag{13}$$

We only need to calculate the Ricci scalar, because in $n = 2$ we must have

$$R_{\mu\nu} = \frac{1}{2}Rg_{\mu\nu} \tag{14}$$

(can be computed from expression for $R_{\mu\nu\rho\sigma}$ or from fact that given R is the only independent component of curvature, it must have this form). Let's just work out each term using the fact the metric only depends on r and the above Christoffel symbols:

$$g^{\mu\nu}\partial_\rho\Gamma_{\mu\nu}{}^{\rho} = g^{tt}\partial_r\Gamma_{tt}{}^{r} = -\frac{f''}{2f} \tag{15}$$

$$-g^{\mu\nu}\partial_\mu\Gamma_{\nu\rho}{}^{\rho} = -g^{rr}\partial_r\Gamma_{rt}{}^{t} = -\partial_r\frac{f'}{2f} = -\frac{f''}{2f} + \frac{(f')^2}{2f^2} \tag{16}$$

$$g^{\mu\nu}\Gamma_{\rho\lambda}{}^{\rho}\Gamma_{\mu\nu}{}^{\lambda} = g^{tt}\Gamma_{rt}{}^{t}\Gamma_{tt}{}^{r} = -\frac{(f')^2}{4f^2} \tag{17}$$

$$
\begin{aligned}
-g^{\mu\nu}\Gamma_{\mu\lambda}{}^{\rho}\Gamma_{\rho\nu}{}^{\lambda} &= -g^{tt}\Gamma_{t\lambda}{}^{\rho}\Gamma_{t\rho}{}^{\lambda} - g^{rr}\Gamma_{r\lambda}{}^{\rho}\Gamma_{r\rho}{}^{\lambda} \\
&= -g^{tt}\Gamma_{tt}{}^{r}\Gamma_{tr}{}^{t} - g^{tt}\Gamma_{tr}{}^{t}\Gamma_{tt}{}^{r} - g^{rr}\Gamma_{rt}{}^{t}\Gamma_{rt}{}^{t} \\
&= \frac{2}{f}\frac{(f')^2}{4f^2} - \frac{(f')^2}{4f^2} \\
&= \frac{(f')^2}{4f^2}
\end{aligned}
\tag{18}
$$

Thus using expressions for derivatives given previously

$$
R = -\frac{f''}{f} + \frac{(f')^2}{2f^2} = \frac{4}{\cosh^2 r} \tag{19}
$$

while

$$
R_{tt} = -\frac{2\sinh^2}{\cosh^4 r} \quad R_{rr} = \frac{2}{\cosh^2 r}\,, \quad R_{tr} = 0 \tag{20}
$$

The only possible curvature invariant is R, and this is finite at $r = 0$, so in fact this is a coordinate singularity (corresponding to the black hole horizon).

(c) *(2 pts)* Define a tortoise coordinate

$$
r_* = r + \ln(1 - e^{-2r}) \tag{21}
$$

and new coordinates

$$
2v = e^{t+r_*}\,, \quad 2u = -e^{-t+r_*}\,. \tag{22}
$$

What is the metric in these coordinates? What is the Ricci scalar? (Please do not work it out from scratch!) Comment on what you find if you extend the ranges of the coordinates u and v.

Solution: let's first work out

$$
dr_* = dr + \frac{1}{1-e^{-2r}}e^{-2r}(+2dr) = dr\frac{1+e^{-2r}}{1-e^{-2r}} = dr\frac{1}{\tanh r} \tag{23}
$$

Then

$$
2dv = 2v(dt + dr_*) = 2v(dt + \tfrac{dr}{\tanh r})\,, \quad 2du = 2u(-dt + dr_*) = 2u(-dt + \tfrac{dr}{\tanh r}) \tag{24}
$$

We observe that

$$
4dudv = 4\frac{uv}{\tanh^2 r}(-\tanh^2 r\, dt^2 + dr^2) \tag{25}
$$

Then note that

$$
4uv = -e^{2r_*} = -(1 - e^{-2r})^2 e^{2r} = -(e^r - e^{-r})^2 = -(2\sinh r)^2 \Rightarrow uv = -\sinh^2 r \tag{26}
$$

As $\cosh^2 - \sinh^2 = 1$, we can express

$$\tanh^2 r = \frac{\sinh^2}{1+\sinh^2 r} = \frac{-uv}{1-uv} \tag{27}$$

Hence

$$ds^2 = -\tanh^2 r dt^2 + dr^2 = -\frac{dudv}{1-uv} \tag{28}$$

The Ricci scalar is

$$R = \frac{4}{\cosh^2 r} = \frac{4}{1+\sinh^2 r} = \frac{4}{1-uv} \tag{29}$$

Originally we have $u < 0$, $v > 0$ and $uv < 0$, so $1 - uv$ can be never be zero. The horizon corresponds to $u = 0$ or $v = 0$. Say we extend the ranges to $u < 0$ and $v < 0$. There is then a singularity in the curvature when $uv = 1$. The situation is similar to Schwarzschild (see String theory and black holes by Edward Witten, Phys. Rev. D 44, 314 for the Kruskal diagram. However for the purposes of grading commenting on the curvature singularity is sufficient, I do not expect them to draw the Kruskal diagram).

2. *Flat slicing of de Sitter.* The de Sitter (dS) spacetime is defined by the equation

$$-(x^0)^2 + (x^1)^2 + \dots (x^4)^2 = L^2\,, \tag{30}$$

in (a fictitious) $\mathbb{R}^5$ with the Minkowski metric. In the lectures, we used the parametrisation

$$\begin{aligned}
x^0 &= L \sinh \frac{t}{L}\,,\\
x^1 &= L \cosh \frac{t}{L} \sin\chi \sin\theta \cos\phi\,,\\
x^2 &= L \cosh \frac{t}{L} \sin\chi \sin\theta \sin\phi\,,\\
x^3 &= L \cosh \frac{t}{L} \sin\chi \cos\theta\,,\\
x^4 &= L \cosh \frac{t}{L} \cos\chi\,,
\end{aligned} \tag{31}$$

in terms of globally defined coordinates $t \in (-\infty, \infty)$, and (χ, θ, ϕ) which are S^3 coordinates.

(a) *(1 pt)* Show that the following parametrisation

$$\begin{aligned}
x^0 &= L \sinh \frac{\hat{t}}{L} + \frac{r^2}{2L} e^{\frac{\hat{t}}{L}}\,,\\
x^4 &= L \cosh \frac{\hat{t}}{L} - \frac{r^2}{2L} e^{\frac{\hat{t}}{L}}\,,\\
x^i &= e^{\frac{\hat{t}}{L}} \hat{x}^i\,,
\end{aligned} \tag{32}$$

where $r^2 \equiv \delta_{ij}\hat{x}^i\hat{x}^j$, $i,j = 1,2,3$, obeys the defining equation (30), and gives the metric for dS as:

$$ds^2 = -d\hat{t}^2 + e^{2\frac{\hat{t}}{L}}\delta_{ij}d\hat{x}^i d\hat{x}^j \,. \tag{33}$$

This is called the *flat slicing*, because surfaces of constant $\hat{t}$ are flat.

Solution: firstly

$$-(x^0)^2+(x^4)^2+\delta_{ij}x^ix^j = L^2(\cosh^2\frac{\hat{t}}{L}-\sinh^2\frac{\hat{t}}{L})-\frac{r^2}{L}e^{\frac{\hat{t}}{L}}(\cosh\frac{\hat{t}}{L}+\sinh\frac{\hat{t}}{L})+e^{2\frac{\hat{t}}{L}}r^2 = L^2 \tag{34}$$

Secondly

$$\begin{aligned} dx^0 &= \left(\cosh\frac{\hat{t}}{L}+\frac{r^2}{2L^2}e^{\frac{\hat{t}}{L}}\right)d\hat{t}+\frac{rdr}{L}e^{\frac{\hat{t}}{L}} \\ dx^4 &= \left(\sinh\frac{\hat{t}}{L}-\frac{r^2}{2L^2}e^{\frac{\hat{t}}{L}}\right)d\hat{t}-\frac{rdr}{L}e^{\frac{\hat{t}}{L}} \\ dx^i &= e^{\frac{\hat{t}}{L}}\left(\frac{d\hat{t}}{L}\hat{x}^i+d\hat{x}^i\right) \end{aligned} \tag{35}$$

So:

$$\begin{aligned} -(dx^0)^2+(dx^4)^2+\delta_{ij}dx^idx^j &= d\hat{t}^2(-\cosh^2\frac{\hat{t}}{L}-\frac{r^2}{L^2}\cosh\frac{\hat{t}}{L}e^{\frac{\hat{t}}{L}}+\sinh^2\frac{\hat{t}}{L}-\frac{r^2}{L^2}\sinh\frac{\hat{t}}{L}e^{\frac{\hat{t}}{L}}) \\ &\quad -2d\hat{t}dr\frac{r}{L}e^{\frac{\hat{t}}{L}}(\cosh\frac{\hat{t}}{L}+\sinh\frac{\hat{t}}{L}) \\ &\quad +e^{2\frac{\hat{t}}{L}}\left(\frac{d\hat{t}^2r^2}{L^2}+\delta_{ij}d\hat{x}^id\hat{x}^j+2\frac{d\hat{t}}{L}\hat{x}^id\hat{x}_i\right) \\ &= -d\hat{t}^2+e^{2\frac{\hat{t}}{L}}\delta_{ij}d\hat{x}^id\hat{x}^j \end{aligned} \tag{36}$$

after using

$$\cosh\frac{\hat{t}}{L}+\sinh\frac{\hat{t}}{L} = e^{\hat{t}/L}\,,\quad dr = \frac{1}{r}\hat{x}_id\hat{x}^i \tag{37}$$

(b) *(2 pts)* Show that the flat FLRW solution for a universe dominated by a positive cosmological constant corresponds to the flat slicing of de Sitter.

Solution: the Friedmann equation is

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho_0\left(\frac{a}{a_0}\right)^{-3(1+w)}-\frac{k}{a^2}\,. \tag{38}$$

We are asked to consider $w=-1$, $k=0$, so letting $C^2 = \frac{8\pi G\rho_0}{3}$ we have

$$\frac{\dot{a}}{a} = \pm C \tag{39}$$

As the universe is expanding, the $+$ sign is appropriate. Hence $a(t) = Ae^{Ct}$ where A is a constant of integration. We can absorb this into a shift of the origin of the time coordinate, hence $a(t) = e^{Ct}$ and

$$ds^2 = -dt^2 + e^{2Ct}\delta_{ij}dx^i dx^j \tag{40}$$

which we can identify exactly with the flat slicing of dS with $L = \frac{1}{C}$. (Relabelling with hats is trivial.)

(c) *(2 pts)* The coordinates (32) are only defined for $x^0 + x^4 \geq 0$. By equating (32) to the original coordinates used, (31), (using spherical polar coordinates for the $\hat{x}^i$), show that on the Penrose diagram, the flat slicing corresponds to the left upper triangular region $\eta \geq \chi - \frac{\pi}{2}$.

Solution: the condition $x^0 + x^4 \geq 0$ implies

$$L \sinh \frac{t}{L} + \cosh \frac{t}{L} \cos\chi \geq 0 \Rightarrow \tanh\frac{t}{L} \geq -\cos\chi \tag{41}$$

Now, the Penrose diagram of dS was obtained by defining η via $\cosh\frac{t}{L} = \frac{1}{\cos\eta}$. Hence

$$\sinh\tfrac{t}{L} = \pm\sqrt{\cosh\tfrac{t}{L} - 1} = \pm\sqrt{\frac{1-\cos^2\eta}{\cos^2\eta}} = \pm\frac{\sin\eta}{\cos\eta} \tag{42}$$

The range of η is $(-\pi/2, \pi/2)$. When $t > 0$, $\eta \in (0, \pi/2)$, hence both $\sin\eta$ and $\cos\eta$ are positive, as is $\sinh\frac{t}{L}$. When $t < 0$, $\eta \in (-\pi/2, 0)$, where $\cos\eta$ is positive and $\sin\eta$ is negative, while $\sinh\frac{t}{L}$ is negative. Hence the plus sign is correct: $\sinh\frac{t}{L} = \frac{\sin\eta}{\cos\eta}$. Thus $\tanh\frac{t}{L} = \sin\eta$. We therefore find that $x^0 + x^4 \geq 0$ corresponds to

$$\sin\eta \geq -\cos\chi \tag{43}$$

Now, $\chi \in (0,\pi)$. To make sense of this inequality, it is convenient to shift the arguments:

$$\sin(\chi - \frac{\pi}{2}) = \frac{1}{2i}\left(e^{i\chi}e^{-i\pi/2} - e^{-i\chi}e^{i\pi/2}\right) = -\cos\chi \tag{44}$$

using $e^{\pm i\pi/2} = \cos\frac{\pi}{2} \pm i\sin\frac{\pi}{2} = \pm i$. So we have a condition:

$$\sin\eta \geq \sin(\chi - \frac{\pi}{2}) \tag{45}$$

It can then be checked that sin is increasing on $\eta \in (-\pi/2, \pi/2)$ and $\chi - \pi/2$ in the same interval, thus we have

$$\eta \geq \chi - \frac{\pi}{2} \tag{46}$$