General Relativity 2020/21 - Homework 2

Given: 06/11, due: 20/11.

1. Consider the two-sphere with coordinates (θ, ϕ) and metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

   (a) *(1 pt)* Show that lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).
   Solution: The geodesic equations (from e.g. Problem Sheet 6) are

$$\ddot{\theta} = \sin\theta\cos\theta\dot{\phi}^2\,, \quad \ddot{\phi} = 2\frac{\cos\theta}{\sin\theta}\dot{\theta}\dot{\phi}\,. \tag{1}$$

   For $\phi = \phi_0$ constant, we have just $\ddot{\theta} = 0$ so $\phi(\lambda) = \phi_0$, $\theta(\lambda) = a\lambda + \theta_0$ is indeed a geodesic. Now suppose $\theta = \theta_0$ constant. Then:

$$0 = \sin\theta_0\cos\theta_0\dot{\phi}^2\,, \quad \ddot{\phi} = 0 \tag{2}$$

   One solution is $\dot{\phi} = 0$ but this is means that both ϕ and θ are constant, which we can disregard as a trivial solution to the geodesic equation. The only alternative is that $\ddot{\phi} = 0$ and either $\sin\theta_0 = 0$ or $\cos\theta_0 = 0$ i.e. $\theta = 0, \pi$ or $\theta = \frac{\pi}{2}$. We disregard the former as we working on the chart with $\theta \in (0, \pi)$, hence the only solution is $\phi(\lambda) = a\lambda + \phi_0$, $\theta(\lambda) = \frac{\pi}{2}$.

   (b) *(1 pt)* Take a vector with components $V^\mu = (1, 0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of θ?
   Solution: We need to solve the parallel transport equation $\nabla_X V = 0$ where X is the tangent to the curve $x^\mu(\lambda) = (\theta(\lambda), \phi(\lambda)) = (\theta_0, \lambda + \phi_0)$, with $\lambda \in [0, 2\pi]$, thus $X^\mu = (0, 1)$. (Note: you could rescale $\lambda \to a\lambda$ but then we would also take $2\pi \to 2\pi/a$ so no effect on the final answer).) This equation is

$$\nabla_X V^\mu = X^\nu \partial_\nu V^\mu + X^\nu \Gamma_{\nu\rho}{}^\mu V^\rho = 0 \tag{3}$$

   or more usefully on the curve we have

$$\frac{d}{d\lambda} V^\mu(\lambda) + X^\nu(\lambda)\Gamma_{\nu\rho}{}^\mu V^\nu(\lambda) = 0 \tag{4}$$

   Thus

$$0 = \frac{d}{d\lambda}V^\theta + \Gamma_{\phi\phi}{}^\theta V^\phi + \Gamma_{\phi\theta}{}^\theta V^\theta = \frac{d}{d\lambda}V^\theta - \sin\theta\cos\theta V^\phi \tag{5}$$

$$0 = \frac{d}{d\lambda}V^\phi + \Gamma_{\phi\phi}{}^\phi V^\phi + \Gamma_{\phi\theta}{}^\phi V^\theta = \frac{d}{d\lambda}V^\phi + \frac{\cos\theta}{\sin\theta}V^\theta \tag{6}$$

Differentiating again (θ is a constant)

$$\frac{d^2V^\theta}{d\lambda^2} = -\cos^2\theta V^\theta\,, \quad \frac{d^2V^\phi}{d\lambda^2} = -\cos^2\theta V^\phi\,, \tag{7}$$

So

$$V^\theta = A\cos(\cos\theta\lambda) + B\sin(\cos\theta\lambda)\,, \quad V^\phi = C\cos(\cos\theta\lambda) + D\sin(\cos\theta\lambda) \tag{8}$$

Initially, $V^\theta(\lambda = 0) = 1$ and $V^\phi(\lambda = 0)$ so $A = 1$, $C = 0$. We also have $\frac{d}{d\lambda}V^\theta(\lambda = 0) = 0$ and $\frac{d}{d\lambda}V^\phi(\lambda = 0) = \cos\theta/\sin\theta$. Hence $B = 0$ and $D = -1/\sin\theta$. Thus:

$$V^\mu(\lambda) = \left(\cos(\cos\theta\lambda), -\frac{\sin(\cos\theta\lambda)}{\sin\theta}\right) \tag{9}$$

Result of parallel transporting around a full circle is that $V^\mu(\lambda = 0) = (1, 0)$ is mapped to

$$V^\mu(2\pi) = \left(\cos(2\pi\cos\theta), -\frac{\sin(2\pi\cos\theta)}{\sin\theta}\right) \tag{10}$$

2. A good approximation to the metric outside the surface of the Earth is

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{11}$$

where $\Phi = -GM/r$ may be thought of as the usual Newtonian gravitational potential (assumed to be small), where G is Newton's constant and M is the mass of the Earth.

(a) *(2 pts)* Compute the geodesic equations. Consider a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

Solution: Geodesic eqns can be obtained from

$$L = \frac{1}{2}g_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} = -\frac{1}{2}(1 + 2\Phi)\dot{t}^2 + \frac{1}{2}(1 - 2\Phi)\dot{r}^2 + \frac{1}{2}r^2(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{12}$$

Euler-Lagrange equations

$$\frac{d}{d\tau}\frac{\partial L}{\partial\dot{x}} = \frac{\partial L}{\partial x} \tag{13}$$

so

$$\frac{d}{d\tau}\left((1 + 2\Phi)\dot{t}\right) = 0 \Rightarrow (1 + 2\Phi)\dot{t} = \text{constant} \tag{14}$$

$$\frac{d}{d\tau}\left(r^2\sin^2\theta\dot{\phi}\right) = 0 \Rightarrow r^2\sin^2\theta\dot{\phi} = \text{constant} \tag{15}$$

$$\frac{d}{d\tau}\left(r^2\dot{\theta}\right) = r^2\sin\theta\cos\theta\dot{\phi}^2 \tag{16}$$

$$\frac{d}{d\tau}\left((1-2\Phi)\dot{r}^2\right) = -\frac{d\Phi}{dr}(\dot{t}^2+\dot{r}^2) + r(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) \tag{17}$$

Circular orbit: $\theta = \pi/2$, $r =$ constant. The last equation gives

$$0 = -\frac{d\Phi}{dr}\dot{t}^2 + r\dot{\phi}^2 \Rightarrow \left(\frac{d\phi}{dt}\right)^2 = \frac{1}{r}\frac{GM}{r^2} \Rightarrow \frac{d\phi}{dt} = \pm\sqrt{\frac{GM}{r^3}} \tag{18}$$

(b) *(2 pts)* How much proper time elapses while a satellite at radius R completes one such orbit. How does this compare to the proper time elapsed on a clock stationary at radius R?

Solution: Proper time on circular orbit with $r = R$ and $\theta = \pi/2$ is:

$$d\tau^2 = -ds^2 = (1+2\Phi)dt^2 - R^2 d\phi^2 \tag{19}$$

Use $dt^2 = \frac{R^3}{GM}d\phi^2$ then

$$d\tau^2 = \left((1+2\Phi)\frac{R^3}{GM} - R^2\right)d\phi^2 = \left(\frac{R^3}{GM} - 3R^2\right)d\phi^2 \tag{20}$$

Proper time on one orbit means integrate from 0 to 2π, hence

$$\Delta\tau_{\text{sat}} = 2\pi R\sqrt{\frac{R}{GM} - 3} = 2\pi R\sqrt{\frac{R}{GM}}\sqrt{1 - \frac{3GM}{R}} \tag{21}$$

Meanwhile the amount of coordinate time that elapses is $t = \sqrt{\frac{R^3}{GM}}2\pi$. For a stationary observer

$$d\tau^2 = -ds^2 = (1+2\Phi)dt^2 \Rightarrow \Delta\tau_{\text{stat}} = 2\pi R\sqrt{\frac{R}{GM}}\sqrt{1 - \frac{2GM}{R}} \tag{22}$$

(Note $|\Phi| = GM/R$ is small so the square root is well-defined.) We see that

$$\Delta\tau_{\text{sat}} < \Delta\tau_{\text{stat}} \tag{23}$$

3. Consider the geodesic equation

$$\ddot{x}^\mu + \Gamma_{\nu\rho}{}^{\mu}(x)\dot{x}^\nu\dot{x}^\rho = 0\,, \tag{24}$$

for an affine geodesic $x^\mu(\lambda)$, with $\dot{x}^\mu \equiv dx^\mu/d\lambda$, where $\Gamma_{\mu\nu}{}^{\rho}$ are the components of the Levi-

Civita connection. Take as initial conditions

$$x^\mu(\lambda=0)=\bar{x}^\mu\,,\quad \dot{x}^\mu(\lambda=0)=X^\mu\,. \tag{25}$$

(a) *(1 pt)* By Taylor expanding $x^\mu(\lambda)$ in the parameter λ, show that the solution to the geodesic equation has the form

$$x^\mu(\lambda)=\bar{x}^\mu+\lambda X^\mu-\frac{\lambda^2}{2}\bar{\Gamma}_{\nu\rho}{}^\mu X^\nu X^\rho-\frac{\lambda^3}{6}(\partial_\sigma\bar{\Gamma}_{\nu\rho}{}^\mu-\bar{\Gamma}_{\lambda\rho}{}^\mu\bar{\Gamma}_{\sigma\nu}{}^\lambda-\bar{\Gamma}_{\nu\lambda}{}^\mu\bar{\Gamma}_{\sigma\rho}{}^\lambda)X^\sigma X^\nu X^\rho+O(\lambda^4) \tag{26}$$

where $\bar{\Gamma}_{\mu\nu}{}^\rho\equiv\Gamma_{\mu\nu}{}^\rho(\lambda=0)$.

Solution: Taylor expanding with the given initial conditions gives

$$\begin{aligned}
x^\mu(\lambda)&=\bar{x}^\mu+X^\mu\lambda+\frac{1}{2}c_2^\mu\lambda^2+\frac{1}{6}c_3^\mu\lambda^3+O(\lambda)^4\\
\dot{x}^\mu(\lambda)&=X^\mu+c_2^\mu\lambda+\frac{1}{2}c_3^\mu\lambda^2+O(\lambda)^3\\
\ddot{x}^\mu(\lambda)&=c_2^\mu+c_3^\mu\lambda+O(\lambda)^2\\
\Gamma_{\nu\rho}{}^\mu(x)&=\bar{\Gamma}_{\nu\rho}{}^\mu+\partial_\sigma\bar{\Gamma}_{\nu\rho}{}^\mu\lambda X^\sigma+O(\lambda)^2
\end{aligned} \tag{27}$$

Plugging in to the geodesic equation and keeping terms only up to order λ gives

$$c_2^\mu+\bar{\Gamma}_{\nu\rho}{}^\mu X^\nu X^\rho+\lambda\left(c_3^\mu+\bar{\Gamma}_{\nu\rho}{}^\mu(c_2^\nu X^\rho+c_2^\rho X^\nu)+\partial_\sigma\bar{\Gamma}_{\nu\rho}{}^\mu X^\sigma X^\nu X^\rho\right)=0 \tag{28}$$

hence

$$c_2^\mu=-\bar{\Gamma}_{\nu\rho}{}^\mu X^\nu X^\rho \tag{29}$$

and thus

$$c_3^\mu=\left(-\partial_\sigma\bar{\Gamma}_{\nu\rho}{}^\mu+\bar{\Gamma}_{\lambda\rho}{}^\mu\bar{\Gamma}_{\sigma\nu}{}^\lambda+\bar{\Gamma}_{\nu\lambda}{}^\mu\bar{\Gamma}_{\sigma\rho}{}^\lambda\right)X^\sigma X^\nu X^\rho \tag{30}$$

(b) *(1 pt)* The above solution is valid in any coordinate system. Suppose we take $x^\mu(\lambda=0)$ to be some point p on the manifold M, and use Riemann normal coordinates around p. In this case, the coordinates of the points in the neighbourhood of p correspond to tangent vectors $X\in T_pM$, by following a geodesic through p with initial tangent vector X from $\lambda=0$ to $\lambda=1$. In this case we know that geodesics have the form $x^\mu(\lambda)=\lambda X^\mu$. Hence show that in Riemann normal coordinates that

$$\bar{\Gamma}_{\mu\nu}{}^\rho=0\,,\quad \partial_{(\sigma}\bar{\Gamma}_{\nu\rho)}{}^\mu=0\,, \tag{31}$$

where barred quantities are evaluated at $\lambda=0$ i.e. at the origin ($\bar{x}^\mu=0$) in Riemann normal coordinates. Show further that

$$\partial_\rho \bar{\Gamma}_{\mu\nu}{}^\sigma = \frac{1}{3}(\bar{R}^\sigma{}_{\mu\rho\nu} + \bar{R}^\sigma{}_{\nu\rho\mu})\,. \tag{32}$$

Solution: as the geodesic solution is exact to order λ in RNCs, the higher order terms must all vanish. So in particular from order λ^2

$$\bar{\Gamma}_{(\mu\nu)}{}^\rho = \bar{\Gamma}_{\mu\nu}{}^\rho = 0 \tag{33}$$

and hence from order λ^3

$$\partial_{(\sigma}\bar{\Gamma}_{\nu\rho)}{}^\mu = 0 \tag{34}$$

Now consider the Riemann tensor evaluated at the origin:

$$\begin{aligned} \bar{R}^\sigma{}_{\nu\rho\mu} &= \partial_\rho \bar{\Gamma}_{\mu\nu}{}^\sigma - \partial_\mu \bar{\Gamma}_{\rho\nu}{}^\sigma\,, \\ \bar{R}^\sigma{}_{\mu\rho\nu} &= \partial_\rho \bar{\Gamma}_{\mu\nu}{}^\sigma - \partial_\nu \bar{\Gamma}_{\rho\mu}{}^\sigma\,, \end{aligned} \tag{35}$$

Hence

$$\bar{R}^\sigma{}_{\nu\rho\mu} + \bar{R}^\sigma{}_{\mu\rho\nu} = 2\partial_\rho \bar{\Gamma}_{\mu\nu}{}^\sigma - \partial_\nu \bar{\Gamma}_{\rho\mu}{}^\sigma - \partial_\mu \bar{\Gamma}_{\mu\rho}{}^\sigma = 3\partial_\rho \bar{\Gamma}_{\mu\nu}{}^\sigma \tag{36}$$

(c) *(2 pts)* By Taylor expanding in Riemann normal coordinates about the origin, show that an arbitrary second rank tensor $T_{\mu\nu}$ can be expressed as

$$T_{\mu\nu}(X) = \bar{T}_{\mu\nu} + X^\rho \bar{\nabla}_\rho \bar{T}_{\mu\nu} + \frac{1}{2}X^\rho X^\sigma (\bar{\nabla}_\rho \bar{\nabla}_\sigma \bar{T}_{\mu\nu} - \frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\mu}\bar{T}_{\lambda\nu} - \frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\nu}\bar{T}_{\mu\lambda}) + \ldots \tag{37}$$

Explain how to relate this result to the statement that a non-vanishing Riemann curvature tensor prevents one from choosing coordinates such that the metric $g_{\mu\nu}$ is everywhere equal to the Minkowski metric.

Solution: the Taylor expansion $x^\mu = 0^\mu + X^\mu$ gives

$$\begin{aligned} T_{\mu\nu}(X) &= \bar{T}_{\mu\nu} + X^\rho \partial_\rho \bar{T}_{\mu\nu} + \frac{1}{2}X^\rho X^\sigma \partial_\rho \partial_\sigma \bar{T}_{\mu\nu} \\ &= \bar{T}_{\mu\nu} + X^\rho \bar{\nabla}_\rho \bar{T}_{\mu\nu} + \frac{1}{2}X^\rho X^\sigma \partial_\rho \partial_\sigma \bar{T}_{\mu\nu} \end{aligned} \tag{38}$$

using that $\bar{\nabla} = \partial$. Next, consider

$$\begin{aligned} \bar{\nabla}_\rho \bar{\nabla}_\sigma \bar{T}_{\mu\nu} &= \partial_\rho \partial_\sigma \bar{T}_{\mu\nu} + \partial_\rho \bar{\Gamma}_{\sigma\mu}{}^\lambda T_{\lambda\nu} + \partial_\rho \bar{\Gamma}_{\sigma\nu}{}^\lambda T_{\mu\lambda} \\ &= \partial_\rho \partial_\sigma \bar{T}_{\mu\nu} + \frac{1}{3}\left(\bar{R}^\lambda{}_{\sigma\rho\mu} + \bar{R}^\lambda{}_{\mu\rho\sigma}\right) T_{\lambda\nu} + \frac{1}{3}\left(\bar{R}^\lambda{}_{\sigma\rho\nu} + \bar{R}^\lambda{}_{\nu\rho\sigma}\right) T_{\mu\lambda} \end{aligned} \tag{39}$$

thus

$$\begin{aligned} X^\rho X^\sigma \partial_\rho \partial_\sigma \bar{T}_{\mu\nu} &= X^\rho X^\sigma \left(\bar{\nabla}_\rho \bar{\nabla}_\sigma \bar{T}_{\mu\nu} - \frac{1}{3}\left(\bar{R}^\lambda{}_{\sigma\rho\mu} + R^\lambda{}_{\mu\rho\sigma}\right) T_{\lambda\nu} - \frac{1}{3}\left(\bar{R}^\lambda{}_{\sigma\rho\nu} + R^\lambda{}_{\nu\rho\sigma}\right) T_{\mu\lambda} \right) \\ &= X^\rho X^\sigma \left(\bar{\nabla}_\rho \bar{\nabla}_\sigma \bar{T}_{\mu\nu} - \frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\mu} T_{\lambda\nu} - \frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\nu} T_{\mu\lambda} \right) \end{aligned} \tag{40}$$

using symmetry/antisymmetry contraction. Hence the desired expression.
For the special case $T_{\mu\nu} = g_{\mu\nu}$ one obtains

$$g_{\mu\nu}(X) = \bar{g}_{\mu\nu} + \frac{1}{2} X^\rho X^\sigma \left(-\frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\mu}\bar{g}_{\lambda\nu} - \frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\nu}\bar{g}_{\mu\lambda}\right) + \ldots \tag{41}$$

so then although $g_{\mu\nu} = \bar{g}_{\mu\nu}$ can be taken to be the Minkowski metric in RNCs, and its first derivative vanishes, its second derivative will be non-zero if the Riemann tensor is non-zero at the origin in these coordinates. Thus the metric will not be constant in RNCs everywhere as long as Riemann tensor is non-zero somewhere.