General Relativity 2021/22 - Homework 1

1. In four-dimensional Minkowski spacetime, define new coordinates (t', x', y', z') by

$$t = \left(\frac{c}{g} + \frac{x'}{c}\right) \sinh\left(\frac{gt'}{c}\right) , \quad x = c\left(\frac{c}{g} + \frac{x'}{c}\right) \cosh\left(\frac{gt'}{c}\right) - \frac{c^2}{g} , \quad y = y' , \quad z = z' , \qquad (1)$$

where g is a constant with the dimensions of acceleration.

(a) *(1 pt)* Compute the line element of special relativity in terms of these new coordinates.

Solution: We have

$$\begin{aligned} dt &= \left(1 + \frac{gx'}{c^2}\right) \cosh(gt'/c)dt' + \sinh(gt'/c)\frac{dx'}{c} \\ dx &= c\left(1 + \frac{gx'}{c^2}\right) \sinh(gt'/c)dt' + \cosh(gt'/c)dx' \end{aligned} \qquad (2)$$

hence remembering that $\cosh^2 - \sinh^2 = 1$

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 = -c^2\left(1 + \frac{gx'}{c^2}\right)^2 (dt')^2 + (dx')^2 + (dy')^2 + (dz')^2 \quad (3)$$

(b) *(1 pt)* For $gt'/c << 1$ show that this corresponds to a transformation to a uniformly accelerated frame in Newtonian mechanics.

Solution: for small ϵ one has $\cosh \epsilon \approx 1 + \frac{1}{2}\epsilon^2$, $\sinh \epsilon \approx \epsilon$. Hence

$$t \approx \left(\frac{c}{g} + \frac{x'}{c}\right) \frac{gt'}{c} = t' + \frac{x'}{c}\frac{gt'}{c} \qquad (4)$$

$$x \approx x' + (c^2/g + x')\frac{1}{2}\frac{(gt')^2}{c^2} = x' + \frac{1}{2}gt'^2 + \frac{1}{2}x'\left(\frac{gt'}{c}\right)^2 \qquad (5)$$

so dropping the terms involving gt'/c we have

$$t \approx t' , \quad x \approx x' + \frac{1}{2}gt^2 \qquad (6)$$

which describes a transformation to a frame uniformly accelerating with acceleration g.

(c) *2 ps)* Show that an at-rest clock in this frame at $x' = h$, $h > 0$, runs fast compared to a clock at $x' = 0$ by a factor $(1 + gh/c^2)$. How is this related to the equivalence principle?

Solution: for the clock at-rest at $x' = 0$ we have

$$c^2(d\tau)^2 = c^2(dt')^2 \qquad (7)$$

while for the clock at $x' = h$ we have

$$c^2(d\tau)^2 = c^2(1 + gh/c^2)^2(dt')^2 \tag{8}$$

In the same coordinate time interval $\Delta t'$ (corresponding to the coordinate time between light signals sent from one position to the other - geometry is time independent observer sends/receives signals at same intervals as measured in coordinate time) we have

$$\Delta\tau\Big|_{x=0} = \Delta t' \,, \quad \Delta\tau\Big|_{x=h} = (1 + gh/c^2)\Delta t' \tag{9}$$

so indeed

$$\Delta\tau\Big|_{x=h} = (1 + gh/c^2)\Delta\tau\Big|_{x=0} \tag{10}$$

For $h > 0$, we see that the time elapsed on the clock at $x = h$ is greater than that on the clock at $x = 0$, so it runs faster (i.e. records more time).

According to the Weak Equivalence Principle in Newtonian mechanics, we cannot distinguish between being in a gravitational field at rest and being in an accelerating frame in a situation without a gravitational field. Meanwhile, the Einstein Equivalence Principle asserts that the physics in locally inertial frames is indistinguishable from physics in Minkowski spacetime. In the lectures, we used this idea to show that starting with observers in a gravitational field, and passing to an inertial frame, that there is a gravitational time dilation effect. In the above example, the same time dilation effect is shown to be present for an accelerating observer in Minkowski spacetime. Thus we have the same physics in both cases, in accordance with the equivalence principle.

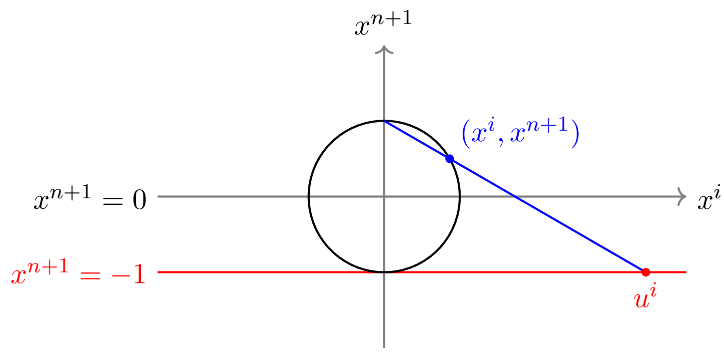
2. Consider the n-sphere defined in $\mathbb{R}^{n+1}$ by the equation $(x^1)^2 + \cdots + (x^{n+1})^2 = 1$.

   (a) *(2 pts)* Show that the stereographic projection from the north pole $(x^1, \ldots, x^n, x^{n+1}) = (0, \ldots, 0, 1)$ to the plane $x^{n+1} = -1$ leads to the map

$$\phi_S : (x^1, \ldots, x^n, x^{n+1}) \mapsto u^i \equiv \frac{2x^i}{1 - x^{n+1}} \,, \tag{11}$$

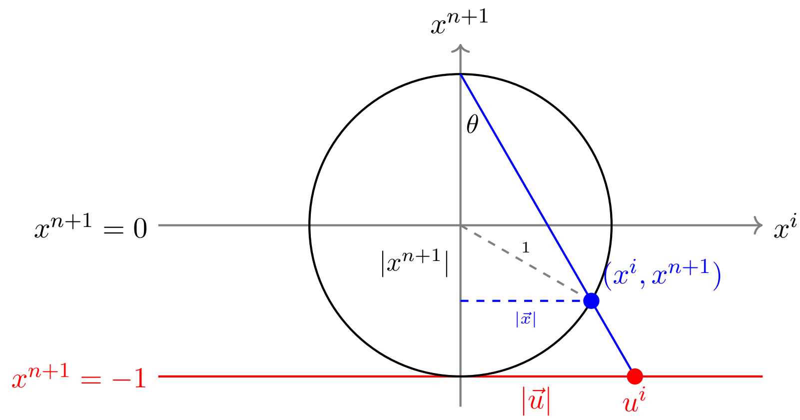
   defined on $U_S \equiv S^n \backslash \{0, \ldots 0, 1\}$, where u^i are the Cartesian coordinates on the plane.

   For a given point on the sphere with coordinates $(\vec{x}, x^{n+1})$, where $\vec{x} = (x^1, \ldots, x^n)$, the below diagram shows the geometry in the two-dimensional plane corresponding to the directions $(\hat{n}, x^{n+1})$, where $\hat{n} = \vec{x}/|\vec{x}|$ is the n-dimensional unit vector in the direction picked out by $\vec{x}$.



Solution: let $\vec{x} = (x^1, \ldots, x^n)$ and $\hat{n} = \vec{x}/|\vec{x}|$ denote the n-dimensional unit vector in the direction of $\vec{x}$. What we consider is the two-dimensional geometry in the $(\hat{n}, x^{n+1})$ plane. The vector $\vec{u} = (u^1, \ldots, u^n)$ must point in the same n-dimensional direction as $\vec{x}$ hence it is proportional to $\hat{n}$ and lies in this plane. Let $\vec{u} = \gamma\vec{x}$.

For points on the sphere with $x^{n+1} < 0$, we have the following picture:

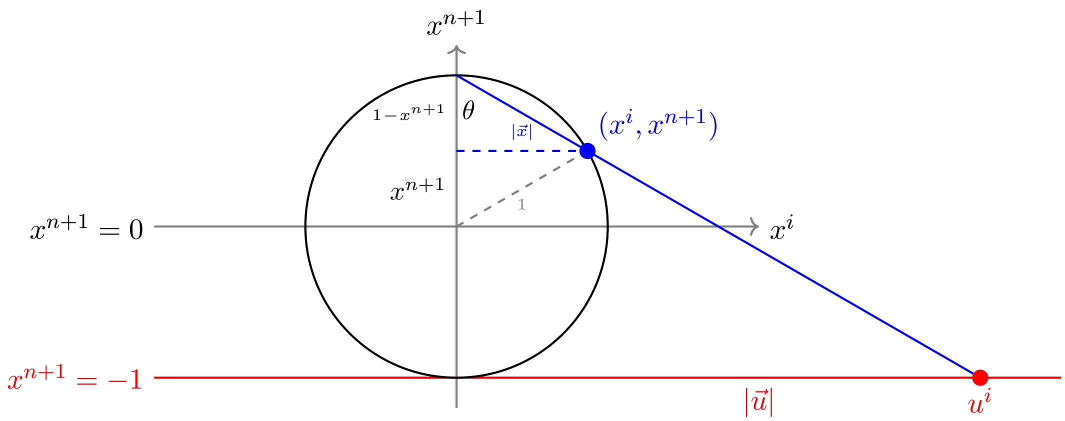


We have

$$\tan\theta = \frac{|\vec{u}|}{2} = \frac{|\vec{x}|}{1 + |x^{n+1}|} = \frac{|\vec{x}|}{1 - x^{n+1}} \tag{12}$$

$$\Rightarrow \frac{1}{2}\gamma|\vec{x}| = \frac{|\vec{x}|}{1 - x^{n+1}} \Rightarrow \gamma = \frac{2}{1 - x^{n+1}} \tag{13}$$

For points on the sphere with x^{n+1} positive we have the following picture:



This time we also have

$$\tan\theta = \frac{|\vec{u}|}{2} = \frac{|\vec{x}|}{1-x^{n+1}} \tag{14}$$

so in all cases

$$\vec{u} = \frac{2\vec{x}}{1-x^{n+1}} \tag{15}$$

(b) (*2 pts*) Similarly, the stereographic projection from the south pole $(x^1,\ldots,x^n,x^{n+1}) = (0,\ldots,0,-1)$ to the plane $x^{n+1}=+1$ leads to the map

$$\phi_N : (x^1,\ldots x^n, x^{n+1}) \mapsto \tilde{u}^i \equiv \frac{2x^i}{1+x^{n+1}}\,, \tag{16}$$

defined on $U_N \equiv S^n\backslash\{0,\ldots 0,-1\}$ (no need to show this). Show that on the overlap $U_N\cap U_S$ that the transition function $\phi_N\circ\phi_S^{-1}$ defines a smooth map

$$\tilde{u}^i(u) = \frac{4u^i}{(u^1)^2+\ldots(u^n)^2}\,, \tag{17}$$

and hence these give an atlas for the n-sphere.

Solution: we can write (on the overlap)

$$\tilde{u}^i = \frac{2x^i}{1+x^{n+1}} = \frac{1-x^{n+1}}{1+x^{n+1}}u^i \tag{18}$$

and taking the norm of the u^i we have

$$|\vec{u}|^2 = \frac{4|\vec{x}|^2}{(1-x^{n+1})} = \frac{4(1-x^{n+1})^2}{(1-x^{n+1})^2} = \frac{4(1+x^{n+1})}{1-x^{n+1}} \tag{19}$$

from which one derives

$$\tilde{u}^i(u) = \frac{4u^i}{|\vec{u}|^2} \tag{20}$$

This is only in danger of not being differentiable for $|\vec{u}| = 0$ which we see will happen for $x^{n+1} = -1$, but this is obviously not in the overlap. As we have two compatible charts whose union covers the whole sphere, we have an atlas.

(c) (*2 pts*) Express the embedding coordinates (x^i, x^{n+1}) of the sphere in $\mathbb{R}^n$ in terms of the u^i and hence show that the metric on the sphere in the chart U_S is

$$ds^2 = \frac{\delta_{ij} du^i du^j}{(1 + \frac{1}{4}\delta_{kl} u^k u^l)^2} \tag{21}$$

Solution: from (19) we have

$$\frac{1}{4}|\vec{u}|^2(1 - x^{n+1}) = 1 + x^{n+1} \Rightarrow x^{n+1} = \frac{\frac{1}{4}|\vec{u}|^2 - 1}{\frac{1}{4}|\vec{u}|^2 + 1} \tag{22}$$

hence

$$x^i = \frac{1}{2}u^i \left(1 - \frac{\frac{1}{4}|\vec{u}|^2 - 1}{\frac{1}{4}|\vec{u}|^2 + 1}\right) = \frac{u^i}{\frac{1}{4}|\vec{u}|^2 + 1} \tag{23}$$

and so

$$dx^i = \frac{du^i}{\frac{1}{4}|\vec{u}|^2 + 1} - \frac{1}{2}\frac{u^i u_j du^j}{\left(\frac{1}{4}|\vec{u}|^2 + 1\right)^2} \tag{24}$$

We also have

$$x_i dx^i + x^{n+1} dx^{n+1} = 0 \tag{25}$$

Then

$$\begin{aligned}
ds^2 &= dx^i dx_i + (dx^{n+1})^2 \\
&= dx^i dx_i + x_i x_j dx^i dx^j \frac{1}{(x^{n+1})^2} \\
&= \left(\delta_{ij} + \frac{u_i u_j}{\left(\frac{1}{4}|\vec{u}|^2 - 1\right)^2}\right) dx^i dx^j \\
&= \left(\delta_{ij} + \frac{u_i u_j}{\left(\frac{1}{4}|\vec{u}|^2 - 1\right)^2}\right)\left(\frac{du^i du^j}{\left(\frac{1}{4}|\vec{u}|^2 + 1\right)^2} - \frac{u^i du^j u_k du^k}{\left(\frac{1}{4}|\vec{u}|^2 + 1\right)^3} + \frac{1}{4}\frac{u^i u^j (u_k du^k)^2}{\left(\frac{1}{4}|\vec{u}|^2 + 1\right)^4}\right)
\end{aligned} \tag{26}$$

Product of first term in first bracket with first term in second bracket gives the desired result. We want everything else to cancel. Adopting some obvious shorthands, everything else is:

$$-\frac{(u\cdot du)^2}{\left(\frac{1}{4}u^2 + 1\right)^3} + \frac{1}{4}\frac{u^2 (u\cdot du)^2}{\left(\frac{1}{4}u^2 + 1\right)^4} + \frac{1}{\left(\frac{1}{4}u^2 - 1\right)^2}\left(\frac{(u\cdot du)^2}{\left(\frac{1}{4}u^2 + 1\right)^2} - \frac{u^2 (u\cdot du)^2}{\left(\frac{1}{4}u^2 + 1\right)^3} + \frac{1}{4}\frac{(u^2)^2 (u\cdot du)^2}{\left(\frac{1}{4}u^2 + 1\right)^4}\right) \tag{27}$$

Taking out the common $(u \cdot du)^2$ factor we have

$$-\frac{1}{\left(\frac{1}{4}u^2+1\right)^3}+\frac{\frac{1}{4}u^2}{\left(\frac{1}{4}u^2+1\right)^4}+\frac{1}{\left(\frac{1}{4}u^2-1\right)^2\left(\frac{1}{4}u^2+1\right)^2}\left(1-\frac{1}{2}\frac{u^2}{\frac{1}{4}u^2+1}\right)^2 \tag{28}$$

which is

$$-\frac{1}{\left(\frac{1}{4}u^2+1\right)^4}+\frac{1}{\left(\frac{1}{4}u^2-1\right)^2\left(\frac{1}{4}u^2+1\right)^4}\left(\frac{1}{4}u^2-1\right)^2=0 \tag{29}$$