General Relativity - Exercises - Lecture 9
VUB MSc in Physics & Astronomy, 2021-22

1. Suppose you are sufficiently interested in physics that you decide to fall into a black hole to find out what happens beyond the event horizon. Let τ denote your proper time. At $\tau = 0$ you start at rest at $r = r_0 > 2GM$, and then follow a radial geodesic into the black hole. Write down a first-order ordinary differential equation determining your trajectory $r(\tau)$. What is the value of the conserved quantity E? Compute the proper time you experience before hitting the singularity at $r = 0$. You may need the integral:

$$\int_0^1 du \sqrt{\frac{u}{1-u}} = \frac{\pi}{2} \tag{1}$$

2. Suppose you find yourself just inside the black hole horizon at $r = 2GM$. Compute the maximal proper time you can possibly experience before hitting the singularity. (You can assume that $\theta = \pi/2$.)

3. The Schwarzschild metric in ingoing Eddington-Finkelstein coordinates is:

$$ds^2 = -f(r)dv^2 + 2dvdr + r^2(d\theta^2 + \sin^2\theta d\phi)\,, \tag{2}$$

where $v = t + r + 2GM \ln\left|\frac{r}{2GM} - 1\right|$ and $f(r) = 1 - \frac{2GM}{r}$.

   (a) Consider the Killing vector $\xi \equiv \frac{\partial}{\partial t}$ associated to the time independent nature of the Schwarzschild solution. Express this in terms of the ingoing EF coordinates. Is it everywhere timelike?

   (b) Compute the covariant derivative

$$\nabla_\xi \xi^\mu \tag{3}$$

   and show that at $r = 2GM$

$$\nabla_\xi \xi^\mu = -\kappa \xi^\mu \tag{4}$$

   for some constant κ.

   (c) Compute the area A of the horizon at $r = 2GM$. Viewing A as a function of the mass M, show that

$$dM = \frac{\kappa}{8\pi G} dA \tag{5}$$

   Aside: this is the first law of black hole thermodynamics. Recall the first law of thermodynamics is $dE = TdS$. Here the energy is $E = M$, the temperature may be identified with $T = \kappa/2\pi$ and the entropy with $S = A/4G$. The fact that black holes have an entropy leads to many fundamental puzzles, some of which may have recently been *partially solved. See the following article for a taste:* `https://www.quantamagazine.org/the-black-hole-information-paradox-comes-to-an-end-20201029` *Other members of the VUB theoretical physics group are working on these issues!*

4. The following metric is a solution to Einstein's equations:

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2\,, \tag{6}$$

where $f(r) = 1 - \frac{2GM}{r} + \frac{GQ^2}{r^2}$, with $M > 0$, $r \geq 0$, $t \in \mathbb{R}$ and (θ, ϕ) are the usual two-sphere coordinates.

(a) What are the possible singularities of this metric?

(b) Suppose I claimed this was a vacuum solution. Without doing any explicit calculations, would you believe me? How could you be sure?

(c) Suppose $Q^2 = GM^2$. Find an explicit expression for a function $r^*(r)$ such that $t \pm r^*$ is constant on radial null geodesics. Explain what this tells you about the possible singularities of part (a).