General Relativity - Exercises - Lecture 8
VUB MSc in Physics & Astronomy, 2021-22

1. The *Schwarzschild radius* of a body of mass M is given by $r_s = 2GM/c^2$. Work this out for the Sun. How does it compare to the radius of the Sun itself?

2. Suppose that our communicative friends, Alice and Bob, have now made their way to Schwarzschild spacetime. Alice is at rest at $r = r_A$, and Bob is at rest at $r = r_B$. Alice sends signals at intervals $\Delta\tau_A$ to Bob, who receives them at intervals $\Delta\tau_B$. Show that

$$\Delta\tau_B = \left(1 - \frac{2GM}{r_B}\right)^{1/2} \left(1 - \frac{2GM}{r_A}\right)^{-1/2} \Delta\tau_A \,, \tag{1}$$

and hence that if Alice positions herself at $r_A = 2GM$ that $\Delta\tau_B \to \infty$: Bob will see Alice as being *frozen* at the Schwarzschild radius (equivalently, there is an infinite red shift).

3. Recall the energy of a photon is related to its frequency by $\mathcal{E} = \hbar\omega$. Consider a photon moving on an affine geodesic with parameter λ such that the tangent to the geodesic is identified with the photon's four-momentum, $p^\mu = \frac{dx^\mu}{d\lambda}$. An observer with four-velocity U^μ measures the energy as $\mathcal{E} = -g_{\mu\nu}U^\mu p^\nu$. In the Schwarzschild spacetime, use this expression along with the fact that $E = -K_\mu \frac{dx^\mu}{d\lambda}$ is conserved on geodesics to reproduce the result of exercise 2.

4. (*Hartle 9.8*) A spaceship is moving in a circular orbit at $r = 7M$. What is the period of the orbit as measured at infinity? What is the period of the orbit as measured in the spaceship?

5. Consider null geodesics in the Schwarzschild spacetime in the equatorial plane, $\theta = \frac{\pi}{2}$.

   (a) Show that
$$\left(\frac{dr}{d\phi}\right)^2 = r^4 \left(\frac{1}{b^2} - \frac{f(r)}{r^2}\right) . \tag{2}$$

   (b) Consider an observer sitting at radius $r = r_O$. Consider all the light that this observer sees, i.e. the null geodesics which arrive at the observer from the past. Show that the angle ψ between such a null geodesic and the radial direction is such that
$$\sin^2\psi = \frac{b^2 f(r_O)}{r_O^2} . \tag{3}$$

   (Hint: suppose that under an infinitesimal shift in the radial direction $(r_O, \phi_0) \to (r_O + dr, \phi_0)$, the geodesic shifts to $(r_O + dr, \phi_0 + d\phi)$. Express this infinitesimal deviation in terms of displacement vectors and use the angle formula from **[Exercises 5, Question 1]**.)

(c) Consider light rays emitted by sources positioned at $r > r_O$. These either fall into the black hole at $r = 2GM$ or bounce off the effective potential and reflect back out to infinity. The observer at $r = r_O$ will only see the latter. The limiting case of light rays that they observe consist of photons which follow orbits asymptotically close towards the unstable circular orbit at $r = 3GM$. The boundary of the *black hole shadow* (see photo on last page) therefore corresponds to light rays on this orbit. Hence show that it is characterised by

$$\sin^2 \psi = \frac{27(GM)^2 f(r_O)}{r_O^2} \tag{4}$$

Discuss the appearance of the black hole shadow as r_O varies. Show that for large r_O,

$$\sin \psi \approx \sqrt{3}\frac{3GM}{r_O} \tag{5}$$

Compare this to the Euclidean formula for the angular diameter of a sphere of radius $3GM$. (The angular diameter is related to the angle ψ by $\delta = 2\psi$.)

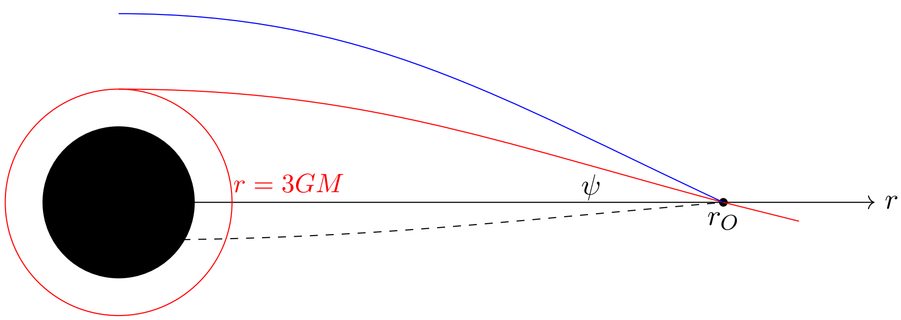

Figure 1: Null geodesics arriving at $r = r_O$: i) coming from infinity after deflection by the black hole (blue line), ii) coming from the unstable photon orbit at $r = 3GM$ (red line), or iii) coming from the Schwarzschild radius at $r = 2GM$ (dotted line). Case i) corresponds to observed light signals, case iii) does not, and case ii) is the border between the two.

6. Consider the 2-d spacetime with metric

$$ds^2 = -x^2 dt^2 + dx^2 \,, \tag{6}$$

where $x > 0$ and $t \in (-\infty, \infty)$. What prevents us from letting $x \in (-\infty, \infty)$?

(a) Compute the general form of null and timelike geodesics.

(b) Let $U = -xe^{-t}$, $V = xe^{t}$. Compute the metric in the new coordinates. What spacetime is this?

Figure 2: Actual image of a black hole, with the shadow corresponding to the black disc at centre. Source: the Event Horizon Telescope.