General Relativity - Exercises - Lecture 7
VUB MSc in Physics & Astronomy, 2021-22

1. Explain why the Maxwell equation $\partial_{[\mu}F_{\nu\rho]} = 0$ is unchanged on minimal coupling $\partial_\mu \to \nabla_\mu$.

2. (Carroll 3.12) Show that any Killing vector K^μ obeys

$$\nabla_\mu \nabla_\nu K^\rho = R^\rho{}_{\nu\mu\sigma} K^\sigma \,, \quad K^\mu \nabla_\mu R = 0 \tag{1}$$

3. The metric on S^2 is $ds^2 = (d\theta)^2 + \sin^2\theta (d\phi)^2$. Show that the following vector fields

$$\begin{aligned} K_{(1)} &= -\sin\phi \partial_\theta - \cot\theta \cos\phi \partial_\phi \,, \\ K_{(2)} &= \cos\phi \partial_\theta - \cot\theta \sin\phi \partial_\phi \,, \\ K_{(3)} &= \partial_\phi \,, \end{aligned} \tag{2}$$

are indeed Killing vectors of the sphere, and that they obey

$$[K_{(i)}, K_{(j)}] = -\epsilon_{ijk} K_{(k)} \tag{3}$$

under the commutator of vector fields. What is this algebra?

4. (Carroll 3.13) Find a complete set of Killing vectors for the spacetime with metric

$$ds^2 = -2du dv + a^2(u) dx^2 + b^2(u) dy^2 \tag{4}$$

where a, b are unspecified functions of u. Hint: there are 5 in total, and all have $K^u = 0$.

5. Consider again the metric for the expanding universe:

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij}(x) dx^i dx^j \,, \tag{5}$$

for which the Christoffel symbols are:

$$\Gamma_{ij}{}^t = a\dot{a}\gamma_{ij} \,, \quad \Gamma_{tj}{}^i = \frac{1}{a}\dot{a}\delta^i_j \,, \quad \Gamma_{jk}{}^i = \frac{1}{2}\gamma^{il}\left(\partial_j \gamma_{kl} + \partial_k \gamma_{jl} - \partial_l \gamma_{jk}\right) \,, \tag{6}$$

where $\dot{a} \equiv \frac{da}{dt}$. Now take coordinates $x^i = (r, \theta, \phi)$ such that

$$\gamma_{ij} dx^i dx^j = \frac{dr^2}{1 - kr^2} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \,, \tag{7}$$

where k is either ± 1 or 0. Compute the Ricci tensor and Ricci scalar. (The result is in the course notes, eqns (10.22) and (10.23), and we will need it when we study cosmology.)