General Relativity - Exercises - Lecture 6
VUB MSc in Physics & Astronomy, 2021-22

1. Perhaps the easiest way to compute the components of the Levi-Civita connection is to start with the following alternative action principle for particle trajectories $x^\mu(\lambda)$:

$$S[x] = \int d\lambda \frac{1}{2} g_{\mu\nu}(x(\lambda)) \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} \,. \tag{1}$$

Show that the resulting equation of motion is the geodesic equation of the Levi-Civita connection.

As an example, consider the sphere metric for which this gives

$$S[\theta, \phi] = \int d\lambda \frac{1}{2} (\dot{\theta}^2 + \sin^2\theta \dot{\phi}^2) \tag{2}$$

where dots denote λ derivatives. Show that the equations of motion are:

$$\ddot{\theta} - \sin\theta\cos\theta\dot{\phi}^2 = 0 \,, \quad \ddot{\phi}\sin^2\theta - 2\sin\theta\cos\theta\dot{\theta}\dot{\phi} = 0 \tag{3}$$

and hence the non-zero components of $\Gamma_{\mu\nu}{}^\rho$ are:

$$\Gamma_{\phi\phi}{}^\theta = -\sin\theta\cos\theta \,, \quad \Gamma_{\phi\theta}{}^\phi = \Gamma_{\theta\phi}{}^\phi = \frac{\cos\theta}{\sin\theta} \,. \tag{4}$$

2. Verify that

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - T_{\mu\nu}{}^\sigma \nabla_\sigma V^\rho \,, \tag{5}$$

where

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma_{\nu\sigma}{}^\rho - \partial_\nu \Gamma_{\mu\sigma}{}^\rho + \Gamma_{\mu\lambda}{}^\rho \Gamma_{\nu\sigma}{}^\lambda - \Gamma_{\nu\lambda}{}^\rho \Gamma_{\mu\sigma}{}^\lambda \,, \quad T_{\mu\nu}{}^\rho = 2\Gamma_{[\mu\nu]}{}^\rho \,. \tag{6}$$

Hence or similarly show that

$$[\nabla_\mu, \nabla_\nu] V_\rho = -R^\lambda{}_{\rho\mu\nu} V_\lambda - T_{\mu\nu}{}^\lambda \nabla_\lambda V_\rho \,. \tag{7}$$

3. Verify that the Riemann curvature tensor measures the failure of parallel transport to commute when transporting a vector between two points via two separate paths, by working through the calculations in the course notes leading to eqn (6.13) (page 71 to 73).

4. Verify the geodesic deviation equation, by working through the calculations in the course notes leading to eqn (6.20) (page 73 to 74).