General Relativity - Exercises - Lecture 5
VUB MSc in Physics & Astronomy, 2021-22

1. Given two curves that intersect at a point p, with tangent vectors U and V at p, we can define the angle between the curves using the (assumed Euclidean) metric g

$$\cos\theta = \frac{g(U,V)}{|U||V|} \tag{1}$$

where $|U|^2 \equiv g(U,U)$, $|V|^2 \equiv g(V,V)$. Consider the stereographic projection from the sphere to the plane described in **[Homework 1, Question 2]**. Using the metric on the sphere given by **[Homework 1, Eqn (5)]**, show that stereographic projection from the sphere to the plane preserves angles.

2. We define the area (in two-dimensions) or volume (in higher-dimensions) of a region U in a chart with coordinates u by

$$A(U) = \int_U d^d u \sqrt{|\det g|}\,. \tag{2}$$

Consider the stereographic projection from the sphere to the plane described in **[Homework 1, Question 2]**. Using the metric on the sphere given by **[Homework 1, Eqn (5)]**, show that stereographic projection from the sphere to the plane does not preserve areas (i.e. the area of a region on a sphere is not the same as the area of the region it projects to on the plane).

3. A physically relevant metric is

$$ds^2 = -dt^2 + a(t)^2 \gamma_{ij}(x) dx^i dx^j\,, \tag{3}$$

which describes an expanding (or contracting) universe with scale factor $a(t)$. Here γ_{ij} is a three-dimensional Euclidean metric which depends on the three-dimensional coordinates x^i, $i = 1,2,3$, and is independent of the time coordinate t.

Show that the Christoffel symbols are (where $\dot{a} \equiv \frac{da}{dt}$):

$$\Gamma_{ij}{}^{t} = a\dot{a}\gamma_{ij}\,, \quad \Gamma_{tj}{}^{i} = \frac{1}{a}\dot{a}\delta^i_j\,, \quad \Gamma_{jk}{}^{i} = \frac{1}{2}\gamma^{il}\left(\partial_j\gamma_{kl} + \partial_k\gamma_{jl} - \partial_l\gamma_{jk}\right)\,, \tag{4}$$

4. For the metric (3), assume $\gamma_{ij} = \delta_{ij}$ and obtain the form of null geodesics $x^\mu(\lambda)$ travelling in the (t, x^1) directions i.e. with $dx^2/d\lambda = dx^3/d\lambda = 0$, for

   (a) $a(t) = t^q$, with $0 < q < 1$. In this case we restrict $t \in (0, \infty)$.

   (b) $a(t) = e^{t/L}$, where L is some lengthscale. In this case we have $t \in (-\infty, \infty)$.

Sketch the null geodesics in both cases. The set of null geodesics through any particular point defines the past and future lightcones at that point. What can you infer about the causal structure of the above two cases. What happens in the far past and far future?

5. Show explicitly that the partial derivative of a covector, $\partial_\mu \omega_\nu$, does not transform as a tensor would under a charge of coordinates $x^\mu \mapsto x'^\mu(x)$. Viewing the covector as a 1-form, show that exterior derivative, $(d\omega)_{\mu\nu} = 2\partial_{[\mu}\omega_{\nu]}$ does transform as a tensor.

6. Let ∇ denote a connection/covariant derivative. Define the connection components $\Gamma_{\mu\nu}{}^\rho$ via

$$\nabla_{e_{(\mu)}} e_{(\nu)} \equiv \Gamma_{\mu\nu}{}^\rho e_\rho \,. \tag{5}$$

Show that under a general change of basis, $e'_{(\mu)} = (A^{-1})^\nu{}_\mu e_{(\nu)}$ we have

$$\Gamma'_{\nu\rho}{}^\mu = A^\mu{}_\kappa (A^{-1})^\lambda{}_\nu (A^{-1})^\sigma{}_\rho \Gamma_{\lambda\sigma}{}^\kappa + A^\mu{}_\kappa (A^{-1})^\sigma{}_\nu e_{(\sigma)}((A^{-1})^\kappa{}_\rho) \,. \tag{6}$$

7. Verify explicitly that the Levi-Civita connection transforms as a connection should under change of coordinates.

8. Show using the Leibniz rule on $\nabla_\mu(\omega_\nu X^\nu)$ that the covariant derivative on covectors is

$$\nabla_\mu \omega_\nu = \partial_\mu \omega_\nu - \Gamma_{\mu\nu}{}^\rho \omega_\rho \,. \tag{7}$$

Hence extrapolate to a general tensor:

$$\begin{aligned} \nabla_\mu T^{\nu_1 \dots \nu_r}{}_{\rho_1 \dots \rho_s} =& \partial_\mu T^{\nu_1 \dots \nu_r}{}_{\rho_1 \dots \rho_s} \\ &+ \Gamma_{\mu\sigma}{}^{\nu_1} T^{\sigma \dots \nu_r}{}_{\rho_1 \dots \rho_s} + \dots + \Gamma_{\mu\sigma}{}^{\nu_r} T^{\nu_1 \dots \sigma}{}_{\rho_1 \dots \rho_s} \\ &- \Gamma_{\mu\rho_1}{}^\sigma T^{\nu_1 \dots \nu_r}{}_{\sigma \dots \rho_s} - \dots - \Gamma_{\mu\rho_s}{}^\sigma T^{\nu_1 \dots \nu_r}{}_{\rho_1 \dots \sigma} \,. \end{aligned} \tag{8}$$

9. If ∇_μ is a metric compatible covariant derivative, show that $\nabla_\mu g^{\nu\rho} = 0$.

10. Let $|g| \equiv |\det g|$ denote the absolute value of the determinant of the metric. For $\Gamma_{\mu\nu}{}^\rho$ the Levi-Civita connection, show that $\Gamma_{\mu\nu}{}^\nu = \frac{1}{2}\partial_\mu \ln |g|$.

    If $|g|$ was a scalar, then $\nabla_\mu |g| = \partial_\mu |g|$, and this would not be zero, even though the metric itself is covariantly constant. Why is $|g|$ not a scalar?

    The covariant derivative of a scalar density ϕ of weight ω is defined by

$$\nabla_\mu \phi = \partial_\mu \phi + \omega \Gamma_{\mu\nu}{}^\nu \phi \,. \tag{9}$$

Check that this definition guarantees $\nabla_\mu |g| = 0$ for a natural choice of ω.