**General Relativity - Exercises - Lecture 4**
**VUB MSc in Physics & Astronomy, 2021-22**

1. (Carroll 2.6) Consider $\mathbb{R}^3$ in spherical polar coordinates.

   (a) Consider the curve

   $$x(\lambda) = \cos\lambda\,, \quad y(\lambda) = \sin\lambda\,, \quad z(\lambda) = \lambda\,, \tag{1}$$

   given in Cartesian coordinates. Express the path of the curve in spherical coordinates.

   (b) Calculate the components of the tangent vector to the curve in both Cartesian and spherical polar coordinates.

2. By definition, vectors, and hence vector fields, act linearly on functions: if $X$ is a vector field, and $f, g$ are functions, then $X(\alpha f + \beta g) = \alpha X(f) + \beta X(g)$, for $\alpha, \beta$ constants, and $X(fg) = X(f)g + fX(g)$. Verify that these properties are obeyed by the commutator $[X, Y]$ of two vector fields, and also show that the commutator obeys the Jacobi identity:

   $$[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\,, \tag{2}$$

   where $X, Y, Z$ are vector fields.

3. We defined the contraction of $\mathcal{T}$ a $(r, s)$ tensor field by the expression

   $$\begin{aligned}(\mathcal{T})'&(\omega_1, \ldots, \omega_{r-1}, X_1, \ldots X_{s-1}) \\ &= \mathcal{T}(\omega_1, \ldots, \omega_{i-1}, \theta^{(\mu)}, \omega_{i+1}, \ldots, \omega_r, X_1, \ldots, X_{j-1}, e_{(\mu)}, X_{j+1} \ldots, X_{s-1})\,,\end{aligned} \tag{3}$$

   for arbitrary covector fields $\omega_1, \ldots, \omega_{r-1}$ and vector fields $X_1, \ldots, X_{s-1}$, using a basis for vectors $e_{(\mu)}$ and the corresponding dual basis $\theta^{(\mu)}$. Show that this definition is basis independent.

4. Give an example of two linearly independent, nowhere-vanishing vector fields in $\mathbb{R}^2$ such that their commutator does not vanish. At each point, these provide a basis for the tangent space at that point, however it is not the coordinate basis. Construct and sketch the integral curves for the vector fields you have chosen.

5. Verify explicitly that $d(d\omega) = 0$, for $\omega$ a $p$-form

6. Verify explicitly that $\omega \wedge \eta = (-1)^{pq} \eta \wedge \omega$ for $\omega$ a $p$-form and $\eta$ a $q$-form.

*Supplementary reading:* In the lectures I did not give the full verification that the tangent space $T_pM$ is a vector space. Please look at the course notes pages 36 - 38 and work through these details yourselves.