General Relativity - Exercises - Lecture 3
VUB MSc in Physics & Astronomy, 2021-22

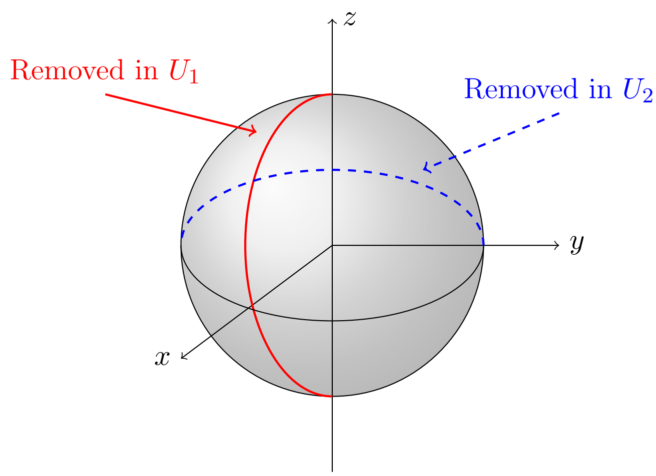
1. Consider the unit sphere defined as the set of all points (x, y, z) in $\mathbb{R}^3$ obeying $x^2+y^2+z^2 = 1$. It can be parameterised by $(x, y, z) = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$, with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi)$. These naive coordinates are problematic, as the coordinate intervals are not open and furthermore at $\theta = 0$ or π (the north and south poles) ϕ is arbitrary.

   Consider instead the following two charts.

   We define a first chart (U_1, φ_1) by restricting to $\theta \in (0, \pi)$ and $\phi \in (0, 2\pi)$. This means that U_1 is the sphere with the arc given by $(x \geq 0, y = 0)$ removed (corresponding to $\theta = 0, \pi$ and $\phi = 0$), and $\varphi_1 : U_1 \to (\theta, \phi) \in \{(0, \pi) \times (0, 2\pi)\} \subset \mathbb{R}^2$.

   A second chart (U_2, φ_2) is then give by $(x, y, z) = (-\sin\theta'\cos\phi', \cos\theta', \sin\theta'\sin\phi')$ with $\theta' \in (0, \pi)$, $\phi' \in (0, 2\pi)$. This means that U_2 is the sphere with the arc $(x \leq 0, z = 0)$ removed (corresponding to $\theta' = 0, \pi$ and $\phi' = 0$), and and $\varphi_2 : U_2 \to (\theta', \phi') \in \{(0, \pi) \times (0, 2\pi)\} \subset \mathbb{R}^2$.

   Show that these two charts provide an atlas for the sphere: firstly convince yourself that $S^2 = U_1 \cup U_2$ (see figure below), and then check that the changes of coordinates $\varphi_1 \circ \varphi_2^{-1}$, $\varphi_2 \circ \varphi_1^{-1}$ are smooth on $U_1 \cap U_2$.



2. (Carroll 2.3) Show that the two-dimensional torus is a manifold by explicitly constructing an appropriate atlas.

3. (Carroll 2.1) Show that the infinite cylinder $\mathbb{R} \times S^1$ can be covered with just one chart, in contrast to the circle S^1.

Supplementary reading: To compensate for the absence of physics on both this and the following exercise sheet, and to whet your appetite for the rest of the course, you may like to read about last year's Nobel Prize in Physics, starting with the popular account:

https://www.nobelprize.org/uploads/2020/10/popular-physicsprize2020-1.pdf
and then the more detailed scientific background:

https://www.nobelprize.org/uploads/2020/10/advanced-physicsprize2020.pdf
We will learn about black holes, orbits and singularities soon enough!