General Relativity - Exercises - Lecture 2
VUB MSc in Physics & Astronomy, 2021-22

1. (Carroll 1.6) In three spatial dimensions, let p be the point with Cartesian coordinates $(x, y, z) = (1, 0, -1)$. Consider the following curves through p:

$$x^i(\lambda) = (\lambda, (\lambda-1)^2, -\lambda)\,, \quad x^i(\mu) = (\cos\mu, \sin\mu, \mu - 1)\,, \quad x^i(\sigma) = (\sigma^2, \sigma^3 + \sigma^2, \sigma)\,. \tag{1}$$

   (a) Calculate the components of the tangent vectors to these curves at p in the coordinate basis $\{\partial_x, \partial_y, \partial_z\}$.

   (b) Let $f(x, y, z) = x^2 + y^2 - yz$. Calculate $df/d\lambda$, $df/d\mu$ and $df/d\sigma$.

2. (Carroll 1.7) Let $X^{\mu\nu}$ and V^μ have components

$$X^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}, \quad V^\mu = (-1, 2, 0, -2)\,. \tag{2}$$

   Find the components of

   (a) $X^\mu{}_\nu$

   (b) $X_\mu{}^\nu$

   (c) $X^{(\mu\nu)}$

   (d) $X_{[\mu\nu]}$

   (e) $X^\lambda{}_\lambda$

   (f) $V^\mu V_\mu$

   (g) $V_\mu X^{\mu\nu}$

3. (Carroll 1.10) Using the tensor transformation of $F_{\mu\nu}$, write down how the electric and magnetic field 3-vectors $\vec{E}$ and $\vec{B}$ transform under

   (a) a rotation about the y-axis,

   (b) a boost along the z-axis.

4. Show that $\vec{B}^2 - \vec{E}^2$ and $\vec{E} \cdot \vec{B}$ are invariant under proper Lorentz transformations.

5. Let $A_{\mu\nu}$ be antisymmetric and $S^{\mu\nu}$ be symmetric. Show that $A_{\mu\nu}S^{\mu\nu} = 0$. If $V_{\mu\nu}$ and $V^{\mu\nu}$ are tensors of no fixed symmetry, show that $A_{\mu\nu}V^{\mu\nu} = A_{\mu\nu}V^{[\mu\nu]}$ and $S^{\mu\nu}V_{\mu\nu} = S^{\mu\nu}V_{(\mu\nu)}$.

6. Let $A_{\mu\nu}$ and $A_{\mu\nu\rho}$ be antisymmetric tensors, and B_μ a covector. Show that

$$3A_{[\mu\nu}B_{\rho]} = A_{\mu\nu}B_\rho + A_{\nu\rho}B_\mu + A_{\rho\mu}B_\nu\,, \tag{3}$$

$$4A_{[\mu\nu\rho}B_{\sigma]} = A_{\mu\nu\rho}B_\sigma - A_{\nu\rho\sigma}B_\mu + A_{\rho\sigma\mu}B_\nu - A_{\sigma\mu\nu}B_\rho\,, \tag{4}$$

Now suppose μ is a four-dimensional index and $A_{\mu\nu\rho\sigma}$ an antisymmetric tensor. Show that

$$A_{\mu\nu\rho\sigma}B_\lambda + A_{\nu\rho\sigma\lambda}B_\mu + A_{\rho\sigma\lambda\mu}B_\nu + A_{\sigma\lambda\mu\nu}B_\rho + A_{\lambda\mu\nu\rho}B_\sigma = 0\,. \tag{5}$$