**General Relativity - Exercises - Lectures 11 & 12**
**VUB MSc in Physics & Astronomy, 2021-22**

1. Verify that $\delta_\xi R_{\mu\nu\rho\sigma} = 0$.

2. The solutions to the wave equation $\partial^\rho \partial_\rho \bar{h}_{\mu\nu} = 0$ are

$$\bar{h}_{\mu\nu}(x) = \mathrm{Re}(H_{\mu\nu} e^{ik\cdot x}) \,, \quad k \cdot x \equiv k_\mu x^\mu \,, \tag{1}$$

where $H_{\mu\nu}$ is a constant symmetric complex tensor and $k_\mu$ is a null vector, $k_\mu k^\mu = 0$. The gauge condition $\partial^\nu \bar{h}_{\mu\nu} = 0$ implies $k^\nu H_{\mu\nu} = 0$. The residual gauge freedom given by transformations with parameter $\xi_\mu$ obeying $\partial^\nu \partial_\nu \xi_\mu = 0$ can be described by writing these explicitly has $\xi_\mu(x) = \mathrm{Re}(X_\mu e^{ik\cdot x})$. Under such a transformation,

$$H_{\mu\nu} \to H_{\mu\nu} + i(k_\mu X_\nu + k_\nu X_\mu - \eta_{\mu\nu} k^\rho X_\rho) \,. \tag{2}$$

Show that we can always choose a transformation $X_\mu$ to ensure that:

$$H_{0\mu} = 0 \,, \quad H^\mu{}_\mu = 0 \,. \tag{3}$$

3. Let $(t, x^i)$ denote the coordinates used in transverse traceless gauge. Define new coordinates by

$$\hat{t} = t \,, \quad \hat{x}^i = x^i + \frac{1}{2} \delta^{ik} h^{\mathrm{TT}}_{kj}(t) x^j \,. \tag{4}$$

Given that the positions $x^i$ are unchanged in the presence of the wave, work out the behaviour of test particles described by the new coordinates $\hat{x}^i$, and compare it to the results obtained via solving the geodesic deviation equation.

Show that the metric in terms of these new coordinates is:

$$ds^2 \approx -d\hat{t}^2 + \delta_{ij} d\hat{x}^i d\hat{x}^j \tag{5}$$

neglecting terms of second order in the perturbation, and assuming that the wavelength $\lambda \sim 1/\omega$ of the gravitational wave is much bigger than the coordinate distances involved so as to also neglect terms involving $|x^i|/\lambda << 1$.

4. Here is a selection of notable gravitational wave detection events:

- GW150914
- GW170817
- GW190814

- GW190521

Consulting the LIGO website `https://www.ligo.caltech.edu/`, or other sources, discuss what physics we can learn from each of these observations. The detection papers can be found at: `https://www.ligo.caltech.edu/page/detection-companion-papers`