General Relativity - Exercises - Lecture 10
VUB MSc in Physics & Astronomy, 2021-22

1. *Global coordinates on de Sitter.* The de Sitter (dS) spacetime is defined by the equation

$$-(x^0)^2 + (x^1)^2 + \ldots (x^4)^2 = R^2\,, \tag{1}$$

in (a fictitious) $\mathbb{R}^5$ with the Minkowski metric. Define coordinates on the curve (1) by

$$\begin{aligned} x^0 &= L \sinh \frac{t}{L}\,, \\ x^1 &= L \cosh \frac{t}{L} \sin\chi \sin\theta \cos\phi\,, \\ x^2 &= L \cosh \frac{t}{L} \sin\chi \sin\theta \sin\phi\,, \\ x^3 &= L \cosh \frac{t}{L} \sin\chi \cos\theta\,, \\ x^4 &= L \cosh \frac{t}{L} \cos\chi\,, \end{aligned} \tag{2}$$

where $t \in (-\infty, \infty)$, and (χ, θ, ϕ) are S^3 coordinates. Verify that this parametrisation obeys (1) and show that the metric of de Sitter spacetime is then:

$$ds^2 = -dt^2 + L^2 \cosh^2 \tfrac{t}{L} (d\chi^2 + \sin^2\chi d\Omega^2)\,. \tag{3}$$

2. Consider two metrics related by a conformal rescaling $\bar{g}_{\mu\nu}(x) = \Omega^2(x) g_{\mu\nu}(x)$. Compute the Christoffel symbols of the metric $\bar{g}_{\mu\nu}$ in terms of those of the metric $g_{\mu\nu}$. Hence show that the two metrics have the same null geodesics.

3. Starting with $(1+3)$-dimensional Minkowski spacetime in polar coordinates, make the coordinate transformations:

$$u = t - r\,, \quad v = t + r\,, \tag{4}$$

and

$$u' = \tan^{-1} u\,, \quad v' = \tan^{-1} v \tag{5}$$

and finally

$$u' = \frac{1}{2}(t' - r')\,, \quad v' = \frac{1}{2}(t' + r') \tag{6}$$

and hence draw the Penrose diagram for flat spacetime. What changes in $(1+1)$-dimensions?

4. The Friedmann equations are:

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2}\,, \tag{7}$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p)\,. \tag{8}$$

The conservation of the energy-momentum tensor gives:

$$\dot{\rho} + 3\frac{\dot{a}}{a}(\rho + p) = 0\,, \tag{9}$$

Show that (8) is implied by (7) and (9).

Define conformal time η such that $d\eta = dt/a$. With $p = w\rho$, show that in terms of conformal time η the Friedmann equation (7) is:

$$\left(\frac{da}{d\eta}\right)^2 = \frac{8\pi G\rho_0}{3} a_0^{3(1+w)} a^{1-3w} - ka^2\,. \tag{10}$$

5. *Matter dominated FLRW universes.* Consider the Friedmann equation (10) in terms of conformal time for a universe dominated by matter, $w = 0$. Letting $C^2 = \frac{8\pi G\rho_0}{3}a_0^3$ and taking $t = \eta = 0$ to correspond to the big bang, with $a(\eta = 0) = 0$, show that we have:

$$\begin{array}{lll} \text{Closed } (k=1) & a(\eta) = \frac{1}{2}C^2(1-\cos\eta) & t = \frac{1}{2}C^2(\eta - \sin\eta) \\ \text{Open } (k=-1) & a(\eta) = \frac{1}{2}C^2(\cosh\eta - 1) & t = \frac{1}{2}C^2(\sinh\eta - \eta) \\ \text{Flat } (k=0) & a(\eta) = \frac{1}{4}C^2\eta^2 & t = \frac{1}{12}C^2\eta^3\,. \end{array} \tag{11}$$

For each case, graph the scale function as a function of conformal time and comment on the future fate of each universe.

6. *Radiation dominated FLRW universes.* Consider the Friedmann equation (10) in terms of conformal time for a universe dominated by radiation, $w = 1/3$. Letting $C^2 = \frac{8\pi G\rho_0}{3}a_0^4$ and taking $t = \eta = 0$ to correspond to the big bang, with $a(\eta = 0) = 0$, show that we have:

$$\begin{array}{lll} \text{Closed } (k=1) & a(\eta) = C\sin\eta & t = C(1-\cos\eta) \\ \text{Open } (k=-1) & a(\eta) = C\sinh\eta & t = C(\cosh\eta - 1) \\ \text{Flat } (k=0) & a(\eta) = C\eta & t = \frac{1}{2}C\eta^2\,. \end{array} \tag{12}$$

For each case, graph the scale function as a function of conformal time and comment on the future fate of each universe.