General Relativity - Exercises - Lecture 1
VUB MSc in Physics & Astronomy, 2021-22

1. Let $\Lambda^{\mu}{}_{\nu}$ be a Lorentz transformation. Show that $(\Lambda^{0}{}_{0})^2 \geq 1$.

2. Verify explicitly that the following are Lorentz transformations:

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \theta \in [0, 2\pi) , \tag{1}$$

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \cosh\phi & -\sinh\phi & 0 & 0 \\ -\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \quad \phi \in (-\infty, \infty) , \tag{2}$$

3. Show that the boost transformation (2) is equivalent to

$$\begin{aligned} t' &= \gamma(t - vx/c^2) , \\ x' &= \gamma(x - vt) , \\ y' &= y \\ z' &= z , \end{aligned} \tag{3}$$

where $\gamma = 1/\sqrt{1-(v/c)^2}$ and $v = c\tanh\phi$. What do these transformations reduce to for $v/c << 1$?

4. Spherical coordinates are defined by

$$x = r\sin\theta\cos\phi , \quad y = r\sin\theta\sin\phi , \quad z = r\cos\theta . \tag{4}$$

Show that the components of the flat Minkowski metric in these coordinates are $\eta_{tt} = -1$, $\eta_{rr} = 1$, $\eta_{\theta\theta} = r^2$, $\eta_{\phi\phi} = r^2\sin^2\theta$ and otherwise zero.

5. Show that for two timelike separated events, there always exists an inertial frame in which $\Delta t \neq 0$, $\Delta x^i = 0$, and that for two spacelike separated events there always exists an inertial frame in which $\Delta t = 0$, $\Delta x^i \neq 0$.

6. Resolve the garage paradox: a very fast car drives towards (and through) a garage of smaller proper length. A (classical) mechanic at rest in the frame of the garage observes the car being Lorentz contracted and so thinks that it fit inside the garage. The driver of the car on the

other hand observes that the garage Lorentz contracted and was too small to contain the car. Draw a spacetime diagram to account for these observations.