General Relativity 2021/22 - Homework 3

Given: 10/12, due: 24/12.

1. *A toy model of black holes.* The following metric is a solution to a two-dimensional version of general relativity:
$$ds^2 = -\tanh^2 r\, dt^2 + dr^2\,. \tag{1}$$
(More precisely, this is a non-vacuum solution with matter present in the form of a scalar field; but this is not important for this question.)

   (a) *(1 pt)* Recall (from your exercise classes, or from Carroll section 2.8) that the Levi-Civita symbol in n-dimensions, $\tilde{\epsilon}_{\mu_1\ldots\mu_n}$, is related to the Levi-Civita tensor, $\epsilon_{\mu_1\ldots\mu_n} = \sqrt{|g|}\tilde{\epsilon}_{\mu_1\ldots\mu_n}$. Explain why any rank n antisymmetric tensor in n dimensions is necessarily proportional to the Levi-Civita tensor.

   Now let $n = 2$. Hence explain why the Riemann tensor (of the Levi-Civita connection) in two-dimensions must be of the form
$$R_{\mu\nu\rho\sigma} \propto R g_{\mu[\rho} g_{\sigma]\nu}\,, \tag{2}$$
   where R is the Ricci scalar, and determine the constant of proportionality.

   (b) *(2 pts)* For the metric (1), compute the Christoffel symbols, Ricci tensor, and Ricci scalar. Is $r = 0$ a coordinate or curvature singularity?

   (c) *(2 pts)* Define a tortoise coordinate
$$r_* = r + \ln(1 - e^{-2r}) \tag{3}$$
   and new coordinates
$$2v = e^{t+r_*}\,, \quad 2u = -e^{-t+r_*}\,. \tag{4}$$
   What is the metric in these coordinates? What is the Ricci scalar? (Please do not work it out from scratch!) Comment on what you find if you extend the ranges of the coordinates u and v.

2. *Flat slicing of de Sitter.* The de Sitter (dS) spacetime is defined by the equation
$$-(x^0)^2 + (x^1)^2 + \ldots (x^4)^2 = L^2\,, \tag{5}$$

in (a fictitious) $\mathbb{R}^5$ with the Minkowski metric. In the lectures, we used the parametrisation

$$\begin{aligned}
x^0 &= L\sinh\frac{t}{L}\,,\\
x^1 &= L\cosh\frac{t}{L}\,\sin\chi\sin\theta\cos\phi\,,\\
x^2 &= L\cosh\frac{t}{L}\,\sin\chi\sin\theta\sin\phi\,,\\
x^3 &= L\cosh\frac{t}{L}\,\sin\chi\cos\theta\,,\\
x^4 &= L\cosh\frac{t}{L}\,\cos\chi\,,
\end{aligned} \tag{6}$$

in terms of globally defined coordinates $t \in (-\infty, \infty)$, and (χ, θ, ϕ) which are S^3 coordinates.

(a) *(1 pt)* Show that the following parametrisation

$$\begin{aligned}
x^0 &= L\sinh\frac{\hat{t}}{L} + \frac{r^2}{2L}e^{\frac{\hat{t}}{L}}\,,\\
x^4 &= L\cosh\frac{\hat{t}}{L} - \frac{r^2}{2L}e^{\frac{\hat{t}}{L}}\,,\\
x^i &= e^{\frac{\hat{t}}{L}}\hat{x}^i\,,
\end{aligned} \tag{7}$$

where $r^2 \equiv \delta_{ij}\hat{x}^i\hat{x}^j$, $i,j = 1,2,3$, obeys the defining equation (5), and gives the metric for dS as:

$$ds^2 = -d\hat{t}^2 + e^{2\frac{\hat{t}}{L}}\delta_{ij}d\hat{x}^i d\hat{x}^j\,. \tag{8}$$

This is called the *flat slicing*, because surfaces of constant $\hat{t}$ are flat.

(b) *(2 pts)* Show that the flat FLRW solution for a universe dominated by a positive cosmological constant corresponds to the flat slicing of de Sitter.

(c) *(2 pts)* The coordinates (7) are only defined for $x^0 + x^4 \geq 0$. By equating (7) to the original coordinates used, (6), (using spherical polar coordinates for the $\hat{x}^i$), show that on the Penrose diagram, the flat slicing corresponds to the left upper triangular region $\eta \geq \chi - \frac{\pi}{2}$.