General Relativity 2021/22 - Homework 2

Given: 15/11, due: 03/12.

1. Consider the two-sphere with coordinates (θ, ϕ) and metric $ds^2 = d\theta^2 + \sin^2\theta d\phi^2$.

   (a) *(1 pt)* Show that lines of constant longitude ($\phi =$ constant) are geodesics, and that the only line of constant latitude ($\theta =$ constant) that is a geodesic is the equator ($\theta = \pi/2$).

   (b) *(1 pt)* Take a vector with components $V^\mu = (1, 0)$ and parallel transport it once around a circle of constant latitude. What are the components of the resulting vector, as a function of θ?

2. A good approximation to the metric outside the surface of the Earth is

$$ds^2 = -(1 + 2\Phi)dt^2 + (1 - 2\Phi)dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1}$$

   where $\Phi = -GM/r$ may be thought of as the usual Newtonian gravitational potential (assumed to be small), where G is Newton's constant and M is the mass of the Earth.

   (a) *(2 pts)* Compute the geodesic equations, and solve them for the case of a geodesic corresponding to a circular orbit around the equator of the Earth ($\theta = \pi/2$). What is $d\phi/dt$?

   (b) *(2 pts)* How much proper time elapses while a satellite at radius R completes one such orbit. How does this compare to the proper time elapsed on a clock stationary at radius R?

3. Consider the geodesic equation

$$\ddot{x}^\mu + \Gamma_{\nu\rho}{}^\mu(x)\dot{x}^\nu\dot{x}^\rho = 0\,, \tag{2}$$

   for an affine geodesic $x^\mu(\lambda)$, with $\dot{x}^\mu \equiv dx^\mu/d\lambda$, where $\Gamma_{\mu\nu}{}^\rho$ are the components of the Levi-Civita connection. Take as initial conditions

$$x^\mu(\lambda = 0) = \bar{x}^\mu\,, \quad \dot{x}^\mu(\lambda = 0) = X^\mu\,. \tag{3}$$

   (a) *(1 pt)* By Taylor expanding $x^\mu(\lambda)$ in the parameter λ, show that the solution to the geodesic equation has the form

$$x^\mu(\lambda) = \bar{x}^\mu + \lambda X^\mu - \frac{\lambda^2}{2}\bar{\Gamma}_{\nu\rho}{}^\mu X^\nu X^\rho - \frac{\lambda^3}{6}(\partial_\sigma\bar{\Gamma}_{\nu\rho}{}^\mu - \bar{\Gamma}_{\lambda\rho}{}^\mu\bar{\Gamma}_{\sigma\nu}{}^\lambda - \bar{\Gamma}_{\nu\lambda}{}^\mu\bar{\Gamma}_{\sigma\rho}{}^\lambda)X^\sigma X^\nu X^\rho + O(\lambda^4) \tag{4}$$

   where $\bar{\Gamma}_{\mu\nu}{}^\rho \equiv \Gamma_{\mu\nu}{}^\rho(\lambda = 0)$.

(b) *(1 pt)* The above solution is valid in any coordinate system. Suppose we take $x^\mu(\lambda = 0)$ to be some point p on the manifold M, and use Riemann normal coordinates around p. In this case, the coordinates of the points in the neighbourhood of p correspond to tangent vectors $X \in T_pM$, by following a geodesic through p with initial tangent vector X from $\lambda = 0$ to $\lambda = 1$. In this case we know that geodesics have the form $x^\mu(\lambda) = \lambda X^\mu$. Hence show that in Riemann normal coordinates that

$$\bar{\Gamma}_{\mu\nu}{}^{\rho} = 0\,, \quad \partial_{(\sigma}\bar{\Gamma}_{\nu\rho)}{}^{\mu} = 0\,, \tag{5}$$

where barred quantities are evaluated at $\lambda = 0$ i.e. at the origin ($\bar{x}^\mu = 0$) in Riemann normal coordinates. Show further that

$$\partial_\rho \bar{\Gamma}_{\mu\nu}{}^{\sigma} = \frac{1}{3}(\bar{R}^\sigma{}_{\mu\rho\nu} + \bar{R}^\sigma{}_{\nu\rho\mu})\,. \tag{6}$$

(c) *(2 pts)* By Taylor expanding in Riemann normal coordinates about the origin, show that an arbitrary second rank tensor $T_{\mu\nu}$ can be expressed as

$$T_{\mu\nu}(X) = \bar{T}_{\mu\nu} + X^\rho \bar{\nabla}_\rho \bar{T}_{\mu\nu} + \frac{1}{2} X^\rho X^\sigma (\bar{\nabla}_\rho \bar{\nabla}_\sigma \bar{T}_{\mu\nu} - \frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\mu}\bar{T}_{\lambda\nu} - \frac{1}{3}\bar{R}^\lambda{}_{\sigma\rho\nu}\bar{T}_{\mu\lambda}) + \dots \tag{7}$$

Explain how to relate this result to the statement that a non-vanishing Riemann curvature tensor prevents one from choosing coordinates such that the metric $g_{\mu\nu}$ is everywhere equal to the Minkowski metric.